Generalizations of the classical Weyl and Colin de Verdière’s formulas and the orbit method

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The classical Weyl formula expresses the leading term of the asymptotics of the counting function \(N(\lambda, H)\) of the spectrum of a self-adjoint operator \(H\) in an invariant form: one can “hear” the volume of the subset of the cotangent bundle where the symbol of the operator \(H\) is less than \(\lambda\). In particular, it is applicable to Schrödinger operators with electric potentials growing at infinity. The Weyl formula is formulated in an invariant form; however, it gives \(\pm\) for magnetic Schrödinger operators with magnetic tensors growing at infinity. For these operators, Colin de Verdière’s formula is known, but the form of the latter is not invariant. In this article, we suggest an invariant generalization of both Weyl’s and Colin de Verdière’s formulas for wide classes of Schrödinger operators with polynomial electric and magnetic fields. The construction is based on the orbit method due to Kirillov, and it allows one to hear the geometry of coadjoint orbits.

The aim of this article is to provide a unifying framework for various types of spectral asymptotics for Schrödinger operators.

Introduction

1.1 The Weyl Formula and Its Generalizations. Let

\[ H = H(a) + V = -\sum_{j=1}^{n} \left( \frac{\partial}{\partial x_j} + \sqrt{-1} a_j(x) \right)^2 + V(x) \]  

[1.1]

be a Schrödinger operator in \(\mathbb{R}^n\) with a real continuous semi-bounded electric potential \(V\) and magnetic potential \(a(x) = (a_1(x), \ldots, a_n(x)) \in C^2(\mathbb{R}^n, \mathbb{R}^n)\). The operator \(H\) admits a unique realization as a self-adjoint operator in \(L_2(\mathbb{R}^n)\), with the domain containing \(C_0^\infty(\mathbb{R}^n)\) (see ref. 1 and references therein). If the electric potential \(V\) grows regularly at infinity, it is well known that the spectrum of \(H(0) + V\) is discrete, and the counting function of the spectrum obeys the classical Weyl formula

\[ N(\lambda, H(0) + V) \sim (2\pi)^{-n} \int_{a(x, \xi) < \lambda} dxd\xi, \]  

[1.2]

Here, \(a(x, \xi) = \|\xi\|^2 + V(x)\) is the symbol of \(H(0) + V\), and \(f(\lambda) \sim g(\lambda)\) means that \(f(\lambda)/g(\lambda) \to 1\) as \(\lambda \to +\infty\). One easily rewrites 1.2 in the following form:

\[ N(\lambda, H(0) + V) \sim (2\pi)^{-n} |v_n| \int_{\mathbb{R}^n} (\lambda - V(x))^{n/2} dx, \]  

[1.3]

where \(|v_n|\) is the volume of the unit ball of \(\mathbb{R}^n\) and \(a_+ = \max \{0, a\}\) (e.g., refs. 2 and 3). The classical Weyl formula is applicable to many classes of operators and, in its classical form, was related to the (Dirichlet or Neumann) Laplacian on a bounded domain \(\Omega\), with symbol \(a(x, \xi)\). For the Laplacian, Eq. 1.3 is valid with \(V(x) = 0\) and the integration over \(\mathbb{R}^n\) replaced by integration over \(\Omega\); hence, 1.3 allows one “to hear the area of the drum.” If more information about the spectrum is available, then one can “hear” much more about the geometry of a “drum” (see refs. 4 and 5).

Refs. 6–10 show that the spectrum of \(H(0) + V\) can be discrete even if \(V\) does not grow in some directions, and for wide classes of degenerate potentials, the leading term of the asymptotics of \(N(\lambda, H(0) + V)\) is computed. The results of these articles agree with the general “uncertainty principle” stated in ref. 11; it seems that this principle provides upper and lower bounds, but it is difficult to use it to study spectral asymptotics. Note that, in many cases, asymptotic formulas are nonclassical in the sense that they do not agree with the “classical” formula (Eq. 1.2). The following three cases are possible: the classical Weyl formula holds (the so-called “weak degeneration case”); an analog of the classical Weyl formula with the operator-valued symbol parameterized by points of a set with a measure inherited from \(\mathbb{T}^*\mathbb{T}^n\) is valid (“strong degeneration case”); and the classical Weyl formula fails, but the leading term of the asymptotics is expressed in terms of an auxiliary scalar function and no operator-valued symbol is involved (“intermediate degeneration case”). In simple strong degeneration cases, an operator-valued symbol is parameterized by the cotangent bundle over a manifold of degeneration of \(V\), called \(M\), and the operator-valued analog of 1.2 is of the following form:

\[ N(\lambda, a(x, \xi)) dxd\xi, \]  

where \(r = \text{codim}M\), and for each \((x', \xi') \in \mathbb{T}^*M, a(x', \xi')\) is an operator in \(L_2(\mathbb{R}^n)\).

Similar types of asymptotic formulas hold for many other classes of differential operators, pseudodifferential operators, and boundary value problems (see refs. 9 and 12–14 and references therein).

1.2. Colin de Verdière’s Formula. If \(V = 0\) and the magnetic tensor \(B = [b_k], b_k(x) = \partial_x a_i(x) - \partial_x a_j(x),\) grows regularly at infinity, the leading term of the asymptotics was obtained in ref. 15:

\[ N(\lambda, H(a)) \sim \int_{\mathbb{R}^n} v_B(\lambda) dx, \]  

[1.4]

where \(v_B(\lambda)\) is defined as follows. Let rank \(B = 2r\), and let \(b_1 \geq b_2 \geq \cdots \geq b_r > 0\) be the positive eigenvalues of \(iB\). Then

\[ v_B(\lambda) = (2\pi)^{n+r} |v_{n-2r}| b_1 \cdots b_r \sum_{n_1, \ldots, n_r} (\lambda - \sum_j (2n_j + 1)b_j)^{n/2-r}. \]

Note that \(B, r,\) and the \(b_j\)’s values depend on \(x\). However, in the case of a Schrödinger operator with polynomial potentials, there is a dense open subset of \(\mathbb{R}^n\) of full measure on which \(B(x)\) has maximal rank, so one can replace the integral in 1.5 by the integral over this subset. Then, \(r\) will remain constant throughout the integration.

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1.3. The Case of Degenerate Potentials. In the general case, only upper and lower bounds for $N(\lambda, H(a) + V)$ are known (16). They are given in terms of a function $\Psi^* = \Psi^*_a, V$ constructed in ref. 1, for polynomial $V(\equiv 0)$ and $b_a$:

$$\Psi^*(x) = \sum_a |\tilde{\phi} \Psi^*(x)|^{1/(1+2)} + \sum_{\alpha, \beta} |\tilde{\phi} \Psi^*(x)|^{1/((1+2)}$$. \[1.6\]

In the case $B \neq 0$ not growing in some directions, the leading term of the asymptotics is unknown apart from a special case of Schrödinger operator (and Dirac operator) in 2D with homogeneous potentials (14).

The following observations indicate the direction where one should look for such a formula. Note the difference among formulas 1.2, 1.4, and 1.5: the first two are written in an invariant form, whereas the last one is similar to 1.3, which is a realization of the invariant formula 1.2. This observation suggests that there should be an invariant formula of which 1.5 is a realization. Moreover, one should expect that there is a general formula, with 1.2, 1.4, and 1.5 as special cases, and that this formula should work in some cases of degenerate potentials.

The Weyl and Colin de Verdière’s Formulas: A Unifying View

1.3. The Case of Degenerate Potentials. Assume that $\sigma(H) = \sigma(J)$, so $\rho$ is irreducible. The orbit method, due to Kirillov (17), provides a natural one-to-one correspondence between (unitary equivalence classes of) unitary irreducible representations of $G$ and orbits of the coadjoint action of $G$ on $g^*$. In particular, we let $\Omega_p \subset g^*$ denote the coadjoint orbit corresponding to $\rho$. Suppose that the magnetic potential $a = 0$, and that $V(x)$ grows regularly at infinity. The values of the symbol $a(x, \xi)$ appearing in the classical Weyl formula (1.2) can be interpreted as the images of $H^* \equiv H$ in a family of representations of $G$ on the 1D space $L^2(\mathbb{R})$. The family is parameterized by points of the orbit $\Omega_p$, and the measure $(2\pi)^{-d}dx d\xi$ coincides with the canonical (Kostant) measure on $\Omega_p$.

However, assume that $V = 0$ and the magnetic tensor $B(x)$ grows regularly at infinity. It is shown in ref. 18 that the formula of Colin de Verdière (1.5) can be written in the following form:

$$N(\lambda, H) \sim \int_{\Omega} N(\lambda, H_0) \, dv(\Theta) \text{ as } \lambda \to +\infty,$$ \[2.1\]

where $H_0$ is the image of $H^*$ in a certain unitary irreducible representation of $G$ on $L^2(\mathbb{R})$, $Q$ is a manifold parameterizing a family of such representations, and the measure $dv(\Theta)$ can be obtained in the following way. Let $Q \subset g^*$ be the union of the orbits corresponding to the representations parameterized by the points of $Q$. There is a natural “projection map” $p : \Omega_p \to Q$, such that the pushforward $\hat{v}$ of the canonical measure on $\Omega_p$ is a $G$-invariant measure on $Q$. One can decompose $\hat{v}$ as an integral of the canonical measures on the orbits contained in $Q$, with respect to a certain “quotient” measure on $Q/G$. Then, we take $v$ to be this quotient measure.

Let us explain the case $n = 2$ in detail. The magnetic tensor must be of the following form:

$$B(x) = \begin{pmatrix} 0 & b(x) \\ -b(x) & 0 \end{pmatrix},$$

where $b(x)$ is a polynomial, and because $B(x)$ grows regularly at infinity, we may assume without loss of generality that $b(x) > 0$ for $|x| \gg 0$. Note that the eigenvalues of $\sqrt{-1} B(x)$ are $\pm b(x)$.

The Lie algebra $g$ is generated by the operators $L_1 = \partial / \partial x_1 + \sqrt{-1} a_1(x)$ and $L_2 = \partial / \partial x_2 + \sqrt{-1} a_2(x)$, which satisfy $[L_1, L_2] = \sqrt{-1} b(x)$. Let us write $P_0 = \sqrt{-1} b(x)$, and let $P_1, \ldots, P_N$ be an arbitrary basis of the vector space spanned by all mixed partial derivatives of $P_0$ of all positive orders (i.e., not including $P_0$). Thus, $(L_1, L_2, P_0, P_1, \ldots, P_N)$ is a basis of $g$. We can now define a projection map $p : \Omega_p \to g^* \text{ by } p(f)(L_1) = f(L_1), p(f)(L_2) = f(L_2), p(f)(P_0) = f(P_0), p(f)(P_j) = 0 \text{ for } 1 \leq j \leq N$.

We will now show that if $Q$ is taken to be the image of this map, then $Q$ is $G$-stable, and the pushforward measure $\hat{v} = p_*(\mu_\rho)$ is $G$-invariant (where $\mu_\rho$ is the Kostant measure on the orbit $\Omega_p$).

Moreover, if $Q = Q/G$ and $v$ is the measure on $Q$ induced by...
It follows from Proposition 3.6 and Proposition 3.7 that the orbit \( \Omega_0 \) admits a parameterization \( \varphi : [0, 1] \to \mathfrak{g} \) given by \( \phi(\xi_1, \xi_2, \eta_1, \eta_2) = \xi_1 \) for \( t = 1, 2 \), and
\[
\phi(\xi_1, \xi_2, \eta_1, \eta_2)(P_t) = -\sqrt{-1}P_t(y_1, y_2) \quad \text{for } 0 \leq j \leq N.
\]
Moreover, we have \( \mu_c = (2\pi)^{-2}\delta_0(dx dy) \), where \( dx dy \) denotes the Lebesgue measure on \( \mathbb{R}^2 \). Let \( a \) denote the subspace of \( \mathfrak{g} \) spanned by \( P_1, \ldots, P_N \); it is clearly an ideal of \( \mathfrak{g} \). By definition, the image of the map \( p \) is contained in the annihilator of this ideal in \( \mathfrak{g}^* \), which we can identify with \( \mathfrak{g}^* \setminus \{0\} \). Now, \( \mathfrak{g}^* \) has basis \( (X, Y, Z) \), where \( X, Y, Z \) are the images of \( L_1, L_2, \) and \( P_0 \) under the quotient map \( \mathfrak{g} \to \mathfrak{g}/a \). They satisfy the relations \( [X, Y] = Z, [X, Z] = [Y, Z] = 0 \), so we see that \( \mathfrak{g}/a \) is the 3D Heisenberg algebra. Let us use the basis \( (X, Y, Z) \) to identify \( \mathfrak{g}/a^* \) with \( \mathbb{R}^3 \) in the obvious way. Then, the composition \( \psi = p \circ \varphi : [0, 1] \to \mathfrak{g}^* \) is given by \( \psi(\xi, \eta) = (\xi_1, \xi_2, b(y)) \), and we are interested in the measure \( \nu = (2\pi)^{-2}\psi(dx dy) \). It is well known that there are two types of coadjoint orbits in \( \mathfrak{g}/a^* \): the 2D orbits given by \( f(Z) = c \), where \( c \) is a nonzero constant, and the 0D orbits [namely, points of the plane defined by \( f(Z) = 0 \)]. In particular, we see that \( \psi([0, 1]) \) is a union of coadjoint orbits, so the set \( \varphi(\Omega_0) \) is G-stable. Moreover, if \( c \neq 0 \) is fixed, then the functions \( u : f \to f(Z) \) and \( v : f \to f(Y) \) are coordinates on the coadjoint orbit defined by \( f(Z) = c \), and the Kostant measure on this orbit is given by \( \mu_c = (2\pi)^{-2}\delta_0(dx, dy) \). Consequently, the pushforward measure \( \nu \) can be decomposed as an integral of the Kostant measures \( \mu_c \) in the following way:
\[
\nu = \int_R \mu_c dv(c).
\]
where \( \nu \) is the measure on \( R \) obtained as the pushforward of the measure \( (2\pi)^{-2}\delta_0(dy) \) by the map \( \varphi : [0, 1] \to \mathbb{R} \). [4 fortiori, this formula implies that \( \nu \) is G-invariant. Note also that we have ignored the plane \( f(Z) = 0 \) in the computation above, which can be done because it has measure zero with respect to \( \nu \).] Last, the representation of \( \mathfrak{g}/a \) corresponding to the orbit \( f(Z) = c \) can be realized in the space \( L^2(\mathbb{R}) \) such that \( X \to \partial/\partial x \) and \( Y \to \partial/\partial y \). Under this representation, the subalgebra \( (X^2 + Y^2) \) maps to the operator \( -\lambda + \lambda^2 \), whose spectrum can be computed explicitly; it consists of eigenvalues of the form \( 2m + 1\lambda \), each having multiplicity 1, where \( m \) runs over all nonnegative integers. We now have all the ingredients that are needed to make sense of the right side of 2.1, and we see that it becomes
\[
(2\pi)^{-1}\int_{\mathbb{R}^2} N(\lambda, -\lambda + \lambda^2 \gamma^2) dy dx,
\]
which coincides with the right side of Colin de Verdière’s formula.

The classical Weyl formula also can be written in the form 2.1, with \( Q \) parameterizing a family of 1D representations (in this case, \( Q = Q_p \), so one does not need to decompose the pushforward measure).

3. Main Results and Conjectures

3.1. Generalizations: The Main Idea. It is tempting to conjecture that for any magnetic Schrödinger operator with discrete spectrum one can find a family of irreducible representations of \( G \) and the pushforward measure \( dt(\Theta) \) on \( Q \) such that 2.1 holds. As it turns out, this construction can be realized in many, albeit not all, cases, and our first goal is to suggest a general way of construction of the family \( Q \) and the pushforward measure \( dt(\Theta) \). Naturally (cf. refs. 9, 12, and 13 for generalizations of the classical Weyl formula), we have two similar (but a bit different) algorithms: one for the strong degeneration case and one for the weak and intermediate degeneration case. In the intermediate degeneration case, one has to introduce additional logarithmic factors into 2.1. To verify our conjecture for several classes of magnetic Schrödinger operators, we use a modification of the variational technique from refs. 9 and 12–14.

Let us keep the same notation as described above and write \( \mu_\alpha \) for the canonical (Kostant) measure on the orbit \( \Omega_0 \). In trying to turn the vague ideas above into a precise formula that applies to wide classes of Kepler operators, one meets two considerable difficulties. The first difficulty is the fact that there seems to be no natural general way of defining a projection map \( p : \Omega_0 \to Q \subset \mathfrak{g}^* \), such that the pushforward \( p_* (\mu_\alpha) \) will always be a G-invariant measure. The second difficulty, which is more serious, is that in the intermediate degeneration cases, there exists an asymptotic formula of the form 2.1 (with additional logarithmic factors), but the measure \( \nu \) cannot be obtained from a pushforward measure arising from a process described above.

Thus, one has to look for a different construction of the subset \( Q \subset \mathfrak{g}^* \) and the G-invariant measure \( \nu \) on \( Q \). We suggest a construction which has the advantage of being canonical (i.e., independent of any choices). Moreover, the measure \( \nu \) that it introduces is automatically G-invariant. Thus, both problems mentioned above are solved at once. To our knowledge, no similar construction has been used previously in this or any related context.

Let us give a brief description of our idea. For each \( \lambda > 0 \), we let \( \mu_\lambda = \mu_\lambda(\Omega) \), denote the positive Borel measure on \( \mathfrak{g}^* \) defined by \( \mu_\lambda(a) = \int_{\mathfrak{g}/a} \delta_0(dx) \) for every Borel subset \( A \subset \mathfrak{g}^* \). Note that \( \mu_\lambda \) is supported on \( \lambda^-1\mathfrak{g}^* \) with which is another coadjoint orbit in \( \mathfrak{g}^* \). Now, \( \Omega_0 \) is closed in \( \mathfrak{g}^* \), and there is a coordinate system on \( \Omega_0 \) which identifies \( \varphi(\Omega_0) \) with \( \mathbb{R}^3 \), such that \( \mu_\lambda \) corresponds to the usual Lebesgue measure under this identification (both of these statements hold for arbitrary nilpotent Lie algebras). In particular, we see that each \( \mu_\lambda \) can be identified with a positive linear functional on the space \( C_c(\mathfrak{g}^*) \) of compactly supported continuous functions on \( \mathfrak{g}^* \). Note also that, if \( A \) is a neighborhood of \( 0 \) in \( \mathfrak{g}^* \), then, as \( \lambda \to +\infty \), the sets \( \Omega_0 \cap \lambda^-1A \) exhaust all of \( \Omega_0 \), thus, \( \mu_\lambda(A) \to +\infty \). Let us now suppose that there exists a function \( f(\lambda) \) such that the functionals \( f(\lambda)\mu_\lambda \in C_c(\mathfrak{g}^*) \) have a nonzero weak-* limit \( f_0 \in C_c(\mathfrak{g}^*) \). By the Riesz representation theorem, \( f_0 \) corresponds to a positive Borel measure \( \mu_{\lambda_0} \) on \( \mathfrak{g}^* \). We define \( Q = \supp(\mu_{\lambda_0}) \), and \( \nu \) is a conical G-invariant subset of \( \mathfrak{g}^* \), and the G-invariance of \( \nu \) is automatic because each of the measures \( \mu_\lambda \) is G-invariant.

For simplicity, we refer to the construction described above as the “scaling construction.” Because of its “homogeneous” nature, it is not surprising that in applying the construction to the computation of spectral asymptotics of Schrödinger operators, one has to require a certain homogeneity condition on the potentials. We say, somewhat imprecisely, that 1.1 is a Schrödinger operator with quasihomogeneous potentials if \( V(x) \) and \( B(x) \) are quasihomogeneous polynomials of the same weight, i.e., if there exists an \( n \)-tuple of positive rational numbers \( \gamma = (\gamma_1, \ldots, \gamma_n) \) such that for all \( f \in \mathbb{R}^n \), \( t > 0 \), and all \( x \in \mathbb{R}^n \), we have
\[
V(t^{\gamma_1}x_1, \ldots, t^{\gamma_n}x_n) = t^{\gamma}V(x) \quad \text{and} \quad B(t^{\gamma_1}x_1, \ldots, t^{\gamma_n}x_n) = tB(x).
\]
We prove that in the quasihomogeneous situation in which the classical formulas of Weyl and Colin de Verdière are applicable, our construction gives the same result as the pushforward construction described above. However, in the intermediate degeneration examples that we have studied, it also produces the “correct” measure space \( (Q, \nu) \), even though the pushforward construction no longer applies.
We remark that our scaling construction makes sense for any nilpotent Lie algebra. Indeed, let $\mathfrak{g}$ be a finite dimensional nilpotent Lie algebra over $\mathbb{R}$ and $\Omega \subset \mathfrak{g}^*$ a coadjoint orbit. It is known (e.g., see chapter I of ref. [19]) that $\Omega$ is a closed (in fact, Zariski closed) submanifold of $\mathfrak{g}^*$. Moreover, it follows from the explicit parameterization obtained in ref. 20 that there exists a polynomial map $\varphi : \mathbb{R}^{2n} \to \mathfrak{g}^*$ which is a diffeomorphism onto $\Omega$, and such that under this diffeomorphism $\mu_{\Omega}$ corresponds to the standard Lebesgue measure on $\mathbb{R}^{2n}$.

As before, for every $\lambda > 0$, we define a positive Borel measure $\mu_{\lambda}$ on $\mathfrak{g}^*$ as follows:

$$\mu_{\lambda}(A) = \mu_{\Omega}(\Omega \cap \lambda A) = \text{meas}(\varphi^{-1}(\lambda A)),$$

where meas is the Lebesgue measure. Because $\varphi$ is proper, we see that $C_c(\mathfrak{g}^*) \subset L^1(\mu_{\lambda})$ for each $\lambda > 0$. In particular, we can again identify $\mu_{\lambda}$ with a positive linear functional on $C_c(\mathfrak{g}^*)$, and the rest of our construction goes through without any changes. It is apparent from the computations of explicit examples that the scaling construction is closely related to the geometry of the embedding $\Omega \to \mathfrak{g}^*$.

The idea of applying representation-theoretic methods to the study of partial differential operators is not new (e.g., see ref. [21] and references therein). Several authors have studied extensions of the known results about Schrödinger operators to the differential operators arising from unitary representations of general nilpotent Lie groups. In ref. 22, upper and lower bounds for $N(\lambda, H)$ were obtained, where $H$ is the image under an irreducible representation of the “sublaplacian” on a stratified nilpotent Lie algebra. Manchon (23) has generalized the approximate spectral projection method of Tulovskii and Shubin (24) to prove a Weyl-type asymptotic formula for elliptic operators associated to representations of arbitrary nilpotent Lie groups. In refs. 25 and 26, this result was generalized to arbitrary Lie groups (more precisely, to the representations corresponding to closed tempered coadjoint orbits for which Kirillov’s character formula is valid). However, note that refs. 23, 25, and 26 use the initial form of the approximate spectral projection method, which requires the high regularity of the symbol. In particular, if a degeneration of any kind is present, this form of the approximate spectral projection method does not work at all. For a general version of the approximate spectral projection method and applications to various classes of degenerate and hypoelliptic operators, see refs. 9, 12, and 13.

Most of the works relating differential operators to representation theory of nilpotent Lie groups deal only with stratified Lie algebras (21, 22); i.e., Lie algebras $\mathfrak{g}$ admitting a decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_s$, such as a direct sum of vector subspaces, such that $[\mathfrak{g}_j, \mathfrak{g}_k] \subset \mathfrak{g}_{j+k}$ $(\mathfrak{g}_j = (0)$ for $j > s)$, and $\mathfrak{g}$ is generated by $\mathfrak{g}_1$ as a Lie algebra. However, there are situations in which the Lie algebra arising from a Schrödinger operator with polynomial potentials admits no natural grading. The theory that we develop in section 3 makes no use of a grading on $\mathfrak{g}$.

In ref. 18, we use an example of the Schrödinger operator in 2D with zero electric potential and magnetic tensor $b(x) = x_2 \partial_{x_1} - x_1 \partial_{x_2}$ (this is an example of strong degeneration, and there is no natural grading) to illustrate in detail the use of our conjectural formula. We also study the weak degeneration case for operators without either magnetic or electrical potential and deduce from our conjectural classical Weyl formula and Colin de Verdière’s formula, respectively. In particular, we prove that, in the case of a quasihomogeneous electric potential, the classical Weyl formula holds if and only if the integral in this formula converges, and our general conjectural formula also gives the classical Weyl formula if and only if this condition is satisfied. Last, we consider the Schrödinger operator in 2D with magnetic tensor $b(x) = x_2 \partial_{x_1}$ and zero electric potential. In the case $k \neq l$, we have the strong degeneration, and in the case $k = l$, we have the intermediate degeneration. In all cases, we derive the leading term of the asymptotics from our conjectural formula, and we prove them by using the variational method in the form (9, 12–14).

The next subsections contain formulations of our main conjectures and statements of several representation-theoretic results that are necessary for the applications of our conjectures and also interesting in their own right. More details and complete proofs are given in ref. 18.

### 3.2 Preliminary Version of the Conjecture

Let us now formulate a preliminary version of our conjecture. Let $H$ be a Schrödinger operator (1.1) with discrete spectrum and quasihomogeneous polynomial potentials, and let $\mathfrak{g} = \mathfrak{g}(U)$ be the associated Lie algebra. Because we are interested in $\sigma(H)$, we may assume, by Theorem 2.1, that the tautological representation of $\mathfrak{g}$ lifts to a unitary representation of $G$ on $L^2(\mathbb{R}^n)$; moreover, this representation is then irreducible, from which corresponds to a coadjoint orbit $\Omega \subset \mathfrak{g}^*$. Let $\mu_{\Omega}$ be the Kostant measure on $\Omega$; for the precise normalization, see Definition 3.4. Then, we have the “dilates” $\mu_k$ of the measure $\mu_{\Omega}$, as defined in section 2: $\mu_k(\lambda A) = \mu_{\Omega}(\lambda \Omega \cap \lambda A)$, for every Borel subset $A \subset \mathfrak{g}^*$.

Moreover, $H$ naturally defines an element $H^* \in U(\mathfrak{g}(U))^\ast$, and the definition of $\mu_k$ implies that $H^*$ is a sublaplacian for $\mathfrak{g}$. For any coadjoint orbit $\Theta \subset \mathfrak{g}^*$, we denote by $H_{\Theta}$ the image of $H^*$ in the unitary irreducible representation of $G$ that corresponds to $\Theta$ via Kirillov’s theory. By Theorem 2.1, each $H_{\Theta}$ can be naturally realized as a Schrödinger operator with polynomial potentials.

**Conjecture 1.** There exist a positive real number $\alpha$ and a nonnegative integer $\beta$ such that the weak limit $\mu_0 = \lim_{k \to 0} \lambda^{-\alpha}(\log \lambda)^{\beta} \mu_k$ exists and is nonzero. Then $\mu_0$ is automatically $G$-invariant; let $Q = (\text{supp } \mu_0)/G$, and let $p : \text{supp } \mu_0 \to Q$ be the natural projection. Let $\nu$ be the measure on $Q$ such that for every nonnegative Borel-measurable function $F$ on $\mathfrak{g}^*$, we have

$$\int_{\mathfrak{g}^*} F \ d\mu_0 = \int_Q \int_{p^{-1}(q)} F(x) \ d\mu_\lambda(x),$$

where $d\mu_\lambda$ denotes the Kostant measure corresponding to the orbit $p^{-1}(q)$ (the existence of $\nu$ is proved in ref. 18, proposition 2.12). Then there exists a constant $\kappa \geq 1$ such that

$$N(\lambda, H) \sim \kappa(\log \lambda)^\beta \int_Q N(\lambda, H_{\Theta}) \ d\nu(\Theta) \quad \text{as} \quad \lambda \to +\infty. \quad \text{[3.1]}$$

Some motivation for the form of this conjecture, and especially for the appearance of the logarithmic factors in both 3.1 and the definition of $\mu_0$, is provided by a result of Nilsson, which we now recall. It is a special case of theorem 1 in ref. 27; the latter is, in turn, based on the results of ref. 28.

**Theorem 3.1 (27, 28).** Let $P(x)$ be a real polynomial on $\mathbb{R}^n$ such that $P(x) \to +\infty$ as $\|x\| \to \infty$, and set

$$G(\lambda) = \text{meas}\{x \in \mathbb{R}^n | P(x) \leq \lambda\}.$$

Then, there exist positive reals $c, C, \alpha$ and a nonnegative integer $\beta$ such that

$$C^{-1} \lambda^{-\alpha}(\log \lambda)^{\beta} \leq G(\lambda) \leq C \lambda^{-\alpha}(\log \lambda)^{\beta} \quad \text{for all} \quad \lambda > c.$$

The precise relationship of this theorem to our results is explained in detail in ref. 18. Here, we remark that the explicit formulas for the measure $\mu$ and its dilates $\mu_k$ obtained in section 3.5 imply that the growth of the measures $\mu_k$ as $\lambda \to +\infty$ is closely related to the growth of the function $G(\lambda)$ in Theorem 3.1 for a suitably defined polynomial $P(x)$. 

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3.3 Precise Version of the Conjecture. We now formulate a more precise form of our conjecture, one that essentially provides a formula for the constant $\kappa$ that appears in 3.1. To that end, we introduce the function

$$\Phi^*(x) = \sum_n |\alpha^n V(x)|^{1/2} + \sum_{\alpha, j, k} |\alpha^n b_{jk}(x)|^{1/2}. \quad [3.2]$$

It is to be compared with the function $\Psi^*$ in refs. 1 and 16 (see 1.6). If, for example, $V = 0$ and $B(x)$ grows (as $x$) tends to $\infty$, the terms corresponding to $\alpha = 0$ dominate both $\Psi^*$ and $\Phi^*$, so we see that these two functions have the same asymptotic behavior as $|x| \to \infty$. However, in general, it may happen that the function $\Psi^*(x)$ grows slower than the function $\Phi^*(x)$.

We keep the same notation and assumptions as in Conjecture 1. In particular, because $H$ has discrete spectrum, both $\Phi^*$ and $\Psi^*$ tend to $\infty$ as $|x| \to \infty$, the functions $G_1(\lambda) = \lambda \{ x \in \mathbb{R}^{n^*} \}$ and $G_2(\lambda) = \{ x \in \mathbb{R}^{n^*} \}$ are well-defined (meas stands for the usual Lebesgue measure).

Conjecture 2. Assume that $H$ is a Schrödinger operator on $L^2(\mathbb{R}^n)$ with discrete spectrum and quasihomogeneous potentials. Let $(Q, V)$ be defined as in Conjecture 1. Then, one of the following situations occurs.

(i) We have $G_2(\lambda)/G_1(\lambda) \to \infty$ as $\lambda \to \infty$. This is the “strong degeneration case.” Then, Conjecture 1 is valid with the normalization constant $\kappa = 1$.

(ii) We have $G_2(\lambda)/G_1(\lambda) \to \infty$ as $\lambda \to \infty$. This is the “weak/intermediate degeneration case.” Then there exists a limit $\lim_{\lambda \to \infty} G_2(\lambda)/G_1(\lambda)$, and Conjecture 1 is valid with $\kappa$ equal to the value of this limit.

3.4. Concrete Realization of Representations. Until the end of the section, the quasihomogeneity assumption will play no role. Let $G$ be a real finite dimensional nilpotent Lie algebra such that $G$ is abelian, and let $H \in U(G)$ be a subalgebra. We want to obtain concrete realizations of the representations of $G$ induced by unitary irreducible representations of $G$ that are defined by the connected subgroup $G$ (Schrödinger operators do not appear until the end of the section, so the notation should not cause any confusion.) Fix $x \in \mathbb{C}^*$. We say that $H$ is subordinate to $f$ if $f(0,0) = 0$. Under this condition, $f$ defines a unitary character $\chi_f$ of $H$ by $\chi_f(h) = \exp(fv(h))$. Thus, we may form the induced representation $\pi_f = Ind_H^G(\chi_f)$. Kirillov’s classification (17) of unitary irreducible representations of $G$ can be summarized as follows.

Let us say that $f$ is a polarization of $G$ if $f$ is at least one point and only if $f$ is a polarization at all other points, $f$. Moreover, in this case, $\pi_f$ does not depend on the choice of $f$, up to unitary equivalence. Also, at every $f \in G$, there exists at least one polarization. Thus, we write $\pi_f = \pi_{f,h}$ for any choice of a polarization $f$ at $f$. Last, every unitary irreducible representation of $G$ is unitarily equivalent to $\pi_f$ for some $f \in G$, and $\pi_f$, $\pi_{f_1}$ are unitarily equivalent if and only if $f_1$, $f_2$ lie in the same coadjoint orbit of $G$.

We define the following bilinear form:

$$B_f: G \times G \to \mathbb{R}, \quad B_f(f, g) = \langle f, [X, Y] \rangle. \quad [3.3]$$

Thus, a subalgebra $H \leq G$ is subordinate to $f$ if and only if $H$ is isotropic with respect to $B_f$. One can prove that $f$ is a polarization at $f$ if and only if $f$ is maximally isotropic with respect to $B_f$ as a linear subspace. In particular, all polarizations at $f$ have the same dimension, $\dim f = \dim f_{\lambda} = \dim \pi_{f_{\lambda}}$, where $\pi_{f_{\lambda}} = \pi_{f_{\lambda}}(x) = 0$ if $x \in G$. In our situation, we can give an elementary proof of the existence of polarizations of a special form:

Lemma 3.2. Let $G$ be a complex Lie algebra, and let $\pi_f = Ind_H^G(\chi_f)$ for some $f \in G$. Then, there exists a polarization $H$ of $G$ such that $H$ is an ideal of $G$.

Moreover, $\{ f, H \} \subseteq \pi_f$, so $\pi_f$ is an ideal of $H$.

Let us now fix a subalgebra $H \leq G$ subordinate to $f$, but not necessarily a polarization at $f$, which satisfies the requirement of the lemma: $\pi_f = \pi_{f_{\lambda}}(x) \subseteq H$. Because $L_0, L_1, \ldots, L_N$ generate $G$ as a Lie algebra, we have $G = [G, G] + span(L_0, L_1, \ldots, L_N)$, and hence, $\pi_f = H + span(L_0, L_1, \ldots, L_N)$. After reindexing, we may assume that for some $0 \leq n \leq N$, the elements $L_1, \ldots, L_n$ form a complementary basis to $H$ in $G$. (We allow $n = 0$, which means that $H = G$).

For every element $h \in H$, we define a real polynomial $p_n(x)$ in the variables $x = (x_1, \ldots, x_n)$ by

$$p_n(x) = \sum_{\alpha_1, \ldots, \alpha_n} \frac{1}{\alpha_1! \cdots \alpha_n!} \cdot \prod_{l=1}^n x_l^{\alpha_l} \in \mathbb{R}[x_1, \ldots, x_n]. \quad [3.4]$$

Proposition 3.3. There exists a realization of the representation $\pi_f = Ind_H^G(\chi_f)$ of the Lie group $G$ in the space $L^2(\mathbb{R}^n, dm)$ (where $dm$ is the Lebesgue measure) such that the induced representation of $G$ takes every $h \in H$ to the operator of multiplication by $\chi_h(f)$ and $L_i$, for $1 \leq i \leq n$, to the operator $\delta/\delta x_i + \chi_i(f)$, where $\alpha_i(x) \in \mathbb{R}[x_1, \ldots, x_n]$ is a certain polynomial.

The practical applications of this proposition are based on the obvious analogy between 3.4 and the usual Taylor’s formula.

3.5. Coadjoint Orbits and Kostant Measures. Let $G$ be any connected Lie group, and let $g$ be its Lie algebra. If $f \in G$, we denote by $G(f)$ the stabilizer of $f$ in $G$ (with respect to the coadjoint action), and by $\pi_f$ the Lie algebra of $G(f)$. If $f \in G$ is a coadjoint orbit, then for any $f \in G$, the orbit map $G \to G, g \mapsto (Ad^g(f))$, identifies $G$ with the homogeneous space $G/G(f)$, and identifies the tangent space $T_fG$ with the quotient $g/f$. The notation is consistent with the one used in section 3.4: if $B_f$ is the alternative bilinear form on $G$ given by $B_f(f, g) = (f, [X, Y])$, then it is easy to see that $g/f$ is precisely the kernel of $B_f$. Moreover, $B_f$ induces an alternative bilinear nondegenerate form $\omega_f$ on $g/f$ by $\omega_f(f, g) = \Pi f$. One then proves the following facts (e.g., see chapter II of ref. 19):

1. The forms $\omega_f$ vary smoothly with $f$, thus defining a nondegenerate differential 2-form $\omega_f$ on $\Omega$;
2. The forms $\omega_f$ is closed, and thus a symplectic form on $\Omega$; and
3. The form $\omega_\Omega$ is $G$-invariant.

Definition 3.4. The form $\omega_\Omega$ is called the “canonical symplectic form” on the orbit $\Omega$. The Kostant measure (or the “canonical measure”) on the orbit $\Omega$ is the positive Borel measure $\mu_\Omega$ associated with the volume form

$$(2\pi)^{-\frac{n}{2}} \cdot \omega_\Omega$$

(Note that $\dim \Omega$ is even because $\Omega$ admits a symplectic form.)

It is clear that the Kostant measure is $G$-invariant. In the rest of this subsection, we obtain an explicit parameterization of the coadjoint orbits of the Lie algebras of the type considered in section 2.1, and we derive formulas for the corresponding canonical symplectic forms and Kostant measures. We note that explicit parameterizations of the dual space of a (not necessarily nilpotent) Lie algebra have been studied by various authors (e.g., ref. 29).

More recently, a very fine stratification of $\mathfrak{g}^*$ for nilpotent $\mathfrak{g}$ has been obtained in ref. 20. A result from loco citato is used in ref. 18.

In our subsequent computations (especially the ones that appear in the concrete examples in ref. 18), we implicitly use the following
result. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a} \subset \mathfrak{g}$ an ideal. Write $a^\perp$ for the annihilator of $\mathfrak{a}$ in $\mathfrak{g}^\ast$. The quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$ induces an isomorphism $\mathfrak{g}/\mathfrak{a}$ as a vector space $\mathfrak{g}/\mathfrak{a}^\ast = a^\perp \to \mathfrak{a}^\ast$. Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$ and let $A \subset G$ be the closed connected normal subgroup corresponding to $\mathfrak{a}$. The adjoint action of $G$ on $\mathfrak{g}$ leaves $\mathfrak{a}$ stable, from which $G$ also acts on $\mathfrak{g}/\mathfrak{a}$ and on $(\mathfrak{g}/\mathfrak{a})^\ast$. Then, we have the following:

**Proposition 3.5.** (i) The isomorphism $(\mathfrak{g}/\mathfrak{a})^\ast \to a^\perp$ above is $G$-equivariant, and the action of $G$ on $(\mathfrak{g}/\mathfrak{a})^\ast$ factors through the quotient group $G/A$; thus, the $G$-orbits in $(\mathfrak{g}/\mathfrak{a})^\ast$ are the same as the coadjoint orbits of $G/A$ in $(\mathfrak{g}/\mathfrak{a})^\ast$.

(ii) If $\Omega \subset \mathfrak{g}^\ast$ is any coadjoint orbit, then either $\Omega \cap a^\perp = \emptyset$ or $\Omega \subset a^\perp$. In the latter case, $\Omega$ is the image of a coadjoint orbit in $(\mathfrak{g}/\mathfrak{a})^\ast$. Moreover, the canonical symplectic form and the Kostant measure on $\Omega$ are the same whether we regard $\Omega$ as a coadjoint orbit for $G$ or as a coadjoint orbit for $G/A$.

(iii) If $G$ is simply connected and nilpotent, then the bijection between the coadjoint orbits in $\mathfrak{g}^\ast$ that meet $a^\perp$ and the coadjoint orbits in $(\mathfrak{g}/\mathfrak{a})^\ast$, defined above, corresponds, by Kirillov's theory, to the natural bijection between the unitary irreducible representations of $G$ that are trivial on $A$, and all unitary irreducible representations of $G/A$.

We return to the situation considered in Section 3.4. Thus, $G$ is a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{g}]$ is abelian. Fix a point $0 \in \mathfrak{g}^\ast$. We want to parameterize the $G$-orbit $\mathfrak{g}^\ast f_0 \subset \mathfrak{g}^\ast$. As before, we assume that we have given a sublaplacian $H^\ast = -\sum L_j^2 - \sum L_k^2$, and we let $\mathfrak{h}$ be a real polarization of $f_0$ provided by Lemma 3.2: $\mathfrak{f}(0) = \mathfrak{f}(0) + \mathfrak{R} \mathfrak{L} + [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}$. Furthermore, we suppose that for some $1 \leq n \leq N$, $L_1, \ldots, L_n$ is a complementary basis for $\mathfrak{h}$ in $\mathfrak{g}$.

From now on, we also assume that $\mathfrak{h}$ is an abelian ideal of $\mathfrak{g}$. To justify this assumption, we note that because $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, so is $a := [\mathfrak{h}, \mathfrak{h}]$; however, by the definition of a polarization, $f_0$ annihilates $\mathfrak{a}$. Thus, $f_0$ induces a linear functional $\mathfrak{f}(0) \in \mathfrak{g}/\mathfrak{a}$. By Proposition 3.5, the canonical inclusion $(\mathfrak{g}/\mathfrak{a})^\ast \to \mathfrak{g}^\ast$ gives an isomorphism of the coadjoint orbit of $f_0$ in $(\mathfrak{g}/\mathfrak{a})^\ast$ onto the coadjoint orbit of $f_0$ in $\mathfrak{g}^\ast$; moreover, this isomorphism preserves the canonical symplectic form and the Kostant measure. Last, note that because $a \subset \mathfrak{g}(f)$ by Lemma 3.2, it is clear that $\mathfrak{h}/\mathfrak{a}$ is a maximal isotropic subspace of $\mathfrak{g}/\mathfrak{a}$ with respect to the form $B_{K'}$. Thus, from the point of view of either the coadjoint orbit of $f_0$, or of the corresponding unitary irreducible representation, nothing is lost by passing from $\mathfrak{g}$ to $\mathfrak{g}/\mathfrak{a}$.

**Proposition 3.6.** With the notation above, assume that $\mathfrak{h}$ is abelian. The map $\varphi: \mathbb{R}^n \times \mathbb{R}^n \to \mathfrak{g}^\ast$ defined by $(\varphi(\xi, \eta), L_j) = \xi_j$ for $1 \leq j \leq n$.

$$\langle \varphi(\xi, \eta), Y \rangle = \sum_{\alpha_1 \cdots \alpha_n \geq 0} \frac{1}{\alpha_1! \cdots \alpha_n!} f_0((ad L_1)^{\alpha_1} \cdots (ad L_n)^{\alpha_n}(Y)) \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \text{ for all } Y \in \mathfrak{h},$$

is a diffeomorphism of $\mathbb{R}^{2n}$ onto the coadjoint orbit of $f_0$ in $\mathfrak{g}^\ast$.

By a slight abuse of notation, we identify $\Omega$ with $\mathbb{R}^{2n}$ by using the diffeomorphism $\varphi$, and in particular, we view $(\xi, \eta)$ as coordinates on the orbit $\Omega$. Let us define polynomials $b_k(x)$ by

$$b_k(x) = \langle \varphi(0, x), [L_j, L_k] \rangle;$$

note that if $\mathfrak{g}$ arises from a Schrödinger operator with polynomial potentials, and if $f_0$ restricts to the linear functional $\langle -i \mathcal{L}, \xi \rangle$ on the subspace of $\mathfrak{g}$ consisting of multiplication operators, then the $b_k(x)$ are precisely the components of the magnetic tensor of the operator. The next proposition gives an explicit formula for the Kostant measure.

**Proposition 3.7.** The canonical symplectic form and the Kostant measure on the orbit $\Omega$ are given by

$$\omega_\Omega = \sum_{j=1}^n d\xi_j \wedge dx_j + \sum_{1 \leq k < \alpha \leq n} b_k(x) \, dx_k \wedge dx_{\alpha}, \quad [3.5]$$

and

$$\mu_\Omega = (2\pi)^{-n} d\xi_1 \cdots d\xi_n d\alpha_1 \cdots d\alpha_n \varphi(x, \eta) = \mathfrak{ad} L_1 \cdots \mathfrak{ad} L_n \mathfrak{ad} L_{k_1} \cdots \mathfrak{ad} L_{k_\alpha};$$

where $\varphi$ denotes the pushforward by the map $\varphi: \mathbb{R}^{2n} \to \mathfrak{g}^\ast$.

In other words, if we identify $\mu_\Omega$ with its extension by zero to $\mathfrak{g}^\ast$, then we can write

$$\mu_\Omega = (2\pi)^{-n} \varphi(\xi, \eta) = \mathfrak{ad} L_1 \cdots \mathfrak{ad} L_n \mathfrak{ad} L_{k_1} \cdots \mathfrak{ad} L_{k_\alpha};$$

where $\varphi$ denotes the pushforward by the map $\varphi: \mathbb{R}^{2n} \to \mathfrak{g}^\ast$.