Let $f: X \to Y$ be a morphism of smooth algebraic varieties over a field of characteristic 0. Define $K \subset X \times_Y T^*Y$ by

$$K := \{(x, \xi) \mid x \in X,\ \xi \in T^*_{f(x)} Y,\ \xi(\operatorname{Im} d_x f) = 0\}.$$

Let $\pi: X \times_Y T^*Y \to T^*Y$ denote the projection. Then $\pi(K) \subset T^*Y$ is a constructible subset. The proposition below gives an upper and lower bound for $\overline{\pi(K)}$.

Set $X_r := \{x \in X \mid \operatorname{rank}(d_x f) = r\}$. Let $X_{r,\alpha}$, $\alpha \in \operatorname{Irr}(X_r)$, be the irreducible components of X_r. Set $Y_{r,\alpha} := \overline{f(X_{r,\alpha})}$. Set

$$\operatorname{Irr}_{ess}(X_r) := \{\alpha \in \operatorname{Irr}(X_r) \mid \dim X_{r,\alpha} \geq r\},$$

where "ess" stands for "essential".

Proposition 1. (i) $\overline{\pi(K)}$ is contained in the union of the conormal bundles of the subvarieties $Y_{r,\alpha}$, $r \in \mathbb{Z}_+$, $\alpha \in \operatorname{Irr}_{ess}(X_r)$.

(ii) If $\dim Y_{r,\alpha} = r$ then $\overline{\pi(K)}$ contains the conormal bundle of $Y_{r,\alpha}$.

Remarks. (a) The characteristic 0 assumption implies that $\dim \overline{f(X_r)} \leq r$, so $\dim Y_{r,\alpha} \leq r$.

(b) The conormal bundle of a singular subvariety is defined to be the closure of the conormal bundle of its smooth locus.

The proposition follows immediately from Lemmas 1 and 2 below.

Lemma 1. If $\dim X_{r,\alpha} < r$ then $X_{r,\alpha} \times_X K$ is nowhere dense in K.

Proof. We have

$$\dim(X_{r,\alpha} \times_X K) = \dim X_{r,\alpha} + \dim Y - r < \dim Y.$$

On the other hand, $K \subset X \times_Y T^*Y$ is locally defined by m equations, $m := \dim X$. So the dimension of each irreducible component of K is not less than $\dim(X \times_Y T^*Y) - m = \dim Y$. ∎

Lemma 2. Let $X'_{r,\alpha} \subset X_{r,\alpha}$ denote the open subset of all $x \in X_{r,\alpha}$ such that

(a) x is a nonsingular point of $X_{r,\alpha}$,

(b) $f(x)$ is a nonsingular point of $Y_{r,\alpha}$,

(c) the map $T_x X_{r,\alpha} \xrightarrow{d_x f} T_{f(x)} Y_{r,\alpha}$ is surjective.

Then $X'_{r,\alpha} \neq \emptyset$ and

(*) $\operatorname{Im}(d_x f : T_x X \to T_{f(x)} Y) \supset T_{f(x)} Y_{r,\alpha}$ for all $x \in X'_{r,\alpha}$.

Moreover, if $\dim Y_{r,\alpha} = r$ then the inclusion in (*) is an equality.

Proof. $X'_{r,\alpha} \neq \emptyset$ by the characteristic 0 assumption. The inclusion (*) follows from (c). If $\dim Y_{r,\alpha} = r$ then the inclusion (*) has to be an equality because the l.h.s. of (*) has dimension r. ∎

Example (M. Kashiwara). $X = Y = \mathbb{A}^2$, $f(t,x) = (t, t^n x)$, $n \geq 1$. Then $X_0 = \emptyset$, $X_1 = \{(t,x) \mid t = 0\}$, $X_2 = X \setminus X_1$. If $n = 1$ then $\pi(K)$ is the union of the zero section and $T^*_{y_0} Y$, where $y_0 := (0,0) \in Y$. But if $n > 1$ then $\overline{\pi(K)}$ is the union of the zero section and a 1-dimensional subspace of $T^*_{y_0} Y$.

Proposition 2. If $r \geq \dim Y - 1$ then $\mathrm{Irr}_{ess}(X_r) = \mathrm{Irr}(X_r)$.

Proof. If $r = \dim Y$ then X_r is open in X, so if $X_r \neq \emptyset$ then $\dim X_r = \dim X \geq r$. If $r = \dim Y - 1$ then $X_r \subset X$ is locally defined by $\dim X - r$ equations, so each irreducible component of X_r has dimension $\geq r$. ■

In general, it may happen that $\mathrm{Irr}_{ess}(X_r) \neq$ $\neq \mathrm{Irr}(X_r)$.

Example. $X = \mathbb{A}^{n+1}$, $n \geq 1$, $Y = \mathbb{A}^3$,

$f(t, x_1, \ldots, x_n) = (t, \sum_{i=1}^{n} x_i^2, t x_1)$.

Then $X_1 = \{0\}$, so $\dim X_1 = 0$ and $\mathrm{Irr}_{ess}(X_1) = \emptyset$.