Advanced Studies in Pure Mathematics 36, 2002
Algebraic Geometry 2000, Azumino
pp. 145–151

Independence of ℓ for Intersection Cohomology (after Gabber)

Kazuhiro Fujiwara

In his lecture in the conference Algebraic Geometry 2000, O. Gabber explained his proof of the following Theorem 1 that the intersection cohomology of a proper scheme is independent of ℓ.

Theorem 1. *Let X be a proper equidimensional scheme over a finite field $\mathbb{F}_q$ and let $i \in \mathbf{Z}$. For a prime $\ell \not| q$, let $IH^i(X_{\bar{k}}, \overline{\mathbf{Q}}_\ell)$ be the intersection cohomology of degree i. Then* $\det(1 - t\,\mathrm{Fr}, IH^i(X_{\bar{k}}, \overline{\mathbf{Q}}_\ell))$ *is with coefficients in* $\mathbf{Z}$ *and independent of $\ell \not| q$.*

The aim of this note is to give the proof of Gabber. Some details of the proofs are filled by the author, and he takes the full responsibility for the inaccuracies that may appear in this note. In the talk, Gabber also presented the proofs of other independence of ℓ results which are not contained in this short article.

Acknowledgement. The author thanks the referees for their helpful comments. The author also thanks Gabber for his comments on this article.

§1. Independence for $K(X)$

1.1. Notation

We work over $k = \mathbb{F}_q$. For a prime $\ell \not| q$, choose an algebraic closure $\overline{\mathbf{Q}}_\ell$. For a scheme X separated of finite type over k, $D^b_c(X, \overline{\mathbf{Q}}_\ell)$ denotes the derived category of $\overline{\mathbf{Q}}_\ell$-sheaves defined in Weil II ([De 4]). This notion of derived category is stable under the six operations $f_!$, f_*, f^*, $f^!$, $\otimes$, $R\,Hom$, and also by the Grothendieck-Verdier dualizing functor $D = D_X$ which we normalize by

$$D_X K = R\,Hom(K,\ f^!\overline{\mathbf{Q}}_\ell) \quad \text{for } f : X \to \operatorname{Spec} k.$$

Received April 13, 2001.
2000 *Mathematics Subject Classification*: 14F20. Keywords: étale cohomology, intersection cohomology, ℓ-adic sheaves.

By $K(X, \overline{\mathbf{Q}}_\ell)$, we denote the Grothendieck ring of the category of $\overline{\mathbf{Q}}_\ell$-sheaves. For a $\overline{\mathbf{Q}}_\ell$-sheaf $\mathcal{F}$ on X, we denote by $[\mathcal{F}]$ the class in $K(X, \overline{\mathbf{Q}}_\ell)$. For $K \in D_c^b(X)$, we also denote by $[K]$ the class $\sum_{i \in \mathbf{Z}}(-1)^i [H^i(K)]$. $|X|$ denotes the set of the closed points.

For a $\overline{\mathbf{Q}}_\ell$-sheaf $\mathcal{F}$ and $x \in |X|$, we have a geometric Frobenius element $\mathrm{Fr}_x \in \mathrm{Gal}(\overline{k(x)}/k(x))$ acting on the geometric fiber $\mathcal{F}_{\bar{x}}$ localized at x. The local L-function is defined by

$$L_x(\mathcal{F},\ t) = \det(1 - t^{\deg x}\,\mathrm{Fr}_x,\ \mathcal{F}_{\bar{x}})^{-1}.$$

This definition extends to $K(X, \overline{\mathbf{Q}}_\ell)$ by additivity, and the homomorphism

$$\begin{aligned} K(X, \overline{\mathbf{Q}}_\ell) &\to \prod_{x \in |X|} (1 + t \cdot \overline{\mathbf{Q}}_\ell[[t]])^\times \\ [\mathcal{F}] &\mapsto (L_x(\mathcal{F},\ t))_{x \in |X|} \end{aligned}$$

is injective by the Chebotarev density theorem.

1.2. (E, I)-compatibility

Let E be a field of characteristic 0, and let I be a subset of $\{(\ell, \iota),\ \ell \neq p$ is a prime, $\iota : E \hookrightarrow \overline{\mathbf{Q}}_\ell$ is a field embedding$\}$. For $\alpha = (\ell, \iota) \in I$, we denote the first component by $\ell_\alpha = \ell$, the second by $\iota_\alpha = \iota$.

Definition. *Let X be a separated scheme of finite type over $k = \mathbf{F}_q$. We say that a system $(K_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} K(X,\ \overline{\mathbf{Q}}_{\ell_\alpha})$ is (E, I)-compatible if and only if it satisfies the following* a), b):

a) *$L_x(K_\alpha,\ t)$ is E-rational for any $\alpha \in I, x \in |X|$, i.e., belongs to the image of $E[[t]]$ via embedding ι_α,*
b) *For each $x \in |X|$, $L_x(K_\alpha,\ t)$, viewed as an element in $E[[t]]$, coincide for all $\alpha \in I$.*

It is easy to see

$$\begin{aligned} K_\alpha \in K(X, \overline{\mathbf{Q}}_{\ell_\alpha}),\ \alpha \in I \text{ are } (E, I)\text{-compatible} \\ \iff \text{For all } n \geq 1 \text{ and } x \in X(\mathbf{F}_{q^n}), \\ \mathrm{Tr}(\mathrm{Fr}_x,\ K_{\alpha, \bar{x}}) = \iota_\alpha(t_x)\ (\alpha \in I) \text{ for some } t_x \in E. \end{aligned}$$

1.3. Results

The following Theorem 2, Theorem 3 will be proved in §3.

Theorem 2. *Let $(K_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} K(X,\ \overline{\mathbf{Q}}_{\ell_\alpha})$ be an (E, I)-compatible system. Then for a morphism $f : X \to Y$ of separated schemes of finite* *type over k,* $(f_*K_\alpha)_{\alpha\in I}$, $(f_!K_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} K(Y, \overline{\mathbf{Q}}_{\ell_\alpha})$ *are also* (E,I)*-compatible. Similar results hold for* f^*, $f^!$, $\otimes^{\mathbf{L}}$, $R\,Hom$, D.

For the normalization of perverse sheaves, we follow [BBD].

Theorem 3. *Let* $j : U \hookrightarrow X$ *be an open immersion and let* $(K_\alpha)_{\alpha\in I}$ *be an* (E,I)*-compatible system on* U*. Assume that each* K_α *is pure and perverse. Then the system of middle extensions* $(j_{!*}K_\alpha)_{\alpha\in I}$ *is also* (E,I)*-compatible.*

§2. Linear reccurence sequences

Later we need an elementary lemma on sequences which can be proved by a high-school student. We recall the following definition:

Definition. Let $X \subset \mathbf{Z}$ be a non-empty subset. Assume X is stable under addition by $\mathbf{N}$, or stable under substraction by $\mathbf{N}$. Let E be a field. An E-valued sequence $(a_n)_{n\in X}$ is called a linear reccurence sequence if it satisfies a difference equation of the form

$$\sum_{i=0}^{r} c_i a_{n+i} = 0 \quad (\text{for all } n \text{ such that } n,\ n+r \in X)$$

for some $r \geq 0$, $c_i \in E$, $c_0 c_r \neq 0$. The set of the E-valued linear reccurence sequences is denoted by $Lr(X,\ E)$.

Then the lemma we need is the following:

Key lemma.

a) *Let* E *be an algebraically closed field of characteristic* 0*. Then* $Lr(X,E)$ *has a basis given by the sequences* $e_{\alpha,a} : n \mapsto n^a\alpha^n$ *for* $\alpha \in E^\times$, $a \in \mathbf{N}$.

b) $Lr(X,E)$ *is a vector space over* E *and the canonical map* $Lr(\mathbf{Z},E) \to Lr(X,E)$ *is bijective.*

c) *Fix an embedding of fields* $E \subset E'$*. Assume that* $f \in Lr(X,\ E')$ *takes values in* E*. Then* f *belongs to* $Lr(X,\ E)$.

Proof of Key lemma. a) is well-known, and b) is obvious. We prove c). Assume that X is stable under addition by $\mathbf{N}$. For a field F, let $S : (a_n)_{n\in X} \mapsto (a_{n+1})_{n\in X}$ be the shift operator acting on the set $\mathrm{Map}(X,F)$ of all maps $X \to F$. Then $f = (a_n)_{n\in X} \in \mathrm{Map}(X,F)$ belongs to $Lr(X,F)$ if and only if f satisfies the following conditions (1), (2). Let V be the F-subspace of $\mathrm{Map}(X,F)$ generated by $S^n(f)$ $(n \geq 0)$.

(1) V is finite dimensional over F.

(2) The action of S on V is bijective.

For $f \in \mathrm{Map}(X,E)$, since $\mathrm{Map}(X,E)\otimes_E E' \to \mathrm{Map}(X,E')$ is injective, the conditions (1), (2) with $F=E$ are satisfied if and only if the conditions (1), (2) with $F=E'$ are satisfied. □

§3. Proofs of the theorems

Proof of Theorem 2. For f^* and $\otimes^{\mathbf{L}}$, the claim is obvious. The statement for $f_!$ follows from the Grothendieck trace formula for powers of Frobenius:

$$\mathrm{Tr}(\mathrm{Fr}_y,\ (f_!K_\alpha)_{\bar{y}}) = \sum_{x\in X_y(\mathbf{F}_{q^n})} \mathrm{Tr}(\mathrm{Fr}_x,\ K_{\alpha,\bar{x}}) \quad \text{for all } y \in Y(\mathbf{F}_{q^n}),\ n\geq 1.$$

Since

$$f_* = D_Y f_! D_X, \quad R\,\mathit{Hom}(K,L) = D_X(K\otimes^{\mathbf{L}} D_X(L))$$

by the duality formalism, the claim for f_* and $R\,\mathit{Hom}$ follows from that of D_X. We prove the stability of (E,I)-compatibility under D. It suffices to prove the independence of $\mathrm{Tr}(\mathrm{Fr}_x, D_X K_\alpha)$ at each $x\in X(\mathbf{F}_{q^n})$.

Since our problem is local on X for the Zariski topology, we may assume that X is affine. By taking a closed embedding $i: X\hookrightarrow \mathbf{A}^n_k$, we may also assume that $X=\mathbf{A}^n_k$ is an affine space, since i_* commutes with the duality D. By taking a finite extension of k, we may assume $x\in \mathbf{A}^n_k(k)$, and by translation $x = 0_{\mathbf{A}^n_k}$ is the origin.

Further, we reduce to the case $X=A$, where A is an abelian variety over k, and $x=0$ is the identity element of $A(k)$: by a definition of smoothness, there is an open set $j: U\hookrightarrow A$ containing 0, with étale morphism $p: U\to \mathbf{A}^n_k$ sending 0 to $0_{\mathbf{A}^n_k}$. Since the problem depends only on henselizations at 0 and $0_{\mathbf{A}^n_k}$, it suffices to prove it for $j_!p^*K$ on A.

Now assume $X=A$, an abelian variety over k. Define functions $f_{\alpha,n},\ g_{\alpha,n}: A(\mathbf{F}_q)\to \overline{\mathbf{Q}}_{\ell_\alpha}$ for $n\geq 1$ by

$$f_{\alpha,n}(a) = \sum_{b\in A(\mathbf{F}_{q^n}),\ T_n(b)=a} \mathrm{Tr}(\mathrm{Fr}_b,\ K_{\alpha,\bar{b}}),$$

$$g_{\alpha,n}(a) = \sum_{b\in A(\mathbf{F}_{q^n}),\ T_n(b)=-a} \mathrm{Tr}(\mathrm{Fr}_b,\ (D_A K_\alpha)_{\bar{b}}),$$

where T_n denotes the trace map $A(\mathbf{F}_{q^n})\to A(\mathbf{F}_q)$. Note it is sufficient to prove that $g_{\alpha,1}(0)$ belongs to E via ι_α and is independent of α. We will show that there is a $\overline{\mathbf{Q}}_{\ell_\alpha}$-valued linear reccurence sequence $S_\alpha(a)$ on $\mathbf{Z}$ such that $f_{\alpha,n}(a) = S_\alpha(a)_n$, $g_{\alpha,n}(a) = S_\alpha(a)_{-n}$ for $n \geq 1$. By our assumption, $f_{\alpha,n}(a)$ takes values in E via ι_α for $n \geq 1$ and is independent of a choice of $\alpha \in I$. By Key lemma, the same is true for $g_{\alpha,n}(a)$.

Now we prove the existence of $S_\alpha(a)$.

Let

$$\mathcal{L} : A \to A, \qquad a \mapsto a - \mathrm{Fr}_A(a)$$

denotes the Lang torsor defined by $\mathrm{id}_A - \mathrm{Fr}_A$. The Lang torsor is an $A(\mathbb{F}_q)$-torsor. For a finite dimensional representation

$$\rho : A(\mathbb{F}_q) \to \mathrm{GL}_{\overline{\mathbf{Q}}_{\ell_\alpha}}(V)$$

we have the associated local system $\mathcal{L}_\rho = A \times^{A(\mathbb{F}_q)} \rho$ on A as the contracted product. For $m \geq 1$, it is known that the fiber of $\mathcal{L}_\rho$ at $b \in A(\mathbb{F}_{q^m})$ corresponds to the representation of $\mathrm{Gal}(\overline{\mathbb{F}}_{q^m}/\mathbb{F}_{q^m})$ sending the Frobenius element Fr_b at b to $\rho(T_n(b))$ [De2].

When $\rho : A(\mathbb{F}_q) \to \overline{\mathbf{Q}}_{\ell_\alpha}^\times$ is a character, by the Grothendieck trace formula for $K_\alpha \otimes \mathcal{L}_\rho$ and $D_A(K_\alpha \otimes \mathcal{L}_\rho) = D_A K_\alpha \otimes \mathcal{L}_{\rho^{-1}}$, we have

$$\text{(1)} \quad \mathrm{Tr}(\mathrm{Fr}^n,\ R\Gamma(A_{\bar{k}},\ K_\alpha \otimes \mathcal{L}_\rho)) = \sum_{b \in A(\mathbb{F}_{q^n})} \mathrm{Tr}(\mathrm{Fr}_b,\ K_{\alpha,\bar{b}}) \cdot \rho(T_n(b)),$$

$$\text{(1')} \quad \mathrm{Tr}(\mathrm{Fr}^n,\ R\Gamma(A_{\bar{k}},\ D_A K_\alpha \otimes \mathcal{L}_{\rho^{-1}})) = \sum_{b \in A(\mathbb{F}_{q^n})} \mathrm{Tr}(\mathrm{Fr}_b,\ (D_A K_\alpha)_{\bar{b}}) \cdot \rho(-T_n(b))$$

for $n \geq 1$.

These two identities are rewritten as follows. Let

$$\mathcal{F}(f)(\rho) = \sum_{a \in A(\mathbb{F}_q)} f(a)\rho(a)$$

be the Fourier transform on $A(\mathbb{F}_q)$. Then

$$\mathrm{Tr}(\mathrm{Fr}^n,\ R\Gamma(A_{\bar{k}},\ K_\alpha \otimes \mathcal{L}_\rho)) = \mathcal{F}(f_{\alpha,n})(\rho),$$
$$\mathrm{Tr}(\mathrm{Fr}^n,\ R\Gamma(A_{\bar{k}},\ D_A K_\alpha \otimes \mathcal{L}_{\rho^{-1}})) = \mathcal{F}(g_{\alpha,n})(\rho)$$

follow from formula (1) and (1').

Let $\alpha_{ij} \in \overline{\mathbf{Q}}_{\ell_\alpha}$, $j \in J_i$ be the Fr-eigenvalues on $H^i_{\mathrm{et}}(A_{\bar{k}},\ K_\alpha \otimes \mathcal{L}_\rho)$. By global duality, $R\Gamma(A_{\bar{k}},\ K_\alpha \otimes \mathcal{L}_\rho)$ and $R\Gamma(A_{\bar{k}},\ D_A K_\alpha \otimes \mathcal{L}_{\rho^{-1}})$ are dual to each other since A is proper. Hence

$$\mathcal{F}(f_{\alpha,n})(\rho) = \sum_{i\in\mathbf{Z}}(-1)^i \sum_{j\in J_i} \alpha_{ij}^n \quad (n \geq 1),$$
$$\mathcal{F}(g_{\alpha,n})(\rho) = \sum_{i\in\mathbf{Z}}(-1)^i \sum_{j\in J_i} \alpha_{ij}^{-n} \quad (n \geq 1)$$

hold, and it follows that there are $\overline{\mathbf{Q}}_{\ell_\alpha}$-valued linear reccurence sequences $s(\rho)$ on $\mathbf{Z}$ such that $\mathcal{F}(f_{\alpha,n})(\rho) = s(\rho)_n$, $\mathcal{F}(g_{\alpha,n})(\rho) = s(\rho)_{-n}$ for $n \geq 1$. The existence of $S_\alpha(a)$ follows by the inverse Fourier transform.

Remark (by Gabber). In case of curves, Deligne proves the compatibility for Rj_* for any compatible system of ℓ-adic sheaves on open parts ([De1], Théorème 9.8). To get the compatibility at missing points, he uses twists by highly ramified rank one sheaves. The argument here uses twists by unramified rank one sheaves.

Proof of Theorem 3. We may assume $\sharp I = 1$ or 2, and each K_α is pure of weight w. By Gabber's purity theorem [BBD], $j_{!*}K_\alpha$ is also pure of weight w (cf. [De4] for pure complexes).

We put $Y = X \setminus U$, $i : Y \to X$. We prove the claim by a descending induction on $\dim Y$. So it suffices to prove the compatibility near all maximal points of Y. By shrinking Y if necessary, we may assume that Y is smooth of pure dimension d, and the middle extension is calculated as

$$j_{!*}K_\alpha = \tau_{\leq -d-1} j_* K_\alpha$$

for any α. Moreover, we may assume that all cohomology sheaves $\mathcal{H}^q(i^! j_{!*}K_\alpha)$, $\mathcal{H}^q(i^* j_* K_\alpha)$ ($\alpha \in I$) are smooth on Y. To show the compatibility, it suffices to recover the local L-function $L_y(\tau_{\leq -d-1} j_* K_\alpha, t)$ at any $y \in |Y|$ from $L_y(j_* K_\alpha, t)$, which is already independent of α by Theorem 2.

By purity, for $q \leq -d-1$, $\mathcal{H}^q(j_* K_\alpha) = \mathcal{H}^q(\tau_{\leq -d-1} j_* K_\alpha)$ are punctually mixed of weight $\leq q+w$. By purity again, $i^* D_X j_{!*} K_\alpha = D_Y i^! j_{!*} K_\alpha$ is mixed of weight $\leq -w$, and hence the smooth sheaves $\mathcal{H}^q(i^! j_{!*} K_\alpha)$ on Y are punctually mixed of weight $\geq w+q$. By localization triangle

$$\to i^! j_{!*} K_\alpha \to j_{!*} K_\alpha \to j_* K_\alpha \to i^! j_{!*} K_\alpha[1],$$

we know that the smooth sheaves $\mathcal{H}^q(i^* j_* K_\alpha) = \mathcal{H}^{q+1}(i^! j_{!*} K_\alpha)$ are mixed of weight $\geq w+q+1$ for $q \geq -d$ on Y. This implies that for $y \in |Y|$, $L_y(\tau_{\leq -d-1} j_* K_\alpha, t)$ is extracted from $L_y(j_* K_\alpha, t)$ as the part of weight $\leq w-d-1$. □

Proof of Theorem 1. By definition,

$$IH^i(X_{\bar{k}},\ \overline{\mathbb{Q}}_\ell) = H^i_{\text{et}}(X_{\bar{k}},\ IC^\cdot_X), \quad IC^\cdot_X = (j_{!*}\overline{\mathbb{Q}}_\ell[\dim X])[-\dim X].$$

Here j is the inclusion of a dense smooth subscheme in X. By Theorem 3, $\det(1 - t\,\mathrm{Fr}; R\Gamma(X_{\bar{k}},\ IC^\cdot_X))$ is with coefficients in $\mathbb{Q}$ and is independent of $\ell \nmid q$. The result for the individual cohomology group $IH^i(X_{\bar{k}},\ \overline{\mathbb{Q}}_\ell)$ follows from this by the fact $IH^i(X_{\bar{k}},\ \overline{\mathbb{Q}}_\ell)$ is of weight i. For the integrality of the coefficients, we may assume that X is reduced and irreducible. By [dJ], there is an alteration $\pi : Y \to X$ with Y smooth and projective over $\mathbb{F}_q$. By the decomposition theorem, $IC^\cdot_X$ is a direct summand of $R\pi_* IC^\cdot_Y$, hence the fact that the eigenvalues of Fr on $H^i_{\text{et}}(X_{\bar{k}}, IC^\cdot_X)$ are algebraic integers follows from the corresponding statement for Y, which holds by [De3], or [SGA7], XXI 5.5.3. □

References

[SGA4] M. Artin, A. Grothendieck and J. L. Verdier, *Théorie des Topos et Cohomologie Étale des Schémas*, Lecture Notes in Mathematics, vol. 269, 270, 305, Springer-Verlag.

[SGA7II] P. Deligne and N. Katz (ed), *Groupes de Monodromie en Géométrie Algébrique*, Lecture Notes in Mathematics, vol. 340, Springer-Verlag, 1973.

[BBD] A. A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux pervers*, Astérisque, **100** (1982).

[De1] P. Deligne, *Les constantes des equations fonctionnelles des fonctions L*, Lecture Notes in Mathematics, vol. 349, Springer-Verlag, 1973, pp. 501–597.

[De2] ———, *Applications de la formule des traces aux sommes trigonométriques*, SGA 4 1/2, Lecture Notes in Mathematics, vol. 569, Springer-Verlag, 1977, pp. 168–232.

[De3] ———, *La conjecture de Weil. I*, Inst. Hautes Études Sci. Publ. Math., **43** (1974), 273–307.

[De4] ———, *La conjecture de Weil. II*, Inst. Hautes Études Sci. Publ. Math., **52** (1980), 137–252.

[dJ] A. J. de Jong, *Smoothness, semi-stability and alterations*, Inst. Hautes Études Sci. Publ. Math., **83** (1996), 51–93.

Graduate School of Mathematics
Nagoya University
Nagoya, 464-8602
Japan