Princeton, le 9 juillet 2015

Dear Sacha,

I am intrigued by Drinfeld question : "what are the possible irreducible components of a $SS(\mathcal{F})$?". I could compute only in dimension 2. In that case, not too many different behaviours are allowed by your theorem, and anything you allow is possible.

Let X be a smooth surface over k algebraically closed of char. p. Let Y be an irreducible curve in X. Let $\mathbb{P}(T^*_X|Y)$ be the restriction to Y of the bundle $\mathbb{P}(T^*_X)$ of projective lines. If, on U open and dense in Y, we have $C \subset \mathbb{P}(T^*_X|Y)$ finite over U and irreducible, we want $\mathcal{F}$ such that $SS(\mathcal{F})$ contains the cone corresponding to C, and we need this only on a smaller $V \subset U$ dense in Y.

<u>Reduction</u> : to $C \to Y$ radicial.

Factor (over some U) the projection $C \to Y$ as

$$C \xrightarrow{\text{radicial}} Y' \xrightarrow{\text{étale}} Y .$$

On a neighborhood W of the generic point of Y, extend $Y'_{|W} \to Y \cap W$ to $X' \xrightarrow{f} W$ finite étale. The C we started with is the image by df of $C \to Y'$, mapping to $\mathbb{P}(T^*_{X'|Y'})$ (the pull back of $\mathbb{P}(T^*_X|Y)$). If $\mathcal{F}'$ on X' has $SS(\mathcal{F}')$ containing this $C \hookrightarrow \mathbb{P}(T^*_{X'|Y'})$, $f_*\mathcal{F}'$ will do for C

<u>Reduction</u> : for some n, $C \xrightarrow{\sim} Y^{1/p^n}$.

This will happen over U small enough, because Y is smooth of dimension 1.

<u>Reduction</u> : X retracts to Y.

This will happen over some etale neighborhood of the generic point of Y (with no extension of the residue field at this generic point), and one uses a direct image from this neighborhood

Take a local coordinate systems : an etale map

$$X \xrightarrow{(x,y)} \mathbb{A}^2$$

for which $Y \subset X$ is $y = 0$. As X retracts to Y, we have

$$x, y : \quad X \longrightarrow \mathbb{A}^1 \times Y \longrightarrow \mathbb{A}^1 \times \mathbb{A}^1 .$$

Abuse of notation : we will write $f(y)$ for a function on Y, and $f(y^{1/p^n})$ for a function on Y^{1/p^n}.

Our C is given by

$$dy - \lambda\, dx \qquad \text{at } y \text{ in } Y$$

with λ a function on Y^{1/p^n}. If $n > 0$, we want this function to generate Y^{1/p^n} over Y, meaning that its derivative $d\lambda / dy^{1/p^n}$ should be invertible. Equivalently : $\lambda^{p^n} : Y \to \mathbb{A}^1$ should be etale. If we use this λ^{p^n} instead of y, and rescale x by $d\lambda^{p^n}/dy$, this means, in the new local coordinates, we want to achieve C given by

$$dy - y^{1/p^n} dx .$$

For $x=0$, by an étale change of variables $(x,y) \longmapsto (x, y+\lambda(y)x)$, one sees one want to achieve C given by dy.

<u>A. $n=0$</u> : Here, it suffices to use Artin Schreier sheafs $\mathcal{A}(f)$, for f on the complement of the y axis of $\mathbb{A}^2$; the rank one sheaf $\mathcal{A}(f)$ comes from the $\mathbb{F}_p$-torsor $T^p - T = f$ and a (non trivial) character of $\mathbb{F}_p$. For $p \geqslant 3$, one takes

$$\mathcal{A}(y/x^p) .$$

If we restrict $\mathcal{A}(y/x^p)$ to the curve $y = y_0 + \lambda x$, one gets $\mathcal{A}((y_0+\lambda x)/x^p) \sim \mathcal{A}(y_0^{1/p}/x + \lambda/x^{p-1})$, of Swan conductor $p-1$, except for $\lambda = 0$, where it is smaller. This implies that dy ~~so~~ is in SS : if one sweeps with a family of lines with changing slopes, eg one uses the function

$$\frac{y-y_0}{x+1} ,$$

the Swan conductor of the restriction will drop, indicating a non local acyclicity, on $\frac{y-y_0}{x+1} = 0$: the line with slope 0.

For any p, one could as well use $\mathcal{A}(y/x^{p^n})$ with $p^n \geqslant 3$: the Swan conductor drops from p^n-1 to 1 or 0.

For $p=2$, $\mathcal{A}(y/x^2)$ gives $C = \langle dy + y^{1/2}dx \rangle$

B. $n>0$ It suffices to use the direct image of the constant sheaf on the covering given by

$$T^{p^n} + xT - y = 0 .$$

On $y = y_0 + \lambda x$, and making the change of variable replacing T by $T + y_0^{1/p^n}$, we get the covering given by the equation

$$(T+y_0^{1/p^n})^{p^n} + x(T+y_0^{1/p^n}) - (y_0 + \lambda x) = 0 ,$$

that is

$$T^{p^n} + xT + x(y_0^{1/p^n} - \lambda) = 0 .$$

For $\lambda \neq y_0^{1/p^n}$, this is an Eisenstein equation of degree divisible by p : we have wild ramification and a non zero Swan.

For $\lambda = y_0^{1/p^n}$, we have $T^{p^n} + xT = 0$: this is tame, with solutions $T = 0$ and the $(-x)^{1/p^n-1}$. Swan is zero. The singular support contains, on the y axis, the $dy - y^{1/p^n} dx$.

Best

Pierre