Horospherical transform as a curved version of the Radon transform.

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Horospheres in Riemannian symmetric spaces

Let $X = G/K$ be a Riemannian symmetric space of noncompact type. Here G be a connected semisimple Lie group with a finite center and K be its maximal compact subgroup. Let us fix an Iwasawa decomposition

$$G = KAN,$$

where N, A are transversal to K maximal unpotent and Cartanian subgroups. Let M be the centralizer of A in K, We call the homogeneous space

$$\Xi = G/MN$$

by *the horospheric space*. The spaces X and Ξ have equal dimensions. The natural projections

$$X \to G/M \leftarrow \Xi$$

define an incidence relation between X, Ξ. If $\xi \in \Xi$ we take its preimage in G/M and then its projection in X. Such sets $E(\xi) \subset X$ called *horospheres*.

Horospheres

Their codimension in X are equal the rank l of the symmetric space X - the dimension of the Cartanian subgroup A. The horospheres are orbits on X of subgroups conjugateed to N. Correspondingly points $x \in X$ have on Ξ the incidence subsets $U(x)$ which we call pseudospheres. They are just parameters of horospheres passing through x. Since A normalizes MN we have on Ξ the "left" action of A (which commutes with the "right"action of G); it fibers Ξ on the left orbits of A over the flag space

$$F = G/AMN.$$

Here $P = AMN$ is a parabolic subgroup and F is compact and isomorphic K/M. The pseudospheres $U(x)$ are sections of the fibering $\Xi \to F$.
As the result on Ξ acta the extended group $A \times G$ and the actions of A, G commutate,

Horospherical transform

The horospherical transform is the operator of integration along the horospheres. We want to invert this operator. Using the invariancy it is sufficient to in one point, let $x_0 \in X$. Let K. is its isotropy subgroup. Our parameterization of horospheres will be associated with x_0 and the corresponding Iwasawa decomposition $G = KAN$. The horospheres passing through x_0 has form

$$E(x_0|k)K = \{x_0 \cdot Nk\}k \in K\}.$$

Here we can consider $k \mod M$ so as elment of the flag manifold $F = K/M$ In intermediate computations we will sometime omit x_0.

Horospherical transform2

Transformations by A transform horospheres in the parallel ones:

$$E(x_0|a,k) = \{x_0 \cdot aNk\}, a \in A, k \in F = k/M.$$

For fixed a the parameters give the sections of the fibering $\Xi \to F$. We will work with the Lie algebra $Lie(A)$ and the dual space $Lie(A)^*$ and use the operations $\exp, \log$. Let us fix the system of positive roots $\Sigma \subset Lie(A)^*$ and a subsystem of prime roots $\Pi \subset \Sigma$ and let ρ be the half-sum of positive roots.
For $x \in X$ let $a(x), n(x)$ are projections on the corresponding Iwasawa factors. We have $a(x_0) = e$(the unit in A). Let us fixed the invariant forms da, dn, dk on $A, N, F = K/M$ correspondingly. Then

$$dx = a(x)^{-2\rho} da(x) \wedge dn(x)$$

be the invariant form on X.

Horospherical transform3

Let $f(x) \in C_0^\infty$. We define its horospherical transform as the integral along the horopheres $E(x_0|a,k)$:

$$\mathcal{H}f(a,k) = \int f(x_0 \cdot ank)dn, a \in A, k \in K/M.$$

So we integrate on the horosphere $E(a,k) = \{x_0 \cdot aNk\}$ on dn. This definition is not G-invariant: for the invariance we need to add the factor $a^{-2\rho}$. For our aims is more convenient an intermediate correction:

$$\hat{f}(a,k) = a^{-\rho}\mathcal{H}f(a,k).$$

It is connected with the factor $a^{-2\rho}$. in the connection with the spherical Fourier transform. *Our principal aim is to inverse the horospherical transform: to recostruct $f(x_0)$ through $\mathcal{H}f(a,k)$.*

The principal result

Let us state our principal result.

Theorem

There is the nest formula for the inversion of the horospherical tranform

$$f(x_0) = c \int_F dk \int_A da \bigwedge_{\alpha \in \Sigma} D_\alpha^{m_\alpha} \hat{f}(a,k)) \Pi_{1 \le i \le l} (\sinh(\pi_j(\ln(a) - i0)/2)^{-1}.$$

Here we apply the operators of differentiation along the positive roots D_α counting the multiplicity m_α to $\hat{f}(a,k) = a^{-\rho}\mathcal{H}f(a,k)$. Then we substitute $t = \sinh(\pi_j(\ln(a))$ in the distribution

$$(t - i0)^{-1} = i\pi\delta(t) + t^{-1}$$

for all prime roots π_j.

The principal result2

This inversion formula has an universal structure for all roo systems and use only the subsystem of prime roots. in the difference in other known systems which essentially depend of the type of root system. If to break up the distribution $(t - i0)^{-1}$ in even and odd parts in all factors then the 2nd part of the formula will transform in a sum of many terms with different symmetry relative Weyl's group W. Some od them will disappear after the integration along F. So if all multiplicities m_α but purely local one disappear. Let us emphasize that the structure of the operator in the inversion formula is non unique since there many operators a similar structure which are trivial on the image of the horospherical transform. Suggested here structure is not W-invariant.

The horospherical Cauchy transform

The factor with the hyperbolic sinuses has a deep sense: it is connected with the replace of the δ-function with a Cauchy kernel. Let us consider the characters

$$a^{\pi_j} = \exp(\pi_j(\ln a)), \pi_j \in \Pi$$

and a^{π} for $\pi = \pi_1 + \cdots + \pi_l$. We can rewrite the definition of the horospherical transform as

$$\mathcal{H}f(a, v) = a^{\pi/2} \int_X f(x) \Pi_{1 \leq i \leq l} \delta((a(x)^{\pi_i} - a^{\pi_i})) a(x)^{2\rho + \pi/2} dx,$$

where $a(x)$ is the Iwasawa projection of X on A. Let us remark the identity

$$\exp((u+v)/2)\delta(e^u - e^v) = \delta(u - v).$$

The horospherical Cauchy transform2

We define *the horospherical Cauchy transform* as

$$\mathcal{C}f(b,v) = b^{\pi/2}\int_A \mathcal{H}f(a,v)\Pi_{1\le i\le l}(\chi_j(a)-\chi_j(b)-i0)^{-1}(a)^{\Pi/2+\rho}da,$$

where $v \in F, b \in A$. It is possible to rewrite this definition as the integral on X:

$$\mathcal{C}f(b,v) = b^{\pi/2}\int_X f(x)\Pi_{1\le i\le l}(a(x)^{\pi_i} - b^{\pi_i} - i0)^{-1}a(x)^{2\rho+\pi/2}dx,$$

$$= \int_X f(x)\Pi_{1\le i\le l}(\sinh(\pi_j(\ln(a(x)) - ln(b) - i0)^{-1}dx.$$

where $a, b \in A, v \in F$

The horospherical Cauchy transform3

We use the identity

$$\frac{\exp((u+v)/2}{e^u - e^v} = \frac{1}{\sinh(u-v)/2}.$$

We can give another interpretation of this construction. We identified elements of A with the vectors of their chracters $\{a^{\pi_i}\}$. Let us extend this correspondance in the complexification $\mathbb{C}A$ and take the domain $\mathbb{C}A_-$ of $a \in \mathbb{C}A$ with $-\pi < Im(a^{\pi_j} < 0), j \leq l$. Then A will be the edge of the boundary of $\mathbb{C}A_-$ and we can extend the horospherical Cauchy transform.

$$\mathcal{C}f(a,v) = (a)^{\pi/2} \int_X f(x) \Pi_{1 \leq i \leq l} (a(x)^{\pi_j} - a^{\pi_j}))^{-1} a(x)^{2\rho + \pi/2} dx$$

$$= \int_X f(x) \Pi_{1 \leq i \leq l} (\sinh(\pi_j (\ln(a(x) - ln(a) - i0)^{-1} dx. v \in F, a \in \mathbb{C}A_-.$$

.

The Cauchy version of the principal result

There are no singularities in the kernel and tresult will be holomorphic in $\mathbb{C}A_-$. Our "real" horospherical transform can be interpret as boundary values as $a \in \mathbb{C}A_-$ tends to A.e
Using the language of the horospherical Cauchy transform we we can reformulate the princial result.

Theorem

There is a horospherical Cauchy inversion formyla

$$f(x_0) = c \int_F dv \bigwedge_{\alpha \in \Sigma} D_\alpha^{m_\alpha} \mathcal{C} f(a, v)|_{a=e}.$$

To see it we need to remark that the our inversion frmula has the of convolution (on a) wth $\hat{f}(a, k)$ pf 2 distributions: the differential operator and a"Cauchy" kernel. If we change thir order we will obtain the horospherical Cauchy inversion formula.

The Cauchy version of the principal result2

Let us discuss this construction in a more broad environment. The Radon inversion formula is different for odd and even dimensions: it is local in the first case and non local in the second one. Each of these formulas can be written for all dimensions, but of some dimensions they give zero as a consequence of evenness or oddness of dimensions.

Using the distribution $(t - i0)^{-1}$ we can unify to type of inversion formulas. We can rech the same aim by replacing the δ-function in the definition of the Radon transform on the Cauchy kernel - the Radon-Cauchy transform. In a sense in this construction we destroy the symmetr which transforms potential inversions in zeroes.

It is remarkable that in much more complicaye case of symmetric spaces where instead one dimensions we have many root multiplicities the conception of Cauchy transform continues to work.

Spherical Fourier transform

Informaly the horospherical Fourier transform gives a projection on irreducible components of $L^2(X)$. It can be defined by different ways (f.e. through zonal spherical functions).We will use by the form associated with the horospherical transform. Namely we consider (for fixed x_0) the corrected horospherical transform $\hat{f}(a, v), a \in A, v \in K/M)$ for the fixed v and take the Euclidean Fourier transform on $u = \ln a \in Lie(A) \backsimeq \mathbb{R}^l$:

$$\mathcal{F}f(r, v) = \int_{Lie(A)} \hat{f}(a, v) \exp(i < \ln a, r >) da, v \in K/M, r \in (Lie(A)^*.$$

Here we identify $Lie(A)^*$ with $Lie(A)$ using the Cartanian form. We can rewrite the definition of the spherical Fourier transform:

$$\mathcal{F}f(r, v) = \int_X a(x)^{-\rho} f(x) \exp(i < \ln a(x), r >) dx.$$

Analogue of thee Plancherel formula

The inversion of the spherical Fourier transform - the reconstruction of $f(x_0)$ through $\mathcal{F}f$ is the central problem of harmonic analysis on the symmetric space X - the analogue of Plancherel formula. Apparently, it is equivalent to the inversion of the horospherical transform.

. Namely we compute the Plansherel density $P_X(r)$:

$$f(x_0) = c \int_F dv \int_{\ln A} \mathcal{F}f(r,v) P_X(r) dr, r \in Lie(A).$$

Apparently $P_X(r)$ is exactly the Fourier transform (on A) of the kernel in the horospherical inversion formula which is the operator of convolution and the convolution of 2 distributions.

Analogue of thee Plancherel formula2

We have

Corollary

$$P_X(r) = \prod_{\pi_j \in \Pi} (\tanh(\pi \frac{< \pi_j, +r >}{< \pi_j, \pi_j >}) + 1) \prod_{\alpha \in \Sigma} \frac{< \alpha, ir >}{< \alpha, \alpha >}.$$

We use the formula

$$\frac{1}{2\pi i} \mathcal{F}(\frac{1}{\sinh(u/2)}) = \tanh(\pi \xi)$$

which givs the Fourier transform for the distribution $(\sinh(u/2) - i0)^{-1}$.
Our version of the Plancherel density is different from Harish-Chandra formula through c-function.

The structure of the proof

1.Principal result for the flat(tangential) model.
2. Curved perturbation of Radon's type transforms.
3. Specialization for thr horospherical transform.
However we will start from the detailed illustration the method on the example of the hyperbolic plane.

The curved Radon transform on the plane

Our next step is the construction of a general method of a perturbation of inversion formulas in the flat integral geometry up some curved inversion formulas. We will start from the Radon transform on the plane. Before the consideration of the curved versions of Radon's inversion formula let us remind of the Radon's inversion formula for lines. All our considerations are local and generic. Radon's inversion formula on the plane reconstructs a smooth function $f(x), x \in \mathbb{R}^2$, at a fixed point, let $x = 0$, through its integrals on lines. Let us remind of this formul. Let $f \in C_0^\infty(\mathbb{R}^2)$. We consider lines $L(\theta, p)$ defined by the equations

$$x_1 \cos\theta + x_2 \sin\theta = p, \quad 0 \le \theta \le 2\pi, \quad p > 0.$$

They admit the parametric representation

$$x = \varphi_{\theta,p}(t) = (p\cos\theta - t\sin\theta, p\sin\theta + t\cos\theta), \quad t \in \mathbb{R}.$$

The curved Radon transform on the plane

We define the Radon transform of f:

$$\mathcal{R}f(\theta,p)=\int_{-\infty}^{\infty} f(\varphi_{\theta,p}(t))dt.$$

Then Radon's inversion formula is

$$f(0)=c\int_0^{\infty}\frac{dp}{p}\int_0^{2\pi}\mathcal{R}'_p f(\theta,p)d\theta, \qquad c=-\frac{1}{2\pi^2}.$$

Let us remark that there is no singularity at $p=0$ since $\mathcal{R}'_p f$ is an odd function of p. We will transfer this formula from on arbitrary curves. In reality we will generalized only the interior integral (on θ): it is enough for our applications. Let $\mathcal{E}$ be the family of all smooth curves $E=\{\varphi_E(t)\}, t\in\mathbb{R}$. We define the curved Radon transform as the integrals along curves of $f(x)$:

$$\mathcal{R}f(E)=\int_{-\infty}^{\infty} f(\varphi_E(t))dt.$$

Geometry of curves

For a curve E we call the trivial variation

$$\epsilon_E(t) = \{\varphi_E(t)'\}.$$

It is the tangent vector field on E with the unit norm. *For the Radon transform $\mathcal{R}f(E)$ the variational derivative in the direction ϵ_E is zero.*
For any fixed curve γ on the plane we denote through $\mathcal{E}_\gamma$ the subset of such curves E that $\varphi_E(0)$ is a point of γ and E is tangent to γ in this point. *We include in our considerations the degenerate case $\mathcal{E}_x$ when the curve γ is reduces to a point x. So $\mathcal{E}_X$ consist from curves with $\varphi_E(0) = x$.*

The operator κ

Let us give the basic construction on $\mathcal{E}_\gamma$.For a functional F on this space we define

$$\kappa_\gamma F(E|\delta E) = dF(E|(\delta E - c\epsilon_E)/t), E \in \mathcal{E}_\gamma, \delta E \in T_{E_\gamma},$$

where

$$c = \delta E(0)/\varphi_E(0).$$

It is well defined since we take the quotient of 2 vectors which are collinear as tangent to γ in the same point. So we take the evaluation of the differential dF on the variation

$$\tilde{\delta} E = (\delta E - c\epsilon_E)/t$$

The variation $\tilde{E}$ has no singularities at $t = 0$ sinde the difference iz zero for $t = 0$ and our operation has a sense. Let us remark that the variation $\tilde{\delta} E$ will already not be tangent to $\mathcal{E}_\gamma$ but only to $\mathcal{E}$. In the case of $\mathcal{E}_x$ we do not need to make the correction by ϵ_E snce then $\delta E(0) = 0$.

The operator κ 1

The operator κ_γ will interesting for us for in the case when $F = \mathcal{R}f$. Then

$$\kappa_\gamma(\mathcal{R}f)(E|dE) = \int_{-\infty}^{\infty} D_{\tilde{\delta}E} f(\varphi_E(t)dt, \qquad E \in \mathcal{E}_\gamma.$$

Let us emphasize that the differential operator along the vector field acts for fixed t.

The example of lines

Let us consider the subfamily $\mathcal{L} \subset \mathcal{E}$ of lines and let $\gamma = S_p$ be the circle of the radius $p > 0$ with the center 0. Then $\mathcal{L}_p = \mathcal{S}_\gamma \bigcap \mathcal{L}$ consists from lines $L(\theta, p)$ with this p.
The tangent variation to $\mathcal{L}_p$ at $E = L(\theta, p)$ is

$$\delta E(t) = (-p \sin\theta - t\cos\theta, p\cos\theta - t\sin\theta)d\theta.$$

Then we separate the trivial part $\epsilon_E(t) \equiv (-p\sin\theta, p\cos\theta)d\theta$ and

$$\widetilde{\delta E}(t) \equiv (-\cos\theta, -\sin\theta).$$

As result the operator κ on $\mathcal{L}_p$ wich we denote as κ_p is

$$\kappa_p \mathcal{R} f(\theta, d\theta) =$$

$$\int_0^{2\pi}\int_{-\infty}^{\infty}\{-\cos\theta f'_{x_1} + \sin\theta f'_{x_2}\}(p\cos\theta - t\sin\theta, p\sin\theta + t\cos\theta)dtd\theta,$$

it is exactly $-\mathcal{R}f'_p(\theta, p)d\theta$ and we have exactly the interior integral in the Radon inversion formula for lines.

The basic fact on κ

So the operator κ transfers this interior integral from lines on arbitrary curves. The basic fact is

Proposition

1-form $\kappa_\gamma \mathcal{R} f(E|dE)$ is closed on $\mathcal{E}_\gamma$.

The proof of Proposition is a straightforward verification. The kea is that the operator κ differs from the differential just the factor $1/t$ under the integral and follows from the coincidence of mixed derivatives of $f(x)$. The crucial circumstance is that $\kappa \mathcal{R} f$ in the difference with $d\mathcal{R} f$ is *closed* (but not exact!). The singularity of the factor in $t = 0$ is essential.

Tangency Ptinciple

From Proposal follows *Tangency Principle*. Let us now γ be a cycle in $\mathbb{R}^2_x$. We have the projection $\mathcal{E}_\gamma \to \gamma$ by taking the tangent points to γ. For simplicity, we suppose that these points are unique. Let Γ_γ be sections of this fibering. They are cycles in $\mathcal{E}_\gamma$. Let $\tau(\Gamma_\gamma) \subset \mathcal{E}_\gamma$ be the cycle of tangents $\tau(E)$ to curves $E \in \mathcal{E}_\gamma$ (at the tangency to γ points).

Corollary

$$\int_{\Gamma_\gamma} \kappa_\gamma \mathcal{R} f = \int_{\tau(\Gamma_\gamma)} \kappa_\gamma \mathcal{R} f.$$

We integrate the closed 1-form in $\mathcal{E}_\gamma$ along homolical cycles: the cycle of curves can be contract to the cycle of tangents.

An application of Tangency Ptinciple

Tangency Principle gives a possibility in some cases to reduce the inversion of the curved Radon transform to cases of lines. Let us state one such kind result. Let $\Sigma \subset \mathcal{E}$ be a generic 2-parametric family of curve on the plane such that for generic $p > 0, 0 \leq \theta \leq 2\pi$ there is an unique curve $E(\theta, p)$ tangent to the circle S_p in the point θ.Then

Corollary

$$f(0) = c \int_0^\infty \frac{dp}{p} \int_0^{2\pi} \kappa_{S_p} \mathcal{R} f(E(\theta, p)).$$

Here we just for $\gamma = S_p$ apply Tangecy principle. and use for each p tangency principle reducing the general case to the Radon's inversion formula. Of course , the using circles S_p in this statement makes in non invariant, but we need in this paper just this case,

Horocyclic hyperbolic transform

Let us consider the model of the hyperbolic plane on the 2-dimensional hyperboloid X in $\mathbb{R}^2$:

$$x^2 - y^2 - z^2 = 1, x > 0.$$

The horocycleses $E(a, \theta)$ - isotropic sections of X by the planes $x + \cos\theta y + \sin\theta z = e^a$, parallel to the asymptots of the hyperboloid X:

$$x = \cosh a + \frac{1}{2} e^a t^2,$$

$$y = \sinh a \cos\theta - \sin\theta e^a t - \frac{1}{2} \cos\theta e^a t^2,$$

$$z = \sinh a \sin\theta + \cos\theta e^a t - \frac{1}{2} \sin\theta e^a t^2.$$

Here $0 \leq a < \infty, 0 \leq \theta < 2\pi$. The parametrization is assotiated with the point $(1, 0, 0)$ in which we want to reconstruct functions. This family of parabolas is invariant relative to rotations around the line $y = z = 0$.

Horocyclic hyperbolic transform2

We can consider the problem on the (y,z)-plane taking the projection plane along the x-axis (just considering 2nd and 3rd equations). We preserve the notation $E(a,\theta)$. Let $\mathcal{L}$ is this family of parabolas on the plane and compute the operator κ for it. For each a we have subfamily $\mathcal{L}_a$ of parabolas tangent to circles S_a with the center $(0,0)$ of the radius $\sinh a$. Tangent points $\theta \in S_a$ are vertexes of the parabolas. Let us compute κ_a on $\mathcal{L}_a$. Thanks to the rotation symmetry it is enough to make computations for one θ, let $\theta = 0$. The equations of $E(a,0)$ are

$$y = \sinh a - \frac{1}{2}e^a t^2, z = e^a t.$$

Then on the tangent variation $\delta E((a,0)|d\theta)$ we have

$$dy = -e^a t d\theta, dz = (\sinh a - \frac{1}{2}e^a t^2)d\theta.$$

Horocyclic hyperbolic transform3

The trivial variation is

$$\epsilon(d\theta) = (-e^a t, e^a)d\theta.$$

Now we need to make the correction ov the variation δE by the substruction of the multiple of ϵ such, that the resulting variation $\widetilde{\delta E}$ would be zero at $t = 0$. It means that we need substract $\frac{\sinh a}{e^a}\epsilon(d\theta)$ and the result to divide on t:

$$\widetilde{\delta E}(a, 0)\theta) = (-e^a - \sinh a, -\frac{1}{2}e^a t)d\theta = (-\cosh a, -\frac{1}{2}e^a t)d\theta.$$

So we consider for any fixed a

$$\kappa_a \mathcal{R} f(E(a, 0)|\widetilde{\delta E}) =$$

$$(\int_{-\infty}^{\infty}\{-\cosh a\frac{\partial}{\partial y} - \frac{1}{2}e^a t\frac{\partial}{\partial z}\}f(\sinh - \frac{1}{2}e^a t^2, e^a t)dt)d\theta.$$

Horocyclic hyperbolic transform3

In this point we met the characterictical problem: if we can to express this integral through the curved Radon transform for our family of parabolas. Other words, if we can in a sense to change the order of the differentiation and the integration. Let us consider

$$-\frac{\partial}{\partial a}(e^{a/2}\mathcal{R}f(E(a,0)|d\theta)$$

and use the factor e^a for thr change of the parameter $t \to \tilde{t} = e^{a/2}t$. The result will coincide with

$$e^{a/2}\kappa_a\mathcal{R}f(E(a,0)|\widetilde{\delta E})$$

Using the θ-invariancy - we found how to express $\kappa_a\mathcal{R}f$ rhrough $\mathcal{R}f$ on all S_a.

Horocyclic hyperbolic inversion formula

We can now apply Corollary from Tangency Principle and write down the inversion formula:

$$f(0,0) = c\int_0^\infty \frac{da}{\sinh a}\frac{\partial}{\partial a}(e^{a/2}\int_0^{2\pi} \mathcal{R}f(a,\theta)d\theta)$$

$$f(0,0) =$$

$$c\int_0^\infty \frac{da}{\sinh a}\frac{\partial}{\partial a}(e^{a/2}\int_0^{2\pi} d\theta \int_{-\infty}^\infty f(\sinh a\cos\theta - \sin\theta e^a t$$

$$-\frac{1}{2}\cos\theta e^a t^2, \sinh a \sin\theta + \cos\theta e^a t - \frac{1}{2}\sin\theta e^a t^2)a)dt.$$

We just need to see that the tangential part coincies with Radon's inversion formula up to a changw of variables.

Multidimensional curved Radon's inversion formula

Let us explained how to transfer to the multidimensional situation starting from the construction of the operator κ.By an induction we will generalize our construction from curves to m-dimensional surfaces $E \in \mathcal{E}$:

$$x = \varphi_E(t_1, t_2, \dots, t_m), x \in \mathbb{R}^n.$$

Again everything is smoothed and local. We consider the tangent space of $\delta E(t) \in T_E\mathcal{E}$.For a fixed point y let $\mathcal{E}_y$ and the subspace $T_E\mathcal{E}_y$ be the set with the condition

$$\varphi_E(0) = y.$$

We have $\delta E(0) = 0$ if $\delta E \in T_E\mathcal{E}_y$.
We construct a decomposition of this subpace of tangent variations in the direct sum with components

$$\delta^{(j)}E(t) = \delta E(t_1, t_2, \dots, t_j, 0. \dots, 0) - \delta E(t_1, t_2, \dots, t_{j-1}, 0. \dots, 0).$$

Multidimensional curved Radon's inversion formula2

We have

$$\delta^{(j)}(t)|_{t_j=0} \equiv 0.$$

If$F(E)$ is a functional on $\mathcal{E}_y$ we can consider the partial variational derivatives

$$\frac{\delta^{(j)} F}{\delta^{(j)} E}(E)$$

and the variational differentials

$$dF(E|dE) = \sum_j \frac{\delta^{(j)} F}{\delta^{(j)} E}(E) d^{(j)} E.$$

Then we can define the multidimensional operator κ from functionals to m-forms, through partial operators $\kappa^{(j)}$:

$$\kappa^{(j)} = dF(E|\delta^{(j)}/t_j).$$

This operator is well defined, since $\delta^{(j)} E/t_j$ is a regular variation.

Multidimensional curved Radon's inversion formula 3

Then we define

$$\kappa = \bigwedge_j \kappa^{(j)}.$$

The direct computation shows that the m-form $\kappa \mathcal{R} f(E|\delta E)$ is closed on $\mathcal{E}_y$.

Now we want to transfer this construction to the case of surfaces tangent to a a fixed m-dimensional submanifold. Let γ be such a surface with parameters θ and $\mathcal{E}_\gamma$ be a subset of surfaces which are tangent to γ.

We can define the curved multidimensional Radon transform $\mathcal{R}f$ and then m-form $\kappa \mathcal{R} f$ is closed and the tangency principle holds.

Horospherical transform

Let us consider the specialization of this construction in the case of horospheres. Let us consider a neighborhood of a point x_0 of the symmetric space X and let $\Xi \subset \mathcal{E}$ be the set of horospheres and E_0 be an initial horosphere, passing through x_0. We fixed a system of positive restricted roots $\alpha \in \Sigma$. Let us take a base of their root vectors e_j and let $\alpha(e_j)$ be the corresponding root. It could be $\alpha(e_i) = \alpha(e_j)$.
Let $\bar{N}$ be the opposite unipotent subgroup and $\{e_{-j}\}$ be compatible base of negative root vectors: $\alpha(e_{-j}) = -\alpha(e_j)$. Let

$$f_j = [e_{-j}, e_j], f_j \in Lie(A).$$

We have

$$E_0 = \exp\{t_1 e_1 + \cdots + t_m e_m\}$$

and will use $dt = dt_1 \wedge \cdots \wedge t_p$ for the definition of $\mathcal{H}f(E_0)$.

Horospherical transform 2

Let $E(a,v), a \in A, v \in F = K/M$ be the parameterization of horospheres. For $x \in X$ we denote Ξ_x the set of horospheres passing through x. We have $\Xi_x = F = K/M$. The horospheres $E(a,v)$ give the cycle of horospheres $\Xi(a), a \in A$ which are tangent to the cycle $\gamma(a) = \{x_0 aK\}$; $\gamma(a)$ are flag manifolds.We can use for the parameterization $k \in K/M$; but for computations it is more convenient to use on the open set elements $\zeta \in \bar{N}$ as local coordinates.

We want to compute the action of the operator κ on Ξ_x and $\Xi(a)$. Let us start with Ξ_x; the case $\Xi(a)$ is reduced to it,

We take $\zeta \in \bar{N} = \exp\{s_1 e_{-1} + \cdots + s_m e_{-m}\}$ as parameters. So the variations δE will correspond to $d\zeta = ds_1 e_{-1} + \cdots + ds_m e_{-m}$ and we have

$$\delta E(t|d\zeta) = [\sum t_j e_j, \sum ds_i e_{-i}].$$

Horospherical transform 3

Then

$$\delta^{(j)}E(d\zeta) = t_j[e_j, \sum ds_i e_{-i}]$$

and

$$\kappa^{(j)}F(E, d\zeta) = df(E, \tilde{\delta}^{(j)}E/t_j). = df(E, [e_j, \sum ds_i e_{-i}]),$$

We see that here $\delta^{(j)}E/t_j$ are independent of t.
To investigate these components of the variations chose any order of the positive root vectors such that

$$\alpha(e_p) + \alpha(e_q) = \alpha(e_r) \Rightarrow r > p, r > q.$$

Horospherical transform 4

Then we can present the commutator in $\delta^j E/t_j$ as the sum of 3 terms:

$$\delta^j E/t_j = -ds_j f_j + \delta_1^j E/t_j + \delta_2^j E/t_j.$$

We receive the first term when we take $i = j$. For the next term we collect $i < j$ and for the last one we take $i > j$. The variation $\delta_1^j E/t_j$ just corresponds to an unipotent change of the parameterization on the horosphere and the variational derivative of $\mathcal{H}f$ on it equal zero and we can omit it in the computation of $\kappa\mathcal{H}f$. It exactly corresponds to the trvial deformations.
Let us compute $\kappa\mathcal{H}f = \bigwedge_j \kappa^{(j)}\mathcal{H}f$ by the induction on decreasing j.*For* $j = m$ *it will be only one term with* ds_m. *For* $i = m - 1$ *there will be 2 terms with* ds_m *and* ds_{m-1}. *However we can omit the term with* ds_m *as the result of the symmetry and by induction we see that only the variation* $-ds_j f_j$ *participates in the computation of* $\kappa\mathcal{H}f$.

Similar computations hold for $\Xi(a)$ - the set of horospheres tangent to the cycle $\gamma(a)$. So we have

$$\kappa\mathcal{H}f(a,\zeta,d\zeta)=\bigwedge_{\alpha\in\Sigma}(D_\alpha)^{m_\alpha}\mathcal{H}f(a,\zeta)d\zeta;$$

where we unify root vectors with a joint root α and m_α is the multiplicity; D_α is the derivative in the direction of α (corresponding to f_j). We can replace in this formula parameterization $\zeta\in\bar{N}$ on $k\in K/M$. Then we can write the similar operator for the cycle of tangents and apply the tangency principle:

$$\int_{\gamma(a)}\kappa\mathcal{H}f(a,\zeta,d\zeta)=\int_{\gamma(a)}\kappa\mathcal{H}_{tang}f(a,\zeta,d\zeta).$$

Horospherical Cauchy transform

Next we construct the horospherical Cauchy transform $\mathcal{C}f(a,k)a \in A, k \in F = K/M$ on X and its tangential version $\mathcal{C}f(b,k).b \in B = Lie\, A, k \in K/M)$. We define both by a convolution with some Cauchy kernels on the group A either on its Lie algebra.

We have in the tangential case

$$\mathcal{C}f(k, c - i0) = \int_B \frac{\mathcal{H}f(k,b)}{\prod_{j\leq l}(\pi_j(b) - c_j - i0)} db,$$

where π_J are prime roots. On X we must take as the kernel

$$\frac{1}{\prod_{j\leq l}(\sinh(\pi_j(\ln a)) - i0)}.$$

We calibrate the Cauchy kernels such that the tangency principle holds for the horospherical Cauchy kernels.

Horospherical Cauchy transform 2

Since our Cauchy convolutions in the constructions of the horospherical Cauchy transforms commutate with the operator κ, we have the tangency principle for the horospherical Cauchy transforms , which immediately give the inversion formulas. As a result we have

$$\int_{K/M} \kappa\mathcal{C}(x, k, dk) = \int_{K/M} \kappa\mathcal{C}_{tang}(x, k, dk) = cf(x).$$

We just use here that this inversion was already proven for the tangential horospherical Cauchy transform and apply the tangency principle.

Horospherical Cauchy transform 3

Then it gives the inversion formula

$$f(x) = \int_{S(x)} \Big(\prod_{\alpha\in\Sigma} D_\alpha^{m_\alpha}\Big)\mathcal{C}f(a(x),k)dk,$$

where we integrate along the pseudosphere $S(x)$ parameterizing the horospheres passing through x; correspondingly $a(x)$ chosen.
This formula is true simultaneously for X and its tangent model.
However the tangent version was found a long time ago. It means that the curved version on X holds as well.