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Inventiones mathematicae

Monodromy for the hypergeometric function ${}_nF_{n-1}$

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1. Introduction

The classical hypergeometric function is defined by the series

$$F(\alpha,\beta,\gamma;z)=\sum_{k=0}^{\infty}\frac{(\alpha)_k\,(\beta)_k}{(\gamma)_k\,k!}\,z^k \tag{1.1}$$

using the Pochhammer notation

$$(\alpha)_k=\alpha(\alpha+1)\ldots(\alpha+k-1)=\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}. \tag{1.2}$$

During the last century this function has been the subject of an extensive study, especially in the work of Euler, Gauss, Riemann, Schwarz and Klein. For historical background we refer to Klein's lectures on the hypergeometric function [Kl].

The higher hypergeometric function ${}_nF_{n-1}$ was introduced by Thomae as the series

$${}_nF_{n-1}(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_{n-1}|z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_n)_k\,z^k}{(\beta_1)_k\ldots(\beta_{n-1})_k\,k!}. \tag{1.3}$$

The case $n=2$ corresponds to the expression (1.1) [T]. It is the solution of a linear differential equation on $\mathbb{P}^1(\mathbb{C})$ of order n with regular singularities at the points $z=0, 1, \infty$ (see (2.5) or [E, Chap. 4]). As observed by Riemann the monodromy group plays a crucial role in the study of these differential equations and their solutions [R]. For example the differential Galois group which carries all information about algebraic relations between the solutions and their derivatives is just the Zariski closure of the monodromy group. (see [Kap]).

In this paper we discuss the following problem.

Problem 1.1. *What is the differential Galois group of the function* (1.3) *for the various parameters* $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{n-1}$?

The answer to this problem has a surprisingly simple form

Solution 1.2. *Under a suitable primitivity assumption and up to scalars the differential Galois group of the function* (1.3) *is either one of the classical groups* $SL(n, \mathbb{C})$ $SO(n, \mathbb{C})$, $\mathrm{Sp}(n, \mathbb{C})$ *or a finite primitive reflection group as listed in the table of Shephard and Todd* [ST]. *Moreover, Theorems* 6.5 *and* 7.1 *give an explicit algorithm to decide which groups occur for which parameters.*

In particular, Theorem 7.1 classifies the generalised hypergeometric functions which are algebraic over $\mathbb{C}(z)$. For the case $n=2$ this was already done by H.A. Schwarz [Sc] in 1873, but for the case $n>2$ not much was known. The solution of this problem was the primary goal of this paper. However, it turned out that without too much effort one could also describe the differential Galois group of the hypergeometric differential equation in general. This is carried out in Sect. 6, Theorem 6.5.

An important element in the proof of the above results is a theorem of Levelt, which gives a simple algebraic characterisation of the monodromy group of a hypergeometric differential equation [Le, Thm. 1.1]. The original transcendental problem 1.1 is now reduced to an algebraic problem which we set out to solve in this paper.

There remain some unanswered questions as well, the most important one being the determination of hypergeometric equations whose monodromy group is discrete or arithmetic. In this respect we like to draw attention to the very interesting work of Mostow and Deligne [Mo] which describes the monodromy of certain generalised hypergeometric functions in several variables.

Finally we like to thank Geert Verhagen for verifying our computations, settling some undecided cases and removing a number of errors in previous versions of our tables.

2. The hypergeometric equation

Fix an integer $n \geqq 2$. For $p_1, \ldots, p_n \in \mathbb{C}(z)$ consider the differential operator

$$P=\theta^n+p_1\,\theta^{n-1}+\ldots+p_{n-1}\,\theta+p_n, \qquad \theta=z\frac{d}{dz} \tag{2.1}$$

on $\mathbb{P}^1(\mathbb{C})$. Using the criterion of Fuchs [I, Chap. 15.3] the following proposition is immediate.

Proposition 2.1. *The differential equation $Pu=0$ has regular singularities in the points $z=0, 1, \infty$ and is regular elsewhere if and only if for all $j=1, \ldots, n$*

$$p_j(z)=\sum_{k=0}^{j} p_{jk}(z-1)^{-k} \tag{2.2}$$

for suitable $p_{jk}\in\mathbb{C}$.

Definition 2.2. *The differential equation $Pu=0$ with regular singularities in the points $z=0, 1, \infty$ is called a hypergeometric equation if and only if*

$$p_{jk}=0 \qquad \text{for all } k\geqq 2 \quad \text{and all } j \tag{2.3}$$

i.e. the functions $p_j(z)$ have simple poles at $z=1$.

If $Pu=0$ is a hypergeometric equation then $D=(1-z)P$ has the form

$$D=\theta^n+(p_{10}-p_{11})\,\theta^{n-1}+\ldots+(p_{n0}-p_{n1})-z(\theta^n+p_{10}\,\theta^{n-1}+\ldots+p_{n0}). \tag{2.4}$$

We write

$$\begin{aligned} D&=D(\alpha;\beta)=D(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n)\\ &=(\theta+\beta_1-1)\ldots(\theta+\beta_n-1)-z(\theta+\alpha_1)\ldots(\theta+\alpha_n)\end{aligned}$$

for $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\in\mathbb{C}$. From now on we shall denote the hypergeometric equation by

$$D(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n)\,u=0 \quad \text{or} \quad D(\alpha;\beta)\,u=0. \tag{2.5}$$

Its local exponents read,

$$1-\beta_1,\ldots,1-\beta_n \qquad \text{at } z=0 \tag{2.6}$$

$$\alpha_1,\ldots,\alpha_n \qquad \text{at } z=\infty \tag{2.7}$$

$$0,1,2,\ldots,n-2, \quad \gamma=\sum_1^n\beta_j-\sum_1^n\alpha_i-1 \qquad \text{at } z=1 \tag{2.8}$$

around the points $z=0, \infty$ and 1 respectively. If the numbers $\beta_1, \ldots, \beta_n$ are distinct mod $\mathbb{Z}$, n independent solutions of $D(\alpha;\beta)\,u=0$ are given by

$$\begin{gathered} z^{1-\beta_i}{}_nF_{n-1}(1+\alpha_1-\beta_i,\ldots,1+\alpha_n-\beta_i;1+\beta_1-\beta_i,\overset{\vee}{\ldots},1+\beta_n-\beta_i\,|\,z)\\ (i=1,\ldots,n)\end{gathered} \tag{2.9}$$

where $\vee$ denotes omission of $1+\beta_i-\beta_i$. The following proposition is trivial.

Proposition 2.3. *For $\delta\in\mathbb{C}$ we have*

$$(\theta+\delta-1)\,D(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n)=D(\alpha_1,\ldots,\alpha_n,\delta;\beta_1,\ldots,\beta_n,\delta) \tag{2.10}$$

and

$$D(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n)(\theta+\delta)=D(\alpha_1,\ldots,\alpha_n,\delta;\beta_1,\ldots,\beta_n,\delta+1). \tag{2.11}$$

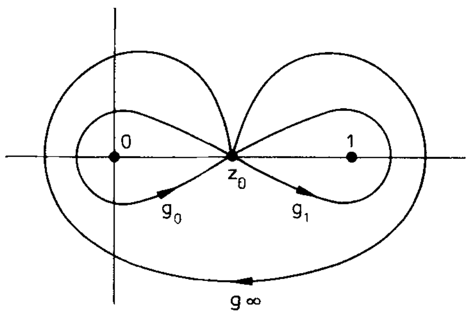
Corollary 2.4. *We have*

$$D(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)(\theta+\alpha_j-1)$$
$$=(\theta+\alpha_j-1)\, D(\alpha_1, \ldots, \alpha_j-1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$$

and

$$D(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)(\theta+\beta_j)$$
$$(\theta+\beta_j-1)\, D(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_j+1, \ldots, \beta_n).$$

Fix a base point $z_0 \in \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, e.g. $z_0 = \frac{1}{2}$. Denote by G the fundamental group $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}, z_0)$. Clearly G is generated by g_0, g_1, g_∞ with a single relation $g_\infty g_1 g_0 = 1$.



Let $V(\alpha; \beta)$ denote the local solution space of the hypergeometric equation $D(\alpha; \beta)\, u = 0$ around z_0. Denote by

$$M(\alpha; \beta)\colon\ G \to GL(V(\alpha; \beta)) \tag{2.14}$$

the monodromy representation of $D(\alpha; \beta)\, u = 0$. The following proposition follows immediately from Corollary 2.4

Proposition 2.5. *The operators*

$$(\theta+\alpha_j-1)\colon\ V(\alpha_1, \ldots, \alpha_j-1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$$
$$\to V(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n) \tag{2.15}$$

and

$$(\theta+\beta_j)\colon\ V(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_j+1, \ldots, \beta_n)$$
$$\to V(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n) \tag{2.16}$$

are intertwining operators for the monodromy representations. The operator (2.15) *has a nontrivial kernel if and only if $\alpha_j = \beta_k$ for some $k = 1, \ldots, n$. Similarly* (2.16) *has a nontrivial kernel if and only if $\alpha_k = \beta_j$ for some $k = 1, \ldots, n$. Moreover, in case the kernel of* (2.15) *or* (2.16) *is nontrivial it has dimension one.*

Corollary 2.6. *If $\alpha_j-\beta_k\notin\mathbb{Z}$ for all j, $k=1,\ldots,n$ then the representations $M(\alpha_1+k_1,\ldots,\alpha_n+k_n;\beta_1+1_1,\ldots,\beta_n+1_n)$ and $M(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n)$ are equivalent for any $k_1,\ldots,k_n,1_1,\ldots,1_n\in\mathbb{Z}$.*

Proposition 2.7. *If $\alpha_j-\beta_k\in\mathbb{Z}$ for some j, $k=1,\ldots,n$ then the monodromy representation* (2.14) *is reducible.*

Proof. Say $\alpha_n-\beta_n=m\in\mathbb{Z}$. If $m=-1$, then $D(\alpha_1,\ldots,\alpha_n;\ \beta_1,\ldots,\beta_n)=D(\alpha_1,\ldots,\alpha_{n-1};\beta_1,\ldots,\beta_{n-1})(\theta+\alpha_n)$ and

$$z^{-\alpha_n}\in V(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n)$$

generates a one dimensional invariant subspace.

If $m\geqq 0$ then consider the sequence

$$\begin{aligned}&V(\alpha_1,\ldots,\alpha_{n-1},\beta_n-1;\beta_1,\ldots,\beta_n)\\&\xrightarrow{\theta+\beta_n-1}V(\alpha_1,\ldots,\alpha_{n-1},\beta_n;\beta_1,\ldots,\beta_n)\xrightarrow{\theta+\beta_n}\cdots\\&\xrightarrow{\theta+\beta_n+m-2}V(\alpha_1,\ldots,\alpha_{n-1},\beta_n+m-1;\beta_1,\ldots,\beta_n)\\&\xrightarrow{\theta+\beta_n+m-1}V(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n).\end{aligned}$$

Clearly $\theta+\beta_n-1$ has a nontrivial kernel. Choose $j\in\{-1,0,\ldots,m-1\}$ maximal such that $\theta+\beta_n+j$ has a nontrivial kernel. Then the image of the map $(\theta+\beta_n+m-1)\ldots(\theta+\beta_n+j)$ is a codimension one invariant subspace in $V(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n)$.

If $m\leqq -2$ then consider the sequence

$$\begin{aligned}&V(\alpha_1,\ldots,\alpha_{n-1},\alpha_n=\beta_n+m;\beta_1,\ldots,\beta_n)\xrightarrow{\theta+\beta_n+m}V(\alpha_1,\ldots,\alpha_{n-1},\\&\beta_n+m+1;\beta_1,\ldots,\beta_n)\xrightarrow{\theta+\beta_n+m+1}\cdots\xrightarrow{\theta+\beta_n-2}V(\alpha_1,\ldots,\alpha_{n-1},\\&\beta_n-1;\beta_1,\ldots,\beta_n)\xrightarrow{\theta+\beta_n-1}V(\alpha_1,\ldots,\alpha_{n-1},\beta_n;\beta_1,\ldots,\beta_n).\end{aligned}$$

Clearly $\theta+\beta_n-1$ has a nontrivial kernel. Choose $j\in\{m,m+1,\ldots,-1\}$ minimal such that $\theta+\beta_n+j$ has a nontrivial kernel. Then the kernel of the map $(\theta+\beta_n+j)\ldots(\theta+\beta_n+m)$ is a one dimensional invariant subspace in $V(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n)$. For the following proposition see also [Po].

Proposition 2.8 (Pochhammer). *If $\gamma\notin\mathbb{N}$ in the notation of* (2.8) *then the hypergeometric equation $D(\alpha,\beta)u=0$ has $n-1$ analytic solutions near $z=1$ of the form*

$$u_j(z)=(z-1)^{j-1}+O((z-1)^{n-1}),\qquad z\to 1 \tag{2.17}$$

for $j=1,\ldots,n-1$ corresponding to the exponents $0,1,\ldots,n-2$.

Proof. If $\gamma-n+2\notin\mathbb{N}$ then the equation $D(\alpha;\beta)u=0$ has an analytic solution near $z=1$ of the form

$$u(\alpha;\beta)(z)=(z-1)^{n-2}+O((z-1)^{n-1}),\qquad z\to 1.$$

Hence the desired solution $u_{n-1}(z)$ can be obtained. The solution $u_j(z)$ can be obtained by a downward induction on j, the case $j=n-1$ being known. Suppose the solutions $u_{j+1}(z), \ldots, u_{n-1}(z)$ have been obtained. Using Corollary 2.4 it follows that $u_j(z)$ can be obtained as a linear combination of $u_{j+1}(z), \ldots, u_{n-1}(z)$ and the solution

$$(\theta+\beta_n)\ldots(\theta+\beta_n+n-2-j)\,u(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_{n-1}, \beta_n+n-1-j)(z).$$

Observe that this solution is well defined since

$$\sum_1^n\beta_j+(n-1-j)-\sum_1^n\alpha_j-1-n+2=\gamma-j+1\notin\mathbb{N}\qquad\text{for } j=1, \ldots, n-1.$$

Definition 2.9. *Let V be a finite dimensional complex vector space. A linear map $g\in GL(V)$ is called a reflection if $g-\mathrm{Id}$ has rank one. The determinant of a reflection is called the special eigenvalue of g.*

Remark. The reflections defined here are often called complex reflections or quasi-reflections to distinguish them from the standard ones of order 2.

Proposition 2.10. *If $\alpha_j-\beta_k\notin\mathbb{Z}$ for all $k, j=1, \ldots, n$, then the monodromy matrix $M(\alpha;\beta)(g_1)$ around $z=1$ is a reflection with special eigenvalue $c=\exp(2\pi i\gamma)$.*

Proof. By Corollary 2.6 we can shift the parameters $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$ by integers such that the condition $\gamma\notin\mathbb{N}$ is satisfied. By Proposition 2.8 we conclude that the rank of the matrix $M(\alpha;\beta)(g_1)-\mathrm{Id}$ is at most one. If $M(\alpha;\beta)(g_1)=\mathrm{Id}$ then $M(\alpha;\beta)(g_\infty)=M(\alpha;\beta)(g_0^{-1})$ and the condition $\alpha_j-\beta_k\notin\mathbb{Z}$ for all $j, k=1, \ldots, n$ becomes violated.

3. The hypergeometric group

Definition 3.1. *Suppose $a_1, \ldots, a_n, b_1, \ldots, b_n\in\mathbb{C}^*$* with $a_j\neq b_k$ *for all $j, k=1, \ldots, n$. A hypergeometric group with numerator parameters $a_1, \ldots, a_n$ and denominator parameters $b_1, \ldots, b_n$ is a subgroup of $GL(n, \mathbb{C})$ generated by elements*

$$h_0, h_1, h_\infty\in GL(n, \mathbb{C}) \tag{3.1}$$

such that

$$h_\infty\, h_1\, h_0=\mathrm{Id} \tag{3.2}$$

and

$$\det(t-h_\infty)=\prod_{j=1}^{n}(t-a_j) \tag{3.3}$$

$$\det(t-h_0^{-1})=\prod_{j=1}^{n}(t-b_j) \tag{3.4}$$

and h_1 is a reflection in the sense of Definition 2.9.

Proposition 3.2. *Suppose $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}^*$* with $a_j \neq b_k$ *for all $j, k=1, \ldots, n$. Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{C}$ be such that $a_j=\exp 2\pi i\alpha_j$ and $b_j=\exp 2\pi i\beta_j$ for $j=1, \ldots, n$. Then the monodromy group of the hypergeometric equation*

$$D(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)\,u=0 \tag{3.5}$$

is a hypergeometric group with parameters $a_1, \ldots, a_n, b_1, \ldots, b_n$.

Proof. Denote by

$$H(a;b)=H(a_1, \ldots, a_n; b_1, \ldots, b_n)=M(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)(G) \tag{3.6}$$

the monodromy group of (3.5). Observe that by Corollary 2.6 this group depends only on the numbers $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ and not on the choice of their logarithms $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$. Also write

$$h_0=M(\alpha;\beta)(g_0), \qquad h_1=M(\alpha;\beta)(g_1), \qquad h_\infty=M(\alpha;\beta)(g_\infty) \tag{3.7}$$

for the corresponding monodromy matrices around $z=0, 1, \infty$. Using formulas (2.6) and (2.7) and Proposition 2.10 it follows that $H(a;b)$ is a hypergeometric group with numerator parameters $a_1, \ldots, a_n$ and denominator parameters $b_1, \ldots, b_n$.

Proposition 3.3. *Any hypergeometric group H generated by h_0, h_1, h_∞ as in Definition* 3.1 *is an irreducible subgroup of $GL(n, \mathbb{C})$.*

Proof. If $V_1 \subset \mathbb{C}^n$ is an H-invariant linear subspace and $V_2 := \mathbb{C}^n/V_1$, then we get induced groups $H_1 \subset GL(V_1)$ and $H_2 \subset GL(V_2)$. Since h_1 is a reflection, either h_1 restricted to V_1 or h_1 restricted to V_2 is the identity. Hence if both $V_1 \neq 0$ and $V_2 \neq 0$ we get a contradiction with the assumption $a_j \neq b_k$ for all $j, k=1, \ldots, n$.

The following theorem was obtained by Levelt in his thesis [Le, Thm 1.1].

Theorem 3.5 (Levelt). *Suppose $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in \mathbb{C}^*$* with $a_j \neq b_k$ *for all $j, k=1, \ldots, n$. Let $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathbb{C}$ be defined by*

$$\prod_{j=1}^{n}(t-a_j)=t^n+A_1 t^{n-1}+\ldots+A_n, \qquad \prod_{j=1}^{n}(t-b_j)=t^n+B_1 t^{n-1}+\ldots+B_n \tag{3.8}$$

and let $A, B \in GL(n, \mathbb{C})$ be given by

$$A=\begin{pmatrix} 0 & 0 & \dots & 0 & -A_n \\ 1 & 0 & \dots & 0 & -A_{n-1} \\ 0 & 1 & \dots & 0 & -A_{n-2} \\ & & \dots & & \\ 0 & 0 & \dots & 1 & -A_1 \end{pmatrix}, \quad B=\begin{pmatrix} 0 & 0 & \dots & 0 & -B_n \\ 1 & 0 & \dots & 0 & -B_{n-1} \\ 0 & 1 & \dots & 0 & -B_{n-2} \\ & & \dots & & \\ 0 & 0 & \dots & 1 & -B_1 \end{pmatrix}. \tag{3.9}$$

Then the matrices $h_\infty = A$, $h_0 = B^{-1}$, $h_1 = A^{-1}B$ generate a hypergeometric group with parameters $a_1, \dots, a_n$, $b_1, \dots, b_n$. Moreover, any hypergeometric group with the same parameters is conjugated inside $GL(n, \mathbb{C})$ to this one.

Proof. An easy calculation shows that

$$\det(t-A) = t^n + A_1 t^{n-1} + \dots + A_n \qquad \det(t-B) = t^n + B_1 t^{n-1} + \dots + B_n$$

and hence conditions (3.3) and (3.4) are satisfied. Also $h_1 - \mathrm{Id} = A^{-1}B - \mathrm{Id} = A^{-1}(B-A)$ has rank one, and the first statement of the theorem follows.

Conversely, suppose we have a hypergeometric group $H \subset GL(n, \mathbb{C})$ with parameters $a_1, \dots, a_n$; $b_1, \dots, b_n$ and generators h_0, h_1, h_∞ as in Definition 3.1. Put $A = h_\infty$, $B = h_0^{-1}$ and let W be the kernel of $B - A$. Since $\dim W = n-1$ there exists a nonzero vector $v \in \bigcap_{j=0}^{n-2} A^{-j} W$. We claim that the vectors $A^j v (j = 0, \dots, n-1)$ form a basis of $\mathbb{C}^n$. If this is not the case, then $\mathrm{span}(A^j v; j \in \mathbb{Z})$ is a nonzero linear subspace of W invariant under A and B, contradicting Proposition 3.3. Moreover, since $A^j v \in W$ $(j = 0, \dots, n-2)$ and $(B-A)x = 0$ for all $x \in W$ we see that $B^j v = A^j v (j = 0, \dots, n-1)$. Thus the matrices of A and B with respect to the basis $A^j v (j = 0, \dots, n-1)$ have the form (3.9) which shows the uniqueness of H.

Corollary 3.6. *Suppose $a_1, \dots, a_n$; $b_1, \dots, b_n \in \mathbb{C}^*$* with $a_j \neq b_k$ *for all $j, k = 1, \dots, n$. Let $A_1, \dots, A_n$, $B_1, \dots, B_n \in \mathbb{C}$ be defined by (3.8). Relative to a suitable basis the hypergeometric group $H(a; b) \subset GL(n, \mathbb{C})$ with parameters $a_1, \dots, a_n$; $b_1, \dots, b_n$ is defined over the ring $\mathbb{Z}[A_1, \dots, A_n, B_1, \dots, B_n, A_n^{-1}, B_n^{-1}]$.*

Remark 3.7. It follows from Proposition 3.2 that the hypergeometric equation $D(\alpha; \beta) u = 0$ can be viewed as an explicit solution of the Riemann monodromy problem [Pl, Sect. 15] for the special case of the hypergeometric group $H(a; b)$.

4. The invariant hermitian form

It is a well-known fact that the second order hypergeometric equation with real parameters has a monodromy group which is either contained in $U(2)$ or $U(1,1) \simeq GL(2, \mathbb{R})$ (see [Kl, p 211]). Surprisingly, it turns out that a similar statement holds for generalised hypergeometric equations as well. The construction of hermitian forms invariant under the monodromy will be the subject of this section.

Lemma 4.1. *Let $P, Q \in M_n(\mathbb{C})$ be two n by n matrices having the same characteristic equation. Suppose there exists a vector v such that $v, Pv, \ldots, P^{n-1}v$ are linearly independent (i.e. P is regular). Consider $W = \{X \in M_n(\mathbb{C});\ XP = QX\}$. Then W is a $\mathbb{C}$-linear vectorspace of dimension at least n.*

Proof. Choose $x \in \mathbb{C}^n$ arbitrarily. Let X be the matrix such that $XP^i v = Q^i x$ for $i = 0, 1, \ldots, n-1$. Then, clearly, $(XP - QX)P^j v = XP^{j+1}v - QXP^j v = Q^{j+1}x - Q^{j+1}x = 0$ for $j < n-1$. Since $P^n + r_1 P^{n-1} + \ldots + r_n = 0$ and $Q^n + r_1 Q^{n-1} + \ldots + r_n = 0$ we also have $(XP - QX)P^{n-1}v = XP^n v - QXP^{n-1}v = XP^n v - Q^n x = -X(r_1 P^{n-1}v + \ldots + r_n v) + r_1 Q^{n-1}x + \ldots + r_n x = 0$. Hence $XP - QX = 0$. The map $\phi\colon \mathbb{C}^n \to W$ which associates X to x is clearly linear and injective, hence $\dim W \geqq n$.

Remark. Let $g \in M_n(\mathbb{C})$ be an n by n matrix with entries in $\mathbb{C}$. In this section g^t will denote the transpose of g and $\bar{g}$ the matrix obtained by complex conjugation of all entries of g.

Lemma 4.2. *Suppose $g \in M_n(\mathbb{C})$ has the form*

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & g_n \\ 1 & 0 & \ldots & 0 & g_{n-1} \\ 0 & 1 & \ldots & 0 & g_{n-2} \\ & & \ldots & & \\ 0 & 0 & \ldots & 1 & g_1 \end{pmatrix} \quad \text{with } g_i \in \mathbb{C} \text{ for all } i \text{ and } g_n \neq 0.$$

Then any solution $X \in M_n(\mathbb{C})$ of $g^t X \bar{g} = X$ has the form $X = (X_{ij})$ where the entries X_{ij} depend only on $i - j$.

Proof. Direct computation.

Theorem 4.3. *Let $H(a; b) \subset GL(n, \mathbb{C})$ denote the hypergeometric group with parameters $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$ as constructed in Theorem 3.5. Suppose the sets $\{a_i\}_i$ and $\{b_i\}_i$ are invariant under the substitution $z \to \bar{z}^{-1}$. Then there exists a nondegenerate hermitian form $F(x, y) = \sum F_{ij} x_i \bar{y}_j$ on $\mathbb{C}^n$ such that*

$$F(hx, hy) = F(x, y) \quad \text{for all } h \in H(a; b) \quad \text{and all } x, y \in \mathbb{C}^n. \tag{4.1}$$

Proof. It suffices to construct a nondegenerate hermitian form F such that (4.1) is satisfied by $h = h_\infty, h_0^{-1}$. Such a form with matrix $F = (F_{ij})$ is solution of

$$h^t F \bar{h} = F \quad \text{for } h = h_\infty, h_0^{-1} \tag{4.2}$$

and

$$F = \bar{F}^t. \tag{4.3}$$

According to Theorem 3.5 the matrices of h_0^{-1}, h_∞ can be given the form required by Lemma 4.2. Hence the entries of $F=(F_{ij})$ depend only on $i-j$, which implies that the solutions of (4.2) are contained in a vector space of dimension $2n-1$. Rewrite (4.2) as $F\bar{h}=(h^t)^{-1}F$, $h=h_\infty, h_0^{-1}$. Since the parameter sets are invariant under $z\to\bar{z}^{-1}$ the matrices $\bar{h}$ and $(h^t)^{-1}$ have the same characteristic equation. Application of Lemma 4.1 now shows that the solutions of (4.2) have dimension at least n for each choice of $h=h_0^{-1}$, h_∞. Since these spaces are contained in a $2n-1$ dimensional space, they have non trivial intersection. So (4.2) has a nontrivial simultaneous solution, say F_0, for $h=h_0^{-1}$, h_∞. Notice that if F is a solution of (4.2) then so is $\bar{F}^t$. In particular, both

$$F_0+\bar{F}_0^t \quad \text{and} \quad i(F_0-\bar{F}_0^t) \tag{4.4}$$

are solutions of (4.2) which, in addition, satisfy the constraint (4.3). Since F_0 is nontrivial, at least one of (4.4) is nontrivial, and this will be the matrix of the required hermitian form. Non-degeneracy of the form F follows from the fact that it is non-trivial and invariant for the group $H(a;b)$, which acts irreducibly on $\mathbb{C}^n$.

In the following Proposition and Theorem we determine the signature of the hermitian form.

Proposition 4.4. *Let $H(a;b)$ be a hypergeometric group as in Theorem 4.3. Let $c=b_1\dots b_n\, a_1^{-1}\dots a_n^{-1}$ and let ζ be a solution of $c\zeta^2=-1$. Consider the rank one linear map $D=\zeta(h_1-\mathrm{Id})$. Then there exists a non-zero vector $u\in\mathbb{C}^n$ such that*

$$D(u)=\pm F(x,u)\,u \quad \textit{for all } x\in\mathbb{C}^n. \tag{4.5}$$

Proof. Using the orthogonality of h_1 with respect to F we see that the adjoint of D with respect to F equals $D^*=\zeta^{-1}(h_1^{-1}-\mathrm{Id})$. Note that c is the special eigenvalue of h_1, hence $(h_1-\mathrm{Id})(h_1-c)=0$, from which one can see in a straightforward manner, that $D=D^*$.

Since D is a rank one map there exists nonzero $v, w\in\mathbb{C}^n$ such that

$$D(x)=F(x,v)\,w \quad \text{for all } x\in\mathbb{C}^n.$$

Clearly, the adjoint D^* of D is given by

$$D^*(x)=F(x,w)\,v \quad \text{for all } x\in\mathbb{C}^n.$$

Because $D^*=D$ we deduce $w=\lambda v$ for some $\lambda\in\mathbb{R}^*$. Now take $u=|\lambda|^{\frac{1}{2}}v$.

Theorem 4.5. *Suppose $a_1,\dots,a_n;\, b_1,\dots,b_n\in\mathbb{C}^*$* with $a_j\neq b_k$ *for all $j,k=1,\dots,n$ and such that $|a_j|=|b_j|=1$ for all $j=1,\dots,n$. Choose α_j, $\beta_j\in[0,1)$ such that $a_j=\exp 2\pi i\alpha_j$ and $b_j=\exp 2\pi i\beta_j$. By renumbering the indices we may assume that $0\leqq\alpha_1\leqq\dots\leqq\alpha_n<1$ and $0\leqq\beta_1\leqq\dots\leqq\beta_n<1$. Let $m_j=\#\{k;\,\beta_k<\alpha_j\}$ for $j=1,\dots,n$. Then the signature (p,q) of the hermitian form for the hypergeometric group $H(a;b)$*

is given by

$$|p-q| = \left|\sum_{j=1}^{n} (-1)^{j+m_j}\right|. \tag{4.6}$$

Proof. We use the notation $A=h_\infty$, $B=h_0^{-1}$ in this proof. First suppose that $a_j \neq a_k$ for all $j \neq k$. Write the vector u, defined by (4.5), as $u=u_1+\ldots+u_n$ with $\mathrm{Au}_j = a_j u_j$ for $j=1, \ldots, n$. Notice that

$$(a_j \bar{a}_k - 1) F(u_j, u_k) = F(\mathrm{Au}_j, \mathrm{Au}_k) - F(u_j, u_k) = 0.$$

When $j \neq k$ we have by assumption $a_j \bar{a}_k \neq 1$ and hence $F(u_j, u_k)=0$ for all $j \neq k$, i.e. the basis $u_1, \ldots, u_n$ is orthogonal. Letting D be as in Lemma 4.4 one easily verifies that

$$\begin{aligned}\prod_{k=1}^{n} (b_k - t)(a_k - t)^{-1} &= \det((B - t\mathrm{Id})(A - t\mathrm{Id})^{-1} \\ &= \det(\mathrm{Id} + \zeta^{-1} D(\mathrm{Id} - tA^{-1})^{-1}).\end{aligned} \tag{4.7}$$

If a rank one n by n matrix M acts on $\mathbb{C}^n$ as $Mx = w(x)u$ for some linear form w, one has $\det(\mathrm{Id}+M) = 1 + w(u)$. Using this fact in (4.7) and Lemma 4.4 we find that

$$\begin{aligned}\prod_{k=1}^{n} (b_k - t)(a_k - t)^{-1} &= 1 \pm \zeta^{-1} F((\mathrm{Id} - tA^{-1})^{-1} u, u) \\ &= 1 \pm \zeta^{-1} F\left(\sum_{j=1}^{n} a_j (a_j - t)^{-1} u_j, \sum_{j=1}^{n} u_j\right) \\ &= 1 \pm \zeta^{-1} \sum_{j=1}^{n} \frac{a_j}{a_j - t} F(u_j, u_j).\end{aligned}$$

Taking residues at $t = a_j$ yields

$$F(u_j, u_j) = \pm \zeta (b_j - a_j) a_j^{-1} \prod_{k \neq j} (b_k - a_j)(a_k - a_j)^{-1}.$$

Writing $\mp \zeta = i a_1^{\frac{1}{2}} \ldots a_n^{\frac{1}{2}} b_1^{-\frac{1}{2}} \ldots b_n^{-\frac{1}{2}}$ we find

$$\begin{aligned}F(u_j, u_j) &= -i(b_j^{\frac{1}{2}} a_j^{\frac{1}{2}} - b_j^{-\frac{1}{2}} a_j^{\frac{1}{2}}) \prod_{k \neq j} (b_k^{\frac{1}{2}} a_j^{-\frac{1}{2}} - b_k^{-\frac{1}{2}} a_j^{\frac{1}{2}})(a_k^{\frac{1}{2}} a_j^{-\frac{1}{2}} - a_k^{-\frac{1}{2}} a_j^{\frac{1}{2}})^{-1} \\ &= 2 \sin \pi(\beta_j - \alpha_j) \prod_{k \neq j} \frac{\sin \pi(\beta_k - \alpha_j)}{\sin \pi(\alpha_k - \alpha_j)}.\end{aligned}$$

Our assertion follows simply by determination of the sign of the latter products for each j. A continuity argument shows that the signature of the hermitian form does not change if we let α_j and β_k vary continuously with the restriction $\alpha_j \neq \beta_k$ for all $j, k = 1, \ldots, n$. Hence the statement also follows if $a_j = a_k$ for some k, j.

Definition 4.6. *Let $a_j = \exp 2\pi i \alpha_j$ and $b_j = \exp 2\pi i \beta_j (j = 1, \ldots, n)$ be two sets of numbers on the unit circle in $\mathbb{C}$. Suppose $0 \leqq \alpha_1 \leqq \alpha_2 \leqq \ldots \leqq \alpha_n < 1$, $0 \leqq \beta_1 \leqq \beta_2 \leqq \ldots \leqq \beta_n < 1$. We say that the sets $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ interlace on the unit circle if and only if either*

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_n < \beta_n \quad \text{or} \quad \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \ldots < \beta_n < \alpha_n.$$

Corollary 4.7. *Let the hypergeometric group $H(a; b)$ have all of its parameters on the unit circle. Then $H(a; b)$ is contained in $U(n, \mathbb{C})$ if and only if the parameter sets $\{a_1, \ldots, a_n\}$ $\{b_1, \ldots, b_n\}$ interlace on the unit circle.*

Theorem 4.8. *Suppose the parameters $a_1, \ldots, a_n$; $b_1, \ldots, b_n$ are roots of unity, and say*

$$\mathbb{Q}(a_1, \ldots, a_n, b_1, \ldots, b_n) = \mathbb{Q}(\exp 2\pi i/h)$$

for some $h \in \mathbb{N}$. Then the hypergeometric group $H(a; b)$ is finite if and only if for each $k \in \mathbb{N}$ with $(k, h) = 1$ the sets $\{a_1^k, \ldots, a_n^k\}$ and $\{b_1^k, \ldots, b_n^k\}$ interlace on the unit circle.

Proof. The Galois automorphisms of $\mathbb{Q}(\exp 2\pi i/h)$ over $\mathbb{Q}$ are given by

$$\sigma_k \colon \exp 2\pi i/h \to \exp 2\pi i k/h$$

for $(k, h) = 1$. It follows from Corollary 3.6 that the hypergeometric group can be represented by matrices whose entries are in the ring of algebraic integers $\mathbb{Z}[\exp 2\pi i/h]$. The Galois automorphism σ_k induces an isomorphism between the matrix group $H(a; b)$ and the hypergeometric group H_k with parameters $a_1^k, \ldots, a_n^k$; $b_1^k, \ldots, b_n^k$. According to Theorem 4.3 each H_k has an invariant form F_k for $(k, h) = 1$.

If $H(a; b)$ is finite, then the group H_k is finite for every k with $(k, h) = 1$. Hence the hermitian forms F_k are all definite and Corollary 4.7 implies that the sets $\{a_1^k, \ldots, a_n^k\}$ and $\{b_1^k, \ldots, b_n^k\}$ interlace on the unit circle.

Conversely, suppose that for each k with $(k, h) = 1$ the sets $\{a_1^k, \ldots, a_n^k\}$ and $\{b_1^k, \ldots, b_n^k\}$ interlace. According to Corollary 4.7 each group is unitary with definite form F_k. The image of $H(a; b)$ under the diagonal embedding

$$\prod_{k \in (\mathbb{Z}/h\mathbb{Z})^*} \sigma_k \colon H(a; b) \to \prod_{k \in (\mathbb{Z}/h\mathbb{Z})^*} H_k$$

is contained (relative to a suitable basis) in $GL(mn, \mathbb{Z})$ and leaves invariant a definite hermitian form on $\mathbb{C}^{mn}$ ($m = \varphi(h)$ is the order of $(\mathbb{Z}/h\mathbb{Z})^*$). Hence $H(a; b)$ is finite.

Remark 4.9. Let $\alpha_1, \ldots, \alpha_n$; $\beta_1, \ldots, \beta_n \in \mathbb{Q}$ with $a_j = \exp 2\pi i \alpha_j$, $b_j = \exp 2\pi i \beta_j$ for $j = 1, \ldots, n$. Using elementary number theoretic techniques one can show that

$$D(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)\, u \equiv 0 \pmod{p}$$

has n solutions in $\mathbb{F}_p[z]$ linearly independent over $\mathbb{F}_p[z^p]$ for almost all primes p if and only if the sets $\{a_1^k \dots, a_n^k\}$ and $\{b_1^k, \dots, b_n^k\}$ interlace on the unit circle for every $k \in \mathbb{N}$ relatively prime to the common denominator of α_j, $\beta_k (j, k = 1, \dots, n)$ (see Katz [Kat] or Landau [La]).

Together with Theorem 4.8 this gives us another verification of Grothendieck's zero p-curvature conjecture for the special case of the hypergeometric equation (see [Ho], [Kat]).

5. The imprimitive case

Definition 5.1. *Let V be a complex vector space of dimension n and let $G \subset GL(V)$ be a subgroup acting irreducibly on V. The group G is called imprimitive if there exists a direct sum decomposition $V = V_1 \oplus V_2 \oplus \dots \oplus V_d$ with* $\dim V_j \geqq 1$ *and $d \geqq 2$, such that G permutes the spaces V_j. If such a decomposition does not exist, G is called primitive.*

Definition 5.2. *Let $H(a; b) \subset GL(n, \mathbb{C})$ be a hypergeometric group with parameters $a_1, \dots, a_n$; $b_1, \dots, b_n$ and generators h_0, h_1, h_∞ as in Definition* 3.1. *The subgroup $H_r(a; b)$ of $H(a; b)$ generated by the reflections $h_\infty^k\ h_1\ h_\infty^{-k}$ for $k \in \mathbb{Z}$ is called the reflection subgroup of $H(a; b)$.*

Theorem 5.3. *Let $H(a; b) \subset GL(n, \mathbb{C})$ be a hypergeometric group with parameters $a_1, \dots, a_n$; $b_1, \dots, b_n$. The reflection subgroup $H_r(a; b)$ acts reducibly on $\mathbb{C}^n$ if and only if there exists a root of unity ζ, $\zeta \neq 1$ such that*

$$\begin{aligned} \{\zeta a_1, \dots, \zeta a_n\} &= \{a_1, \dots, a_n\} \\ \{\zeta b_1, \dots, \zeta b_n\} &= \{b_1, \dots, b_n\} \end{aligned} \tag{5.1}$$

Moreover, $H(a; b)$ is imprimitive in this case.

Proof. Suppose $H_r = H_r(a; b)$ acts reducibly on $V = \mathbb{C}^n$. Let $W \subset V$ be an irreducible invariant subspace for H_r. Let h denote either h_∞ or h_0^{-1}. Since H_r is normal in $H = H(a; b)$ each of the spaces $h^k W$, $k \in \mathbb{Z}$ is an irreducible invariant subspace for H_r. Hence, either $h^k W = h^l W$ or $h^k W \cap h^l W = \{0\}$ for any k, $l \in \mathbb{Z}$. Let d be the smallest positive integer such that $h^d W = W$. Since H acts irreducibly on V and H/H_r is cyclic with generator hH_r, we have $V = \bigoplus_{j=0}^{d-1} h^j W$ with $d \geqq 2$ and $n = dm$, $m = \dim W$. Choose $g \in GL(n, \mathbb{C})$ such that it multiplies the vectors of $h^j W$ with ζ^j, $\zeta = \exp(2\pi i/d)$, $(j = 1, \dots, d-1)$. Then, clearly, $\zeta h = g h g^{-1}$ and ζh has the same eigenvalues as h. Equalities (5.1) follow immediately.

Notice that H permutes the spaces $h^j W$, and thus H is imprimitive, as asserted.

Suppose conversely, that the parameters $a_1, \dots, a_n$; $b_1, \dots, b_n$ have the form (5.1). According to the uniqueness theorem 3.5 the group generated by ζh_∞, ζh_0^{-1} must be conjugated in $GL(n, \mathbb{C})$ to H. Hence there exists $g \in GL(n, \mathbb{C})$ such that $\zeta h_\infty = g h_\infty g^{-1}$, $\zeta h_0^{-1} = g h_0^{-1} g^{-1}$. This implies $r = g r g^{-1}$ for all $r \in H_r$. Hence the eigenspaces of g are invariant under H_r and H_r is thus reducible.

Remark 5.4. Consider the hypergeometric equation

$$D(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)\, u = 0 \tag{5.2}$$

with $\alpha_j - \beta_k \notin \mathbb{Z}$ for all $j, k = 1, \ldots, n$. Then the reflection subgroup of the monodromy group of (5.2) acts reducibly if and only if there exist $d, m \in \mathbb{N}$, $d \geqq 2$ with $n = dm$ and $\lambda_1, \ldots, \lambda_m; \mu_1, \ldots, \mu_m$ such that mod $\mathbb{Z}$ we have the inequalities

$$\{\alpha_1, \ldots, \alpha_n\} \equiv \left\{\lambda_1, \lambda_1 + \frac{1}{d}, \ldots, \lambda_1 + \frac{d-1}{d}, \ldots, \lambda_m, \lambda_m + \frac{1}{d}, \ldots, \lambda_m + \frac{d-1}{d}\right\} (\mathrm{mod}\ \mathbb{Z}),$$

$$\{\beta_1, \ldots, \beta_n\} \equiv \left\{\mu_1, \mu_1 + \frac{1}{d}, \ldots, \mu_1 + \frac{d-1}{d}, \ldots, \mu_m, \mu_m + \frac{1}{d}, \ldots, \mu_m + \frac{d-1}{d}\right\} (\mathrm{mod}\ \mathbb{Z}).$$

Furthermore, solutions of (5.2) are obtained from the hypergeometric equation

$$D(d\lambda_1, \ldots, d\lambda_m; d\mu_1, \ldots, d\mu_m)\, v = 0 \tag{5.3}$$

be the relation $v(z) = u(z^d)$. Following N.M. Katz we say that the hypergeometric group $H(a; b)$ is Kummer induced if its reflection subgroup $H_r(a; b)$ acts reducibly on $\mathbb{C}^n$.

Definition 5.5. *A scalar shift of the hypergeometric group $H(a; b)$ is a hypergeometric group $H(da; db) = H(da_1, \ldots, da_n; db_1, \ldots, db_n)$ for some $d \in \mathbb{C}^*$.*

Remark 5.6. If d has the form $d = \exp(2\pi i\delta)$ for some $\delta \in \mathbb{C}$ then a scalar shift from $H(a; b)$ to $H(da; db)$ is the effect on the monodromy group obtained by multiplying all solutions of the hypergeometric equation by $z^{-\delta}$. Observe that the associated reflection groups $H_r(a; b)$ and $H_r(da; db)$ are naturally isomorphic.

Proposition 5.7. *Let H be a hypergeometric group in $GL(n, \mathbb{C})$ and $n \geqq 3$. If the reflection subgroup H_r of H is irreducible and primitive, then H_r is a scalar shift of H.*

Proof. The element $h_\infty \in H$ normalises H_r. According to a theorem of A.M. Cohen [Co] the primitivity of H_r implies that h_∞ is a scalar times an element of H_r, which establishes our proposition.

Note that the original version of Cohen's theorem contains two exceptions. However, both of them are not really there. For the first exception this was pointed out in [Co, erratum], and for the second it simply follows from $W(M_3) \simeq \{\pm 1\} \times W(L_3)$.

The upshot of Proposition 5.7 is, that if H is primitive then H and H are essentially the same. The remainder of this section is devoted to characterising those hypergeometric groups, whose reflection subgroup is imprimitive.

Theorem 5.8. *Suppose the reflection subgroup of the hypergeometric group $H(a; b) \subset GL(n, \mathbb{C})$ is irreducible. Then H is imprimitive if and only if there exist*

$p, q\in\mathbb{N}$, $p+q=n$, $(p,q)=1$ and $a, b, c\in\mathbb{C}^*$ with $a^n=b^p c^q$ such that

$$\{a_1, \dots, a_n\}=\{a, a\zeta_n, \dots, a\zeta_n^{n-1}\},$$
$$\{b_1, \dots, b_n\}=\{b, b\zeta_p, \dots, b\zeta_p^{p-1}, c, c\zeta_q, \dots, c\zeta_q^{q-1}\}$$

with $\zeta_r=\exp 2\pi i/r$, or the same equalities with the sets $\{a_i\}_i$ and $\{b_i\}_i$ interchanged.

Proof. Letting h denote either h_∞ or h_0^{-1}, we observe that $H_r=H_r(a;b)$ is generated by the reflections $h^k h_1 h^{-k}$ for $k\in\mathbb{Z}$. Let $V=\mathbb{C}^n=V_1\oplus\dots\oplus V_d$ be a system of imprimitivity for H_r. Since H_r acts irreducibly on V there exists for each i an integer k such that $h^k h_1 h^{-k} V_i=V_j$ for some $j\neq i$. Because $h^k h_1 h^{-k}$ is a reflection we deduce that $\dim V_i=1$ for $i=1, \dots, n$. Hence $d=n$ and $V=V_1\oplus\dots\oplus V_n$ is an imprimitive decomposition of V for H into one dimensional subspaces.

Suppose $r\in H_r$ is a reflection. Then either $rV_i=V_i$ for $i=1, \dots, n$ or $r\colon V_i\leftrightarrow V_j$ for some $i\neq j$ and $rV_k=V_k$ for $k\neq i, j$. In the latter case r is a reflection of order two.

We have a natural homomorphism $\sigma\colon H\to S_n$ defined by $gV_i=V_{\sigma(g)(i)}$ for $g\in H$ and $i=1, \dots, n$. The irreducibility of H implies that σ is surjective. Since H is generated by h_1 and h we see that S_n is generated by $\sigma(h_1)$ and $\sigma(h)$. The fact that $\sigma(h_1)$ is a pair exchange forces $\sigma(h)$ to be either a full n-cycle or a product of disjoint p- and q-cycles with $n=p+q$, $(p,q)=1$. Without loss of generality we may assume $\sigma(h_\infty)$ to be an n-cycle. Then $\sigma(h_0^{-1})$ is a product of a disjoint p- and q-cycle with $p+q=n$, $(p,q)=1$. The corresponding eigenvalues of h_∞ and h_0^{-1} follow readily.

Conversely, the imprimitive group generated by

$$\begin{aligned}
h_\infty&\colon\ e_i\to ae_{i+1}\ (1\leqq i<n), && e_n\to ae_1,\\
h_1&\colon\ e_i\to e_i\ (i\neq p, n), && e_p\to a^{-p}b^p e_n, \quad e_n\to a^p b^{-p} e_p,\\
h_0^{-1}=h_\infty h_1&\colon\ e_i\to ae_{i+1}\ (i\neq p, n), && e_p\to a^{-p+1}b^p e_1, \quad e_n\to a^{p+1}b^{-p}e_{p+1}
\end{aligned}$$

is a hypergeometric group with the required parameters, and by the uniqueness theorem 3.5 the group H must be conjugate to it.

Proposition 5.9. *Suppose that the parameters of the hypergeometric group $H\subset GL(n,\mathbb{C})$ have the form*

$$\{a_1, \dots, a_n\}=\{\zeta_{n+1}, \zeta_{n+1}^2, \dots, \zeta_{n+1}^n\},$$
$$\{b_1, \dots, b_n\}=\{1, \zeta_p, \dots, \zeta_p^{p-1}, \zeta_q, \dots, \zeta_q^{q-1}\}$$

with $\zeta_r=\exp 2\pi i/r$ and $p, q\in\mathbb{N}$, $p+q=n+1$, $(p,q)=1$. Then $H\simeq S_{n+1}$ and its reflection subgroup is primitive if $n>3$.

Proof. Consider the representation of S_{n+1} on the space $\mathbb{C}^n \simeq\{(x_1, \dots, x_{n+1})\in\mathbb{C}^{n+1}\mid\sum x_i=0\}$ given by $\sigma\colon\ (x_1, \dots, x_{n+1})\to(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n+1)})$ for every $\sigma\in S_{n+1}$. Choose for h_∞ the $(n+1)$-cycle $(1, 2, \dots, n+1)$ and for h_0^{-1} the product $(1\dots p)\ (p+1\dots n+1)$. Then $h_1=(p, n+1)$ is a reflection of order 2. Note that h_∞ and h_0^{-1} have the required eigenvalues. By the uniqueness theorem 3.5 we obtain $H \simeq S_{n+1}$. Moreover, S_{n+1} is generated by pair exchanges i.e. elements of the form $h_\infty^k\, h_1\, h_\infty^{-k}$, hence $H = H_r$.

Proposition 5.10. *Let K be an algebraic number field and $P(x) \in K[x]$ be irreducible in $K[x]$. Suppose $P(x)$ is not a polynomial in x^r for some $r \geqq 2$. Let $\vartheta_1, \ldots, \vartheta_n$ be the roots of P and suppose ϑ_i/ϑ_j is a root of unity for all i, j. Let the roots of unity in K be generated by $e^{2\pi i/M}$. Write $\mu_M = \{\exp(2\pi i k/M) \mid k = 0, 1, \ldots, M\}$. Then there exists $N \in \mathbb{N}$ odd, square-free with $(N, M) = 1$ and a character χ: $(\mathbb{Z}/N\mathbb{Z})^* \to \mu_M$ such that the set $\vartheta_1, \ldots, \vartheta_n$ is given by either*

$$\text{i)}\quad \alpha S_\chi\, \chi(k)\, e^{2\pi i k/N}, \qquad (k, N) = 1$$

or

$$\text{ii)}\quad (1 \pm i)\, \alpha S_\chi\, \chi(k)\, e^{2\pi i k/N}, \qquad (k, N) = 1$$

where $S_\chi = \sum_{(k,N)=1} \chi^{-1}(k)\, e^{2\pi i k/N}$, $\alpha \in K$ and $n = \varphi(N)$ in case i), *$n = 2\varphi(N)$ in case* ii).

Proof. Let L be the field generated by all ratios ϑ_i/ϑ_j. There exists $N \in \mathbb{N}$ such that $L = K(e^{2\pi i/NM})$. Put $s_m = \vartheta_1^m + \ldots + \vartheta_n^m$ for all $m \in \mathbb{N}$. If $s_m \neq 0$, we have $\vartheta_1^{-m} s_m \in L$ and hence $\vartheta_1^m \in L$. Let r be the greatest common divisor of the elements in $\{m \mid s_m \neq 0\}$. If $r = 1$ then $\vartheta_1 \in L$ and hence $K(e^{2\pi i/MN}) = L = K(\vartheta_1, \ldots, \vartheta_n)$. If $r \geqq 2$ then $P(x)$ is in fact a polynomial in x^r, contradicting our assumption.

The Galois group of L/K is given by elements of the form

$$\sigma_h\colon\ e^{2\pi i/MN} \to e^{2\pi i h/MN}$$

where $(h, MN) = 1$ and $h \equiv 1 \pmod M$.

First we show that we can restrict ourselves to the case when N is odd, square-free and $(N, M) = 1$. Suppose we have a prime p such that either $p^2 \mid M$ or $p \mid (M, N)$. In both cases we can take $h = 1 + NM/p$ and study the action of $\sigma_h \in \mathrm{Gal}(L/K)$ on ϑ_1 say. Notice that $(1 + MN/p)^j \equiv 1 + jMN/p \pmod{MN}$ $\forall j \in \mathbb{Z}$. Suppose $\sigma_h\colon \vartheta_1 \to e^{2\pi i k/MN}\vartheta_1$. Since σ_h has order p, and

$$\sigma_h^p\colon\ \vartheta_1 \to \exp(2\pi i k(1 + h + \ldots + h^{p-1})/MN)\,\vartheta_1$$

we conclude that

$$k(1 + h + \ldots + h^{p-1}) \equiv 0 \pmod{MN}$$

and hence

$$0 \equiv k\left(\sum_{j=0}^{p-1} (1 + jMN/p)\right) \equiv k\left(p + \frac{p(p-1)}{2}\,\frac{MN}{p}\right) \pmod{MN}.$$

If p is odd, then $kp \equiv 0 \pmod{MN}$ i.e. $\exp(2\pi i k/MN)$ is a p-th root of unity. Hence ϑ_1^p is stable under σ_h and $P(x)$ is in fact a polynomial in x^p, contradicting our assumptions.

If $p = 2$, then $k(2 + MN/2) \equiv 0 \pmod{MN}$. If k is even, then observe $2k \equiv 0 \pmod{MN}$ and we have a contradiction as above. If k is odd, then necessarily $4 \| MN$ and we have $4k \equiv 0 \pmod{MN}$, i.e. $\exp(2\pi i k/MN) = \pm i$. Now observe that if σ_h: $\vartheta_1 \to \mp i\vartheta_1$, then σ_h: $\vartheta_1/(1 \pm i) \to \vartheta_1/(1 \pm i)$.

Thus we conclude that neither $p^2 | N$ nor $p | (M, N)$ unless $p = 2$, $4 \| MN$ and $2 \| N$ (note that always $2 | M$ since $-1 \in K$). However, in the latter case we may replace ϑ_1 by $\vartheta_1/(1 \pm i)$ for a suitable $\pm$ sign, note that that the new ϑ_1 has degree $n/2$ and continue our argument. From now on we may assume that N is odd, square-free and $(N, M) = 1$.

To every $\sigma_g \in \mathrm{Gal}(L/K)$ we can associate a $\varphi(g) \in \mathbb{Z}/N\mathbb{Z}$ such that σ_g: $\vartheta_1^M \to \exp(2\pi i \varphi(g)/N)\, \vartheta_1^M$. Notice that $\varphi(hg) \equiv h\varphi(g) + \varphi(h) \pmod N$ for any σ_h, $\sigma_g \in \mathrm{Gal}(L/K)$. Choose h such that $h \equiv 1 \pmod M$ and $h \equiv 2 \pmod N$. Then $\varphi(hg) \equiv 2\varphi(g) + \varphi(h) \pmod N$, but also $\varphi(gh) \equiv g\varphi(h) + \varphi(g) \pmod N$. The equality $\varphi(hg) = \varphi(gh)$ then yields $\varphi(g) \equiv (g-1)\, \varphi(h) \pmod N$. Hence $r = \exp(-2\pi i \varphi(h)/N)\, \vartheta_1^M$ is stable under all $\sigma_g \in \mathrm{Gal}(L/K)$ and thus $r \in K$. We conclude that $\vartheta_1 = r^{1/M}\zeta$, where ζ is an N-th root of unity which is primitive, since the ratios ϑ_i/ϑ_j generate L/K. After conjugation we might as well take $\zeta = e^{2\pi i/N}$.

Since $(M, N) = 1$ we have $\mathrm{Gal}(L/K) \simeq (\mathbb{Z}/N\mathbb{Z})^*$. The Galois element σ corresponding to $h \in (\mathbb{Z}/N\mathbb{Z})^x$ acts as σ: $e^{2\pi i/N} \to e^{2\pi i h/N}$. Moreover, σ: $r^{1/M} \to r^{1/M}\chi(h)$ where χ: $(\mathbb{Z}/N\mathbb{Z})^x \to \mu_M$ is a character. Now notice that σ: $S_\chi \to \chi(h)\, S_\chi$, where S_χ is the charactersum defined in our Proposition. So, $r^{1/M}/S_\chi$ is fixed under $\mathrm{Gal}(L/K)$. Hence $r^{1/M}/S_\chi = \alpha \in K$, which proves our Proposition.

Lemma 5.12. *Let $H \subset GL(4, \mathbb{C})$ be a finite hypergeometric group generated by h_∞, h_0^{-1} such that*

i) *H is primitive,*
ii) *λh_∞, λh_0^{-1} have entries in $\mathbb{Q}$ for suitable $\lambda \in \mathbb{C}^*$,*
iii) $\det h_\infty = -\det h_0^{-1}$.

Then, up to a scalar shift, either $\{a_1, \ldots, a_4; b_1, \ldots, b_4\}$ or $\{b_1, \ldots, b_4; a_1, \ldots, a_4\}$ has one of the following forms

$$\begin{array}{ll} \varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4; 1, i, -1, -i, & \zeta\omega, \zeta\omega^2, \zeta^{-1}\omega, \zeta^{-1}\omega^2; 1, i, -1, -i, \\ \varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4; 1, -1, \omega, \omega^2 & i\omega^2, -i\omega^2, i\omega, -i\omega; 1, -1, \omega, \omega^2 \\ \omega, -\omega, \omega^2, -\omega^2; \zeta, \zeta^3, i, -i & \omega, \omega^2, i\omega, i\omega^2; -1, -i, \zeta, \zeta^5 \end{array}$$

where $\varepsilon = \exp(2\pi i/5)$, $\omega = \exp(2\pi i/3)$, $\zeta = \exp(\pi i/4)$.

Proof. The characteristic polynomial of λh_∞, λh_0^{-1} have degree 4, coefficients in $\mathbb{Q}$ and ratios of their roots are roots of unity. Moreover by Theorem 4.8 these roots are all distinct. Using Prop. 5.10 we can find all such polynomials, whose roots we list here

$$\begin{array}{ll} r^{1/4}(1, i, -1, -i) & r\sqrt{6}(\zeta\omega, -\zeta\omega^2, \zeta^{-1}\omega, -\zeta^{-1}\omega^2), \\ r^{1/2}(\omega^2, -\omega^2, \omega, -\omega) & r(1, -1, \omega, \omega^2), \\ r^{1/2}(-3)^{1/4}(\omega^2, -\omega^2, i\omega, -i\omega) & r(i, -i, \omega, \omega^2), \\ r(\varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4) & r\sqrt{3}(1, -1, i\omega, -i\omega^2), \\ r\sqrt{5}(\varepsilon, -\varepsilon^2, -\varepsilon^2, \varepsilon^4) & r\sqrt{-3}(1, -1, \omega, -\omega^2), \\ r\sqrt{2}(1, -1, \zeta, \zeta^{-1}) & r\sqrt{2}(\zeta\omega, \zeta\omega^2, \zeta^{-1}\omega, \zeta^{-1}\omega^2) \\ r\sqrt{-2}(1, -1, \zeta^{-1}, -\zeta) & r^{\frac{1}{2}}(2i)^{\frac{1}{4}}(1, -1, \zeta^{-1}, -\zeta^{-1}) \end{array}$$

where $r \in \mathbb{Q}$. Using $\det \lambda h_\infty = -\det \lambda h_0^{-1}$ we can find, up to a common factor, all possible combinations for the eigenvalues of h_∞ and h_0^{-1}. To each of these combinations we can apply Theorem 4.7 to see whether the group they generate is finite. Of the remaining possibilities we delete the ones for which H is reducible or imprimitive using Theorems 2.7, 5.3 and 5.8. We are then left with the cases of our assertion.

Lemma 5.13. *Let $H \subset GL(3, \mathbb{C})$ be a finite hypergeometric group generated by h_∞, h_0^{-1} such that*

i) *H is primitive,*
ii) *$\lambda h_\infty, \lambda h_0^{-1}$ have entries in $\mathbb{Q}(\omega)$ for suitable $\lambda \in \mathbb{C}^*$,*
iii) $\det h_\infty = -\det h_0^{-1}$

Then, up to a scalar shift, either $\{a_1, a_2, a_3; b_1, b_2, b_3\} = \{i, -i, 1; -\omega^k i, \omega^k i, -\omega^k\}$ or $\{-\omega^k i, \omega^k i, -\omega^k; i, -i, 1\}$ for $k=1$ or 2.

Proof. We proceed in exactly the same way as in Lemma 5.12. The polynomials we must consider have degree 3 and coefficients in $\mathbb{Q}(\omega)$. Their zeros read

$$\begin{array}{ll} r(1, -\omega^2, \omega) & r(i, -i, \omega^k)\,(k=0,1,2), \\ r(1, -\omega^2, -1) & r^{1/3}(1, \omega, \omega^2), \\ r(1, -\omega^2, \omega^2) & \end{array}$$

where $r \in \mathbb{Q}(\omega)$.

Theorem 5.14. *Let $n \geqq 3$ and let $H \subset GL(n, \mathbb{C})$ be a primitive hypergeometric group with reflection subgroup H_r. Then H_r is imprimitive if and only if, up to a scalar shift, either $\{a_1, \ldots, a_n; b_1, \ldots, b_n\}$ or $\{b_1, \ldots, b_n; a_1, \ldots, a_n\}$ has one of the following forms,*

$$\begin{array}{ll} n=3 & \{i, -i, 1; \omega^k i, -\omega^k i, -\omega^k\} \qquad (k=1,2), \\ n=4 & \{i\omega^2, -i\omega^2, i\omega, -i\omega; 1, -1, \omega, \omega^2\}, \\ n=4 & \{\zeta\omega, \zeta\omega^2, \zeta^{-1}\omega, \zeta^{-1}\omega^2; 1, i, -1, -i\} \\ n=4 & \{\omega, -\omega, \omega^2, -\omega^2; \zeta, \zeta^3, i, -i\} \\ n=4 & \{\omega, \omega^2, i\omega, i\omega^2; -1, -i, \zeta, -\zeta\} \end{array}$$

where $\zeta = \exp \pi i/4$.

Proof. According to Theorem 5.3 H_r is irreducible. Suppose H_r is imprimitive. Just as in the proof of Theorem 5.8 there exists a direct sum decomposition $V = V_1 \oplus \ldots \oplus V_n$, $\dim V_i = 1\,(i=1, \ldots, n)$ and a natural surjective homomorphism $\sigma\colon H_r \to S_n$ given by $r V_i = V_{\sigma(r)(i)}$ for $r \in H_r$ and $i = 1, \ldots, n$. The surjectivity of σ implies that for each $i = 2, \ldots, n$ there exists a reflection $r_i \in H_r$ of order two with $r_i V_1 = V_i$. The image $\sigma(r_i)$ of r_i under σ is the pair exchange $(1\,i) \in S_n$. Conversely, the homomorphism $\tau\colon S_n \to H_r$ defined by $\tau(1\,i) = r_i$ is a section for σ. Choose $e_1 \in V_1$, $e_1 \neq 0$ and $e_i = r_i(e_1)$ for $i = 2, \ldots, n$. Clearly, $e_1, \ldots, e_n$ is a basis for V. The normal subgroup $H_d = \ker \sigma$ of H_r is abelian, since it consists of all diagonal matrices in H_r relative to the basis $e_1, \ldots, e_n$. Rephrasing the above we have a splitting short exact sequence

$$1 \to H_d \to H_r \underset{\sigma}{\overset{\tau}{\leftrightarrows}} S_n \to 1$$

with H_d abelian. The elements $d \in H_d$ will be denoted by $d = \mathrm{diag}(d_1, \ldots, d_n)$ where d_i is given by $d(e_i) = d_i e_i (i = 1, \ldots, n)$.

Suppose H_d consists only of scalars. Then the one-dimensional space spanned by $e_1 + e_2 + \ldots + e_n$ is invariant under H_r, contradicting the irreducibility of H_r.

Hence there exist non-scalar elements $d \in H_d$, i.e. $d = \mathrm{diag}(d_1, \ldots, d_n)$ with $d_i \neq d_j$ for some i, j. Let h be either h_0 or h_∞, to be fixed from now on. Suppose there exists $d \in H_d$, d non-scalar, such that $hdh^{-1} \in H_d$. Let D be the group generated by all rdr^{-1} with $r \in H_r$. Note that if $\sigma(r) = \phi$ then $rdr^{-1} = \mathrm{diag}(d_{\phi(1)}, \ldots, d_{\phi(n)}) \in H_d$. Hence D acts with distinct characters on $V_1, \ldots, V_n$. Moreover, D is normalised by h, and this implies that h permutes the one-dimensional spaces V_i, contradicting the primitivity of H.

So we may finally assume that $hH_d h^{-1} \cap H_d$ consists only of scalars. Note that in this remaining case $\sigma(hH_d h^{-1})$ is a non-trivial abelian normal subgroup of S_n. This leaves us with two possibilities since $n \geqq 3$, i.e. $n = 3$ and $\sigma(hH_d h^{-1}) \simeq \mathbb{Z}/3\mathbb{Z}$, $n = 4$ and $\sigma(hH_d h^{-1}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We also have the natural isomorphism $hH_d h^{-1}/hH_d h^{-1} \cap H_d \simeq \sigma(hH_d h^{-1})$, and since $hH_d h^{-1} \cap H_d$ consists only of scalars we are left with the following possibilities,

$$\text{I)}\ n = 3 \quad \text{and} \quad H_d (\text{mod scalars}) \simeq \mathbb{Z}/3\mathbb{Z},$$
$$\text{II)}\ n = 4 \quad \text{and} \quad H_d (\text{mod scalars}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Bearing in mind, that H_r is generated by reflections of order two and that H_d is normalised by H_r it is straightforward to verify that H_d has one of the following forms,

$$\text{I)}\ n = 3 \quad \text{and} \quad H_d = \{\mathrm{diag}(\omega^k, \omega^1, \omega^m) \mid k + 1 + m \equiv 0 (\mathrm{mod}\ 3)\},$$
$$\text{II)}\ n = 4 \quad \text{and} \quad H_d = \{\mathrm{diag}((-1)^k, (-1)^1, (-1)^m, (-1)^p \mid k + 1 + m + p \equiv 0 (\mathrm{mod}\ 2)\}.$$

We deal with these cases as follows. Note that H_r is finite. Hence there exists $k \in \mathbb{N}$ such that $h^k r h^{-k} = r$ for all $r \in H_r$. So, by Schur's Lemma, h^k is a scalar, and up to a scalar shift H is finite. Denote by aut the automorphism aut: $r \to hrh^{-1}$ of H_r. Then the entries of h satisfy the set of linear equations $hr = \mathrm{aut}(r)\,h$, $\forall r \in H_r$. According to Schur's lemma there is, up to a common factor, a unique solution, which may be chosen in the field of definition of the elements of H_r. Hence there exists $\lambda \in \mathbb{C}^*$ such that λh has entries in $\mathbb{Q}(\omega)$ or $\mathbb{Q}$ in cases I or II respectively.

In case I we invoke Lemma 5.13 to conclude that up to a scalar shift the parameters of H read $i, -i, 1; -\omega^k i, \omega^k i, -\omega^k$ $(k = 1, 2)$, as asserted. Conversely, one easily checks that the group generated by

$$h_\infty = \frac{1}{\omega^k - \omega^{-k}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^k & \omega^{-k} \\ 1 & \omega^{-k} & \omega^k \end{pmatrix} \qquad h_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

satisfies all requirements and has the required parameters.

In case II) we invoke Lemma 5.12 to conclude that up to a scalar shift H has the following parameters,

a) $\varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4; 1, i, -1, -i,$

b) $\varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4; 1, -1, \omega, \omega^2,$

c) $\zeta\omega, \zeta\omega^2, \zeta^{-1}\omega, \zeta^{-1}\omega^2; 1, i, -1, -i,$

d) $i\omega^2, -i\omega^2, i\omega, -i\omega; 1, -1, \omega, \omega^2,$

e) $\omega, \omega^2, i\omega, i\omega^2; -1, -i, \zeta, -\zeta,$

f) $\omega, -\omega, \omega^2, -\omega^2; \zeta, \zeta^3, i, -i.$

According to Proposition 5.9 cases a) and b) give rise to $H \simeq S_5$ and H_r primitive. Cases c), d), e), f) occur in the assertion of our theorem. To show that these cases really correspond to a hypergeometric group with the required properties, we must show that H_r is imprimitive.

Suppose H_r is primitive. In cases c), d), e), f) H_r can be defined over $\mathbb{Q}$. In case d) this is obvious, in case c), e), f) we apply a scalar shift by the factor $\sqrt{2}, 1-i, i\sqrt{2}$ respectively and notice that the shifted hypergeometric group is defined over $\mathbb{Q}$. The only finite primitive reflection group in dimension 4, defined over $\mathbb{Q}$ is F_4 according to Shephard-Todd (see Table 8.1 in Sect. 8). According to Proposition 5.7 $F_4 \simeq H_r$ is a scalar shift of H. So we may as well assume $H = F_4$. However, it is known that the subgroup of F_4 generated by all conjugates of a reflection of F_4 is strictly smaller than F_4, contradicting $H_r = F_4$.

Remark 5.15. Note that the cases I and II discussed in the proof of Theorem 5.14 are precisely the two imprimitive reflection groups $G(3,3,3)$, $G(2,2,4)$ in dimension $n \geqq 3$ which have more than one system of imprimitivity [Co]. The hypergeometric groups containing such imprimitive groups as reflection subgroups permute the various systems of imprimitivity.

6. Differential Galois theory

In this section we determine the differential Galois group of the hypergeometric differential equation (3.5) in case the monodromy modulo scalars is infinite. For a very nice introduction into differential Galois theory we refer to [Kap].

Let V be a complex vector space of dimension n and let $G \subset GL(V)$ be a subgroup. We denote by $\bar{G}$ the closure of G and by G^0 the connected component of the identity of G, both with respect to the Zariski topology. Observe that G^0 is dense in $\bar{G}^0$ and hence the operations $^-$ and ^{0} commute. Note that the natural map $G/G^0 \to \bar{G}/\bar{G}^0$ is an isomorphism of finite groups. The dual group G^* in $GL(V^*)$ is defined by $\{g^*;\ g \in G\}$ and the map $g \to (g^{-1})^*$ is a natural isomorphism of G into G^*.

Proposition 6.1. *The dual map $g \to (g^{-1})^*$ yields a natural isomorphism*

$$H(a_1, \ldots, a_n; b_1, \ldots, b_n) \xrightarrow{\sim} H(a_1^{-1}, \ldots, a_n^{-1}; b_1^{-1}, \ldots, b_n^{-1}). \tag{6.1}$$

In particular the group $H(a;b)$ is self dual if and only if $\{a_1, \ldots, a_n\} = \{a_1^{-1}, \ldots, a_n^{-1}\}$ and $\{b_1, \ldots, b_n\} = \{b_1^{-1}, \ldots, b_n^{-1}\}$. The latter condition implies that the special eigenvalue c of the reflection h_1 is given by $c = \pm 1$. The case $c = +1$ occurs only for n even and implies $H(a;b) \subset \mathrm{Sp}(n, \mathbb{C})$. The case $c = -1$ implies that $H(a;b) \subset O(n, \mathbb{C})$.

Proof. Clearly, the map $g \to (g^{-1})^*$ maps $H(a;b)$ to a hypergeometric group with parameters $a_1^{-1}, \ldots, a_n^{-1}; b_1^{-1}, \ldots, b_n^{-1}$. This implies the first statement.

Self duality of $H(a;b)$ implies the existence of a non-degenerate bilinear form F on $V = \mathbb{C}^n$ which is invariant under $H(a;b)$. This form is either symmetric or anti-symmetric. If $c = -1$ then F must be symmetric, hence $H(a;b) \subset O(n, \mathbb{C})$. If $c = +1$, then F must be anti-symmetric, hence $H(a;b) \subset \mathrm{Sp}(n, \mathbb{C})$ in which case we automatically have n even.

Remark 6.2. The above Proposition is the differential Galois formulation of the quadratic relations of Darling-Bailey for hypergeometric functions [Ba].

Proposition 6.3. *If $H_r = H_r(a;b)$ is a primitive reflection group then either H_r^0 consists of the identity element only or H_r^0 acts irreducibly on $\mathbb{C}^n$.*

Proof. Assume that H_r^0 acts reducibly on $\mathbb{C}^n$. Let $W \subsetneqq \mathbb{C}^n$ be an irreducible invariant subspace for H_r^0. Since H_r acts irreducibly on $\mathbb{C}^n$ there exists a reflection $r \in H_r$ with $rW \neq W$. Since H_r^0 is normal in H_r the intersection $rW \cap W$ is invariant under H_r^0 we conclude that $rW \cap W = 0$. But r is a reflection, hence $\dim W = 1$. Now either H_r^0 consists of scalars only, or the decomposition of $\mathbb{C}^n$ into isotypical components for H_r^0 gives a system of imprimitivity for H_r. The latter possibility is excluded by the assumption that H_r is primitive. Hence H_r^0 is contained in the scalars $\mathbb{C}$. This fact and the fact that H_r/H_r^0 is finite implies that the special eigenvalue c of h_1 is a primitive d-th root of unity for some $d \in \mathbb{N}$, $d \geqq 2$. Hence the image of the map $\det\colon H_r \to \mathbb{C}^*$ consists of all d-th roots of unity. In particular this shows that the scalars in H_r consist of (nd)-th roots of unity. Thus we conclude that H_r^0 is finite and, by connectedness of H_r^0, we see $H_r^0 = \{1\}$.

The group H_r^0 consists of the identity element if and only if H_r is a finite reflection group. We discuss this case in the next section. The following proposition enables one to understand the differential Galois theory in the case that H_r^0 acts irreducibly on $\mathbb{C}^n$.

Proposition 6.4. *Suppose $G \subset SL(V)$ is a connected algebraic group acting irreducibly on V. Let $r \in GL(V)$ be a reflection with special eigenvalue $c \in \mathbb{C}^*$ which normalizes G. Then we have the following three possibilities,*

$$
\begin{array}{llll}
\text{I) } \textit{If } c \neq \pm 1 & \textit{then} & SL(V) = G, & \\
\text{II) } \textit{If } c = +1 & \textit{then} & SL(V) = G & \textit{or} \quad \mathrm{Sp}(V) = G, \\
\text{III) } \textit{If } c = -1 & \textit{then} & SL(V) = G & \textit{or} \quad SO(V) = G
\end{array}
$$

Proof. Clearly the Lie algebra $\mathfrak{g}$ of G is semisimple and acts irreducibly on V. Denote by $\mathrm{Ad}(r)$ the automorphism of $\mathfrak{g}$ induced from conjugation by r.

I. Suppose $c \neq \pm 1$. If $\mathfrak{g}_\lambda$ denotes the eigenspace of $\mathrm{Ad}(r)$ with eigenvalue λ then we have a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_c \oplus \mathfrak{g}_{c^{-1}}$$

with relations

$$\begin{aligned} &[\mathfrak{g}_1, \mathfrak{g}_c] \subset \mathfrak{g}_c, \quad [\mathfrak{g}_1, \mathfrak{g}_{c^{-1}}] \subset \mathfrak{g}_{c^{-1}}, \quad [\mathfrak{g}_c, \mathfrak{g}_{c^{-1}}] \subset \mathfrak{g}_1, \\ &[\mathfrak{g}_c, \mathfrak{g}_c] = 0, \quad [\mathfrak{g}_{c^{-1}}, \mathfrak{g}_{c^{-1}}] = 0. \end{aligned}$$

Also write $V = V_1 \oplus V_c$ where V_λ is the eigenspace of r with eigenvalue λ. Using the formula

$$r(Xv) = \mathrm{Ad}(r)(X)(rv) \qquad X \in \mathfrak{g}, v \in V$$

we get the relations

$$\begin{aligned} &\mathfrak{g}_1(V_1) \subset V_1, \quad \mathfrak{g}_1(V_c) \subset V_c, \\ &\mathfrak{g}_c(V_1) \subset V_c, \quad \mathfrak{g}_c(V_c) = 0, \\ &\mathfrak{g}_{c^{-1}}(V_1) = 0, \quad g_{c^{-1}}(V_c) \subset V_1. \end{aligned}$$

Using these formulas it is easy to see that $W = V_c \oplus \mathfrak{g}_{c^{-1}}(V_c)$ is an invariant linear subspace for $\mathfrak{g}$. The conclusion is that $\dim \mathfrak{g}_{c^{-1}} = n-1$. The same argument applied to the dual representation shows that $\dim \mathfrak{g}_c = n-1$. We claim that in fact $\mathfrak{g} = \mathrm{sl}(V)$. Indeed, let e_1 be an eigenvector of r with eigenvalue c, and $e_2, \ldots, e_n$ a basis of the eigenspace of r with eigenvalue 1. With respect to this basis we identify $\mathrm{gl}(V) \simeq \mathrm{gl}(n, \mathbf{C})$. Denote by $E_{i,j} \in \mathrm{gl}(n, \mathbf{C})$ the matrix with 1 on the place (i,j) and 0 elsewhere. As shown above we have $E_{1,j}, E_{j,1} \in \mathfrak{g}$ for $j = 2, \ldots, n$. Hence also $[E_{1,j}, E_{j,1}] = E_{1,1} - E_{j,j} \in \mathfrak{g}$ for $j = 2, \ldots, n$. In other words $\mathfrak{g}$ contains the full subalgebra of diagonal matrices of trace 0. A semisimple Lie subalgebra of $\mathrm{sl}(n, \mathbf{C})$ of rank $(n-1)$ is equal to $\mathrm{sl}(n, \mathbf{C})$, and the above claim follows.

II. Now suppose $c = +1$. Since r is a unipotent element we have in fact $r \in G$, and $\log(r) = (r - \mathrm{Id}) \in \mathfrak{g}$. By the Jacobson-Morozov theorem the nilpotent element $(r - \mathrm{Id})$ is contained in a subalgebra $\mathfrak{s} \subset \mathfrak{g}$ with $\mathfrak{s} \simeq \mathrm{sl}(2, \mathbf{C})$. Since $\dim(\mathrm{Ker}(r - \mathrm{Id})) = n-1$ we deduce by sl(2)-representation theory that $\mathbf{C}^n \simeq \mathbf{C}^2 \oplus \mathbf{C}^{n-2}$ as an $\mathfrak{s}$-module. Here $\mathbf{C}^2$ is the standard representation of $\mathfrak{s}$, and $\mathbf{C}^{n-2}$ are $(n-2)$ copies of the trivial representation of $\mathfrak{s}$. Suppose V is the irreducible $\mathfrak{g}$-module with highest weight λ (relative to the usual data, cf. [Hu]). Then there exists a dominant root α for $\mathfrak{g}$, such that $(\lambda, \alpha^\vee) = 1$, $(w_0 \lambda, \alpha^\vee) = -1$ and $(\mu, \alpha^\vee) = 0$ for all weights μ with $w_0 \lambda < \mu < \lambda$. (Here w_0 is the longest element in the Weyl group, and $<$ is the usual ordering on the weight lattice.) In particular, λ is a minuscule weight (see [Bou, Chap. VI, § 4, Ex. 15]), and a case by case check gives $\mathfrak{g} = \mathrm{sl}(V)$ or $\mathrm{sp}(V)$.

III. Finally suppose that $c = -1$. As for the case $c \neq \pm 1$ we get

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$$

and

$$V = V_1 \oplus V_{-1}$$

for the eigenspace decomposition of $\mathrm{Ad}(r)$ and r respectively. We claim that $V=V_{-1}\oplus\mathfrak{g}_{-1}(V_{-1})$ is an invariant linear subspace for $\mathfrak{g}$. The invariance for $\mathfrak{g}_1$ is immediate from $\mathfrak{g}_1(V_{-1})\subset V_{-1}$ and $[\mathfrak{g}_1, \mathfrak{g}_{-1}]\subset\mathfrak{g}_{-1}$. The invariance for $\mathfrak{g}_{-1}$ follows from the relation $\mathfrak{g}_{-1}(\mathfrak{g}_{-1}(V_{-1}))\subset V_{-1}$. Since $\mathfrak{g}\subset\mathrm{sl}(V)$ acts irreducibly on V we conclude that $\dim(\mathfrak{g}_{-1})\geqq n-1$. Analogous to the previous cases we get $\mathfrak{g}=\mathrm{sl}(V)$ if $\dim(\mathfrak{g}_{-1})\geqq n$ and $\mathfrak{g}=\mathrm{so}(V)$ if $\dim(\mathfrak{g}_{-1})=n-1$.

We conclude this section with the following theorem.

Theorem 6.5. *Let $H=H(a;b)$ be an infinite primitive hypergeometric group with parameters $a_1, \ldots, a_n$; $b_1, \ldots, b_n$, which is not a scalar shift of a finite group. Let $\bar{H}(a;b)$ be its Zariski closure. Then we have two possibilities,*

I) *There exists $d\in\mathbb{C}^*$ such that $\{da_1, \ldots, da_n\}=\{(da_1)^{-1}, \ldots, (da_n)^{-1}\}$ and $\{db_1, \ldots, db_n\}=\{(db_1)^{-1}, \ldots, (db_n)^{-1}\}$. If $c=+1$ then $\bar{H}(da;db)=\mathrm{Sp}(n,\mathbb{C})$. If $c=-1$ then $\bar{H}(da,db)=O(n,\mathbb{C})$.*

II) *The remaining cases. Then $SL(n,\mathbb{C})\subset\bar{H}(a;b)$.*

Remark. For a classification of hypergeometric groups which are scalar shifts of finite groups we refer to Theorem 7.1

Proof. From Theorem 5.14 it follows that $H_r(a;b)$ is infinite and primitive. By Proposition 6.3 and the infinity of H_r it follows that $\bar{H}_r^0$ and hence $\bar{H}_r^0\cap SL(n,\mathbb{C})$ is irreducible on $\mathbb{C}^n$. Application of Proposition 6.4 with $G=\bar{H}_r^0\cap SL(n,\mathbb{C})$ and $r=h_1$ shows that either $SL(n,\mathbb{C})\subset\bar{H}_r^0$ or $\bar{H}_r^0\cap SL(n,\mathbb{C})=\mathrm{Sp}(n,\mathbb{C})$, $c=+1$ or $\bar{H}_r^0\cap SL(n,\mathbb{C})=SO(n,\mathbb{C})$, $c=-1$.

Suppose we are in case I). By Proposition 6.1 we have either $\bar{H}(da;db)\subset\mathrm{Sp}(n,\mathbb{C})$ (if $c=+1$) or $\bar{H}(da;db)\subset O(n,\mathbb{C})$ (if $c=-1$). Together with the above conclusion of Proposition 6.4 this implies that either $\bar{H}(da;db)=\mathrm{Sp}(n,\mathbb{C})$ (if $c=+1$) or $\bar{H}(da;db)=O(n,\mathbb{C})$ (if $c=-1$).

Suppose we are not in case I, hence in case II. Suppose $\bar{H}_r^0\cap SL(n,\mathbb{C})=SO(n,\mathbb{C})$, $c=-1$. The group H_r is generated by the conjugates of h_1 whose special eigenvalue is -1. Therefore we have $\bar{H}_r=O(n,\mathbb{C})$. The normaliser of $O(n,\mathbb{C})$ in $GL(n,\mathbb{C})$ is $\mathbb{C}^*\cdot O(n,\mathbb{C})$. After a suitable scalar shift we can see to it that $\bar{H}(da;db)=O(n,\mathbb{C})$, i.e. $H(da;db)$ is self dual and by Proposition 6.1 the parameters satisfy $\{da_i\}_i=\{(da_i)^{-1}\}_i$, $\{db_i\}_i=\{(db_i)^{-1}\}_i$. This contradicts the assumption that we are not in case I. The same contradiction occurs if we assume $\bar{H}_r^0\cap SL(n,\mathbb{C})=\mathrm{Sp}(n,\mathbb{C})$. Thus we conclude $SL(n,\mathbb{C})\subset\bar{H}(a;b)$ in case II.

7. Algebraic hypergeometric functions

If the hypergeometric group $H(a;b)$ is not Kummer induced then it follows from Schur's lemma that $H(a;b)$ modulo its center is a finite group if and only if $H_r(a;b)$ is a finite irreducible reflection group. The latter groups have been classified by Shephard and Todd [ST] based on the older classification by Mitchell [Mi] of the primitive collineation groups generated by homologies.

We denote a finite irreducible reflection group by the symbol STk, where $1 \leqq k \leqq 37$ indicates the line of the table of Shephard and Todd. The group ST1 is the symmetric group S_{n+1} and this is the only finite primitive reflection group in dimension $n \geqq 9$. The group ST2 is the finite imprimitive group $G(m, p, n)$. The group ST3 is the cyclic group of order m being a one-dimensional reflection group. There are 19 two dimensional finite primitive reflection group STk with $4 \leqq k \leqq 22$ derived from the tetrahedral $(4 \leqq k \leqq 7)$, the octahedral $(8 \leqq k \leqq 15)$ and the icosahedral group $(16 \leqq k \leqq 22)$. In dimension n with $3 \leqq n \leqq 8$ there remain 15 exceptional finite primitive reflection groups with $23 \leqq k \leqq 37$. In the next section we have reproduced from the table of Shephard and Todd the list of finite primitive reflection groups in dimension $n \geqq 3$ together with some additional information on these groups.

In the following theorem we focus our attention to finite primitive hypergeometric groups in dimension $n \geqq 3$. The algebraic solutions of order $n=2$ were already described by H.A. Scharz [Sc]. The case of an imprimitive hypergeometric group is discussed in Sect. 5.

Theorem 7.1. *Let $n \geqq 3$ and let $H(a; b) \subset GL(n, \mathbb{C})$ be a primitive hypergeometric group whose parameters are roots of unity and generate the cyclotomic field $\mathbb{Q}(\exp 2\pi i/h)$. Then $H(a; b)$ is finite if and only if, up to a scalar shift, the parameters have the form $a_1^k, \ldots, a_n^k; b_1^k, \ldots, b_n^k$ where $(k, h)=1$ and the exponents of either $a_1, \ldots, a_n; b_1, \ldots, b_n$ or $b_1, \ldots, b_n; a_1, \ldots, a_n$ are listed in Table* 8.3.

Proof. Let $H \subset GL(n, \mathbb{C})$ be a finite primitive hypergeometric group. If its reflection group is imprimitive, the parameters are given by Theorem 5.14, and listed in Table 8.3.

Suppose H_r is primitive. Then, by Proposition 5.7, we may as well assume that $H = H_r$. Since H is now a primitive reflection group, it is contained in the list of Shephard and Todd, reproduced in Table 8.1. To determine the eigenvalues of h_∞ and h_0^{-1} we proceed as follows. Suppose H equals, say, ST32. In Table 8.1 we see that this group can be defined over $\mathbb{Q}(\omega)$. So the characteristic polynomials of h_∞, h_0^{-1} are in $\mathbb{Q}(\omega)$ [X] and have degree 4. Moreover, its zeros are roots of unity. There exist finitely many such polynomials and they can be obtained by multiplication of $\mathbb{Q}(\omega)$-irreducible cyclotomic polynomials. In Table 8.2 we have listed the exponents of the roots of the irreducible polynomials for the various fields.

So we have a finite number of possibilities for the eigenvalues of h_∞ and h_0^{-1} and by using Theorem 4.8 we can decide which combinations yield a finite group. Using Theorems 5.3 and 5.8 we can weed out the cases when H is imprimitive and the remaining cases are listed in Table 8.3. This table is made such that if the exponents of $a_1, \ldots, a_n$; $b_1, \ldots, b_n$ occur, then the exponents of $\zeta a_1^k, \ldots, \zeta a_n^k$; $\zeta b_1^k, \ldots, \zeta b_n^k$ and $\zeta b_1^k, \ldots, \zeta b_n^k$; $\zeta a_1^k, \ldots, \zeta a_n^k$ for $\zeta \in \mathbb{C}^*$, $(h, k)=1$ do not occur in the list.

Note also, that an infinite number of cases is given by ST1. In this case however, $H \simeq S_{n+1}$ and the representation is the one described in Proposition 5.9. The eigenvalues, listed in Table 8.3, follow readily.

8. Tables

Table 8.1. The finite primitive complex reflection groups in dimension $n \geqq 3$

The following list has been taken from A.M. Cohen's Utrecht University thesis, 1976.

Shephard-Todd number	Dimension n	Symbol	Order	Order of center	Field of definition
1	$n \geqq 4$	A_n	$(n+1)!$	1	$\mathbb{Q}$
23	3	H_3	120	2	$\mathbb{Q}(\sqrt{5})$
24	3	Klein	336	2	$\mathbb{Q}(\sqrt{-7})$
25	3	Hesse	648	3	$\mathbb{Q}(\omega)$
26	3	Hesse	1296	6	$\mathbb{Q}(\omega)$
27	3	Valentiner	2160	6	$\mathbb{Q}(\sqrt{5}, \omega)$
28	4	F_4	$2^7 \cdot 3^2$	2	$\mathbb{Q}$
29	4		$2^9 \cdot 3 \cdot 5$	4	$\mathbb{Q}(i)$
30	4	H_4	$2^6 \cdot 3^2 \cdot 5^2$	2	$\mathbb{Q}(\sqrt{5})$
31	4		$2^{10} \cdot 3^2 \cdot 5$	4	$\mathbb{Q}(i)$
32	4		$2^7 \cdot 3^5 \cdot 5$	6	$\mathbb{Q}(\omega)$
33	5	Burkhardt	$2^7 \cdot 3^4 \cdot 5$	2	$\mathbb{Q}(\omega)$
34	6	Mitchell	$2^9 \cdot 3^7 \cdot 5 \cdot 7$	6	$\mathbb{Q}(\omega)$
35	6	E_6	$2^7 \cdot 3^4 \cdot 5$	1	$\mathbb{Q}$
36	7	E_7	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	2	$\mathbb{Q}$
37	8	E_8	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	2	$\mathbb{Q}$

Table 8.2. Irreducible cyclotomic polynomials

The construction of all $P(x) \in \mathbb{Q}[x]$, irreducible over $\mathbb{Q}[x]$ of given degree such that all roots of P are roots of unity is simple. One determines $d \in \mathbb{N}$ such that $\phi(d) = \deg P$, where ϕ is Euler's totient function, and put $P(x) = \prod_{(h,d)=1} (x - \exp 2\pi i h/d)$.

Now, let K be an algebraic number field, G its Galois group over $\mathbb{Q}$. Let $P(x) \in K[x]$ be irreducible over $K[x]$ and suppose its roots are roots of unity. Denote by P^σ the polynomial obtained by applying $\sigma \in G$ to all coefficients of P. Then the product of all distinct P^σ is again an irreducible cyclotomic polynomial over $\mathbb{Q}$, and we are back in the former case.

In the following table the notation $(1/4, 3/4) + k/6$ stands for $(1/4 + k/6, 3/4 + k/6)$.

Degree P	K	Exponents of the roots of $P(x)$	
1	$\mathbb{Q}, \mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{-7})$	$k/2$	$(k=0, 1)$
	$\mathbb{Q}(\omega), \mathbb{Q}(\omega, \sqrt{5})$	$k/6$	$(k=0, 1, 2, 3, 4, 5)$
	$\mathbb{Q}(i)$	$k/4$	$(k=0, 1, 2, 3)$
2	$\mathbb{Q}, \mathbb{Q}(\sqrt{-7})$	$1/4, 3/4$	
		$(1/3, 2/3) + k/2$	$(k=0, 1)$
	$\mathbb{Q}(\sqrt{5})$	$1/4, 3/4$	
		$(1/3, 2/3) + k/2$	$(k=0, 1)$
		$(1/5, 4/5) + k/2$	$(k=0, 1)$
		$(2/5, 3/5) + k/2$	$(k=0, 1)$

Table 8.2. (continued)

	$\mathbb{Q}(\omega)$	$(1/4, 3/4)+k/3$	$(k=0, 1, 2)$
	$\mathbb{Q}(i)$	$(1/3, 2/3)+k/4$	$(k=0, 1, 2, 3)$
		$(1/8, 5/8)+k/2$	$(k=0, 1)$
	$\mathbb{Q}(\omega, \sqrt{5})$	$(1/4, 3/4)+k/3$	$(k=0, 1, 2)$
		$(1/5, 4/5)+k/6$	$(k=0, 1, 2, 3, 4, 5)$
		$(2/5, 3/5)+k/6$	$(k=0, 1, 2, 3, 4, 5)$
3	$\mathbb{Q}, \mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(i)$	–	
	$\mathbb{Q}(\sqrt{-7})$	$(1/7, 2/7, 4/7)+k/2$	$(k=0, 1)$
		$(3/7, 5/7, 6/7)+k/2$	$(k=0, 1)$
	$\mathbb{Q}(\omega)$, $\mathbb{Q}(\omega, \sqrt{5})$	$(1/9, 4/9, 7/9)+k/18$	$(k=0, 2, 3, 5)$
4	$\mathbb{Q}$	$1/8, 3/8, 5/8, 7/8$	
		$(1/5, 2/5, 3/5, 4/5)+k/2$	$(k=0, 1)$
		$1/12, 5/12, 7/12, 11/12$	
	$\mathbb{Q}(i)$	$(1/8, 3/8, 5/8, 7/8)+k/16$	$(k=1, 3)$
		$(1/5, 2/5, 3/5, 4/5)+k/4$	$(k=0, 1, 2, 3)$
		$(1/12, 5/12, 7/12, 11/12)+k/8$	$(k=1, 3)$
	$\mathbb{Q}(\omega)$	$(1/5, 2/5, 3/5, 4/5)+k/6$	$(k=0, 1, 2, 3, 4, 5)$
		$(1/8, 3/8, 5/8, 7/8)+k/3$	$(k=0, 1, 2)$
	$\mathbb{Q}(\sqrt{5})$	$(2/15, 7/15, 8/15, 13/15)+k/2$	$(k=0, 1)$
		$(1/15, 4/15, 11/15, 14/15)+k/2$	$(k=0, 1)$
		$1/20, 9/20, 11/20, 19/20$	
		$3/20, 7/20, 13/20, 17/20$	
5	$\mathbb{Q}, \mathbb{Q}(\omega)$	–	
6	$\mathbb{Q}$	$(1/9, 2/9, 4/9, 5/9, 7/9, 8/9)+k/2$	$(k=0, 1)$
		$(1/7, 2/7, 3/7, 4/7, 5/7, 6/7)+k/2$	$(k=0, 1)$
	$\mathbb{Q}(\omega)$	$(1/7, 2/7, 3/7, 4/7, 5/7, 6/7)+k/6$	$(k=0, 1, 2, 3, 4, 5)$
		$1/36, 7/36, 13/36, 19/36, 25/36, 31/36$	
		$5/36, 11/36, 17/36, 23/36, 29/36, 35/36$	
7	$\mathbb{Q}$	–	
8	$\mathbb{Q}$	$1/16, 3/16, 5/16, 7/16, 9/16, 11/16, 13/16, 15/16$	
		$(1/15, 2/15, 4/15, 7/15, 8/15, 11/15, 13/15, 14/15)+k/2$	$(k=0, 1)$
		$1/20, 3/20, 7/20, 9/20, 11/20, 13/20, 17/20, 19/20$	
		$1/24, 5/24, 7/24, 11/24, 13/24, 17/24, 19/24, 23/24$	

Table 8.3. Finite primitive hypergeometric groups

This table essentially contains all parameter sets of finite primitive hypergeometric groups H (see Theorem 7.1). Those groups for which the reflection subgroup is imprimitive are given by Theorem 5.14 and are listed as nrs. 11, 41, 42 in Table 8.3. Of the remaining parameters sets we know that the reflection subgroup is primitive and by Proposition 5.7 the group H is scalar shift of the primitive reflection group H_r. With the possible exception of nrs. 48, 49 the parameters listed are such that $H=H_r$. This can be seen as follows. Let K be the field generated by the coefficients of the characteristic polynomials of h_∞, h_0^{-1}. The parameters listed are such that a scalar shift of H by a root of unity does not change the field of definition of H into a proper subfield of K. Hence

Table 8.3. (continued)

the field of definition of H_r is also K. Given n and K, we can look up the possibilities for H_r in Table 8.1. With the exception of the choices ST 25/26 and ST 29/31 the choice of H_r is unique. Excepting ST 33 and ST 35 we see that the center of the remaining reflection groups is maximal in the sense that they contain all possible scalars contained in $GL(n, K)$. So the transition $H_r \to H$ does not yield any new scalars and hence $H = H_r$. The exceptions will be treated one by one.

ST 25/26

These groups correspond to the numbers 9, 10, 11 of Table 8.3. Note that the determinants of h_∞, h_0^{-1} are cube roots of unity in all these cases. Hence the center of H has order 1 or 3. Since ST 26 has a center of order 6, we conclude $H = H_r = \text{ST } 25$.

ST 29/31

These groups correspond to the numbers 20 to 23 of Table 8.3. We remark that the center of both groups are maximal with respect to $K = \mathbb{Q}(i)$. Hence $H = H_r$ in both cases. It is known that ST 29 contains 40 reflections of order 2 and ST 31 contains 60 such reflections. G. Verhagen actually exhibited 60 reflections for the numbers 22, 23 which implies $H = \text{ST } 31$ for these numbers. For numbers 20, 21 G. Verhagen found that the group can be generated by 4 reflections. This implies that we have ST 29, since ST 31 needs at least 5 generating reflections.

ST 33

This group corresponds to the numbers 41 to 44 of Table 8.3. The determinants of h_∞, h_0^{-1} are ± 1 and since the center of H is defined over $\mathbb{Q}(\omega)$, it has order 1 or 2. The group ST 33 has center of order 2, and hence $H = H_r = \text{ST } 33$

ST 35

This group corresponds to the numbers 45 to 49 of Table 8.3. We either have $H = \text{ST } 35$ or $H = \{\pm 1\} \times \text{ST } 35$. In case the exponents of h_∞ read 1/9, 2/9, 4/9, 5/9, 7/9, 8/9 we see that $h_\infty^9 = \text{Id}$. Notice, $(-h_\infty)^9 = -\text{Id} \notin H_r$, hence $-h_\infty \notin H_r$. So we conclude $h_\infty \in H_r$ and hence $H = H_r$. With respect to the numbers 45, 46 we can follow a similar argument starting from G. Verhagen's observations $(h_\infty h_0^{-4})^3 = \text{Id}$ for number 46 and $h_\infty^3\, h_0^2\, h_\infty^3\, h_0^{-1}\, h_\infty^2\, h_0^{-1}\, h_\infty\, h_0^{-1}\, h_\infty^2\, h_0^{-1} = \text{Id}$ for number 45

No.	Dimension	Parameter set	Field of definition	Group
1	$n \geqq 4$	$\frac{1}{n+1}\,\frac{2}{n+1} \cdots \frac{n-1}{n+1}\,\frac{n}{n+1}$; $0\,\frac{1}{j}\,\frac{2}{j} \cdots \frac{j-1}{j}\,\frac{1}{n+1-j}\,\frac{2}{n+1-j} \cdots \frac{n-j}{n+1-j}$ with $(j, n+1) = 1$	$\mathbb{Q}$	ST 1

Table 8.3. (continued)

No.	Dimension	Parameter set		Field of definition	Group
2	3	$\frac{3}{14}\ \frac{5}{14}\ \frac{13}{14}$;	$0\ \frac{1}{3}\ \frac{2}{3}$		
3			$0\ \frac{1}{4}\ \frac{3}{4}$	$\mathbb{Q}(\sqrt{-7})$	ST 24
4			$\frac{1}{7}\ \frac{2}{7}\ \frac{4}{7}$		
5		$0\ \frac{1}{5}\ \frac{4}{5}$;	$\frac{1}{6}\ \frac{1}{2}\ \frac{5}{6}$	$\mathbb{Q}(\sqrt{5})$	ST 23
6			$\frac{1}{10}\ \frac{1}{2}\ \frac{9}{10}$		
7		$\frac{1}{6}\ \frac{11}{30}\ \frac{29}{30}$;	$0\ \frac{1}{5}\ \frac{4}{5}$	$\mathbb{Q}(\omega, \sqrt{5})$	ST 27
8			$0\ \frac{1}{4}\ \frac{3}{4}$		
9	3	$\frac{1}{6}\ \frac{2}{3}\ \frac{5}{6}$;	$0\ \frac{1}{4}\ \frac{3}{4}$		
10		$\frac{1}{9}\ \frac{4}{9}\ \frac{7}{9}$;	$0\ \frac{1}{6}\ \frac{1}{2}$	$\mathbb{Q}(\omega)$	ST 25
11			$0\ \frac{1}{4}\ \frac{3}{4}$		
12		$\frac{1}{12}\ \frac{7}{12}\ \frac{5}{6}$;	$0\ \frac{1}{4}\ \frac{3}{4}$	$\mathbb{Q}(\omega)$	imprim. H_r
13	4	$\frac{1}{12}\ \frac{5}{12}\ \frac{7}{12}\ \frac{11}{12}$;	$0\ \frac{1}{5}\ \frac{1}{2}\ \frac{4}{5}$		
14		$\frac{1}{30}\ \frac{11}{30}\ \frac{19}{30}\ \frac{29}{30}$;	$0\ \frac{1}{4}\ \frac{1}{2}\ \frac{3}{4}$		
15			$0\ \frac{1}{3}\ \frac{1}{2}\ \frac{2}{3}$		
16			$0\ \frac{1}{5}\ \frac{1}{2}\ \frac{4}{5}$	$\mathbb{Q}(\sqrt{5})$	ST 30
17			$0\ \frac{1}{10}\ \frac{1}{2}\ \frac{9}{10}$		
18		$\frac{1}{20}\ \frac{9}{20}\ \frac{11}{20}\ \frac{19}{20}$;	$0\ \frac{1}{6}\ \frac{1}{2}\ \frac{5}{6}$		
19			$0\ \frac{2}{5}\ \frac{1}{2}\ \frac{3}{5}$		
20		$\frac{3}{20}\ \frac{7}{20}\ \frac{11}{20}\ \frac{19}{20}$;	$0\ \frac{1}{4}\ \frac{3}{8}\ \frac{7}{8}$	$\mathbb{Q}(i)$	ST 29
21			$0\ \frac{1}{3}\ \frac{1}{2}\ \frac{2}{3}$		
22			$\frac{7}{24}\ \frac{11}{24}\ \frac{19}{24}\ \frac{23}{24}$	$\mathbb{Q}(i)$	ST 31
23		$\frac{1}{3}\ \frac{7}{12}\ \frac{2}{3}\ \frac{11}{12}$;	$\frac{1}{5}\ \frac{2}{5}\ \frac{3}{5}\ \frac{4}{5}$		
24	4	$\frac{1}{18}\ \frac{7}{18}\ \frac{13}{18}\ \frac{5}{6}$;	$0\ \frac{1}{9}\ \frac{4}{9}\ \frac{7}{9}$		
25		$\frac{1}{12}\ \frac{1}{6}\ \frac{7}{12}\ \frac{5}{6}$;	$0\ \frac{1}{9}\ \frac{4}{9}\ \frac{7}{9}$		
26		$\frac{1}{12}\ \frac{5}{12}\ \frac{7}{12}\ \frac{11}{12}$;	$0\ \frac{1}{9}\ \frac{4}{9}\ \frac{7}{9}$		
27		$\frac{1}{24}\ \frac{7}{24}\ \frac{13}{24}\ \frac{19}{24}$;	$0\ \frac{1}{4}\ \frac{1}{3}\ \frac{3}{4}$		
28			$0\ \frac{1}{9}\ \frac{4}{9}\ \frac{7}{9}$		
29		$\frac{11}{30}\ \frac{17}{30}\ \frac{23}{30}\ \frac{29}{30}$;	$\frac{1}{5}\ \frac{2}{5}\ \frac{3}{5}\ \frac{4}{5}$		
30			$\frac{1}{4}\ \frac{5}{12}\ \frac{3}{4}\ \frac{11}{12}$	$\mathbb{Q}(\omega)$	ST 32
31			$\frac{1}{8}\ \frac{3}{8}\ \frac{5}{8}\ \frac{7}{8}$		
32			$0\ \frac{1}{2}\ \frac{2}{3}\ \frac{5}{6}$		
33			$0\ \frac{5}{12}\ \frac{2}{3}\ \frac{11}{12}$		
34			$\frac{1}{4}\ \frac{1}{2}\ \frac{3}{4}\ \frac{5}{6}$		
35			$\frac{5}{18}\ \frac{1}{2}\ \frac{11}{18}\ \frac{17}{18}$		
36			$\frac{2}{9}\ \frac{5}{9}\ \frac{2}{3}\ \frac{8}{9}$		
37		$\frac{1}{12}\ \frac{5}{12}\ \frac{7}{12}\ \frac{11}{12}$;	$0\ \frac{1}{3}\ \frac{1}{2}\ \frac{2}{3}$	$\mathbb{Q}$	
38		$\frac{1}{3}\ \frac{7}{12}\ \frac{2}{3}\ \frac{11}{12}$;	$\frac{1}{8}\ \frac{3}{8}\ \frac{5}{8}\ \frac{7}{8}$		imprim- itive H_r
39			$\frac{1}{8}\ \frac{1}{2}\ \frac{5}{8}\ \frac{3}{4}$	$\mathbb{Q}(i)$	
40		$\frac{1}{24}\ \frac{5}{24}\ \frac{13}{24}\ \frac{17}{24}$;	$0\ \frac{1}{8}\ \frac{1}{4}\ \frac{5}{8}$		

Table 8.3. (continued)

No.	Dimension	Parameter set		Field of definition	Group
41	5	$\frac{1}{12}\ \frac{1}{4}\ \frac{7}{12}\ \frac{3}{4}\ \frac{5}{6}$;	$0\ \frac{1}{9}\ \frac{4}{9}\ \frac{2}{3}\ \frac{7}{9}$		
42			$0\ \frac{1}{5}\ \frac{2}{5}\ \frac{3}{5}\ \frac{4}{5}$		
43		$\frac{1}{6}\ \frac{5}{18}\ \frac{1}{2}\ \frac{11}{18}\ \frac{17}{18}$;	$0\ \frac{1}{5}\ \frac{2}{5}\ \frac{3}{5}\ \frac{4}{5}$	$\mathbb{Q}(\omega)$	ST 33
44			$0\ \frac{2}{9}\ \frac{1}{3}\ \frac{5}{9}\ \frac{8}{9}$		
45	6	$\frac{1}{12}\ \frac{1}{3}\ \frac{5}{12}\ \frac{7}{12}\ \frac{2}{3}\ \frac{11}{12}$;	$0\ \frac{1}{8}\ \frac{3}{8}\ \frac{1}{2}\ \frac{5}{8}\ \frac{7}{8}$		
46			$0\ \frac{1}{5}\ \frac{2}{5}\ \frac{1}{2}\ \frac{3}{5}\ \frac{4}{5}$		
47		$\frac{1}{9}\ \frac{2}{9}\ \frac{4}{9}\ \frac{5}{9}\ \frac{7}{9}\ \frac{8}{9}$;	$0\ \frac{1}{6}\ \frac{1}{4}\ \frac{1}{2}\ \frac{3}{4}\ \frac{5}{6}$	$\mathbb{Q}$	ST 35
48			$0\ \frac{1}{8}\ \frac{3}{8}\ \frac{1}{2}\ \frac{5}{8}\ \frac{7}{8}$		
49			$0\ \frac{1}{5}\ \frac{2}{5}\ \frac{1}{2}\ \frac{3}{5}\ \frac{4}{5}$		
50		$\frac{1}{8}\ \frac{1}{4}\ \frac{3}{8}\ \frac{5}{8}\ \frac{3}{4}\ \frac{7}{8}$;	$\frac{1}{6}\ \frac{11}{30}\ \frac{17}{30}\ \frac{2}{3}\ \frac{23}{30}\ \frac{29}{30}$		
51			$\frac{1}{9}\ \frac{1}{6}\ \frac{1}{3}\ \frac{4}{9}\ \frac{2}{3}\ \frac{7}{9}$		
52		$\frac{5}{42}\ \frac{11}{42}\ \frac{17}{42}\ \frac{23}{42}\ \frac{29}{42}\ \frac{41}{42}$;	$0\ \frac{1}{6}\ \frac{5}{18}\ \frac{1}{2}\ \frac{11}{18}\ \frac{17}{18}$		
53			$0\ \frac{2}{9}\ \frac{1}{3}\ \frac{1}{2}\ \frac{5}{9}\ \frac{8}{9}$		
54			$0\ \frac{1}{4}\ \frac{1}{3}\ \frac{1}{2}\ \frac{2}{3}\ \frac{3}{4}$	$\mathbb{Q}(\omega)$	ST 34
55			$0\ \frac{1}{5}\ \frac{2}{5}\ \frac{1}{2}\ \frac{3}{5}\ \frac{4}{5}$		
56			$\frac{1}{12}\ \frac{1}{6}\ \frac{1}{3}\ \frac{5}{12}\ \frac{7}{12}\ \frac{11}{12}$		
57			$0\ \frac{1}{8}\ \frac{3}{8}\ \frac{1}{2}\ \frac{5}{8}\ \frac{7}{8}$		
58	7	$\frac{1}{18}\ \frac{5}{18}\ \frac{7}{18}\ \frac{1}{2}\ \frac{11}{18}\ \frac{13}{18}\ \frac{17}{18}$;	$0\ \frac{1}{12}\ \frac{1}{3}\ \frac{5}{12}\ \frac{7}{12}\ \frac{2}{3}\ \frac{11}{12}$		
59			$0\ \frac{1}{5}\ \frac{1}{3}\ \frac{2}{5}\ \frac{3}{5}\ \frac{2}{3}\ \frac{4}{5}$		
60			$0\ \frac{1}{7}\ \frac{2}{7}\ \frac{3}{7}\ \frac{4}{7}\ \frac{5}{7}\ \frac{6}{7}$	$\mathbb{Q}$	ST 36
61		$\frac{1}{14}\ \frac{3}{14}\ \frac{5}{14}\ \frac{1}{2}\ \frac{9}{14}\ \frac{11}{14}\ \frac{13}{14}$;	$0\ \frac{1}{12}\ \frac{1}{3}\ \frac{5}{12}\ \frac{7}{12}\ \frac{2}{3}\ \frac{11}{12}$		
62			$0\ \frac{1}{5}\ \frac{1}{3}\ \frac{2}{5}\ \frac{3}{5}\ \frac{2}{3}\ \frac{4}{5}$		
63	8	$\frac{1}{30}\ \frac{7}{30}\ \frac{11}{30}\ \frac{13}{30}\ \frac{17}{30}\ \frac{19}{30}\ \frac{23}{30}\ \frac{29}{30}$;	$0\ \frac{1}{18}\ \frac{5}{18}\ \frac{7}{18}\ \frac{1}{2}\ \frac{11}{18}\ \frac{13}{18}\ \frac{17}{18}$		
64			$0\ \frac{1}{12}\ \frac{1}{3}\ \frac{5}{12}\ \frac{1}{2}\ \frac{7}{12}\ \frac{2}{3}\ \frac{11}{12}$		
65			$0\ \frac{1}{8}\ \frac{1}{3}\ \frac{3}{8}\ \frac{1}{2}\ \frac{5}{8}\ \frac{2}{3}\ \frac{7}{8}$		
66			$0\ \frac{1}{5}\ \frac{1}{4}\ \frac{2}{5}\ \frac{1}{2}\ \frac{3}{5}\ \frac{3}{4}\ \frac{4}{5}$		
67			$0\ \frac{1}{5}\ \frac{1}{3}\ \frac{2}{5}\ \frac{1}{2}\ \frac{3}{5}\ \frac{2}{3}\ \frac{4}{5}$		
68			$0\ \frac{1}{7}\ \frac{2}{7}\ \frac{3}{7}\ \frac{1}{2}\ \frac{4}{7}\ \frac{5}{7}\ \frac{6}{7}$		
69			$0\ \frac{1}{12}\ \frac{1}{4}\ \frac{5}{12}\ \frac{1}{2}\ \frac{7}{12}\ \frac{3}{4}\ \frac{11}{12}$		
70			$0\ \frac{1}{8}\ \frac{1}{4}\ \frac{3}{8}\ \frac{1}{2}\ \frac{5}{8}\ \frac{3}{4}\ \frac{7}{8}$	$\mathbb{Q}$	ST 37
71		$\frac{1}{20}\frac{3}{20}\ \frac{7}{20}\ \frac{9}{20}\ \frac{11}{20}\ \frac{13}{20}\ \frac{17}{20}\ \frac{19}{20}$;	$0\ \frac{1}{12}\ \frac{1}{3}\ \frac{5}{12}\ \frac{1}{2}\ \frac{7}{12}\ \frac{2}{3}\ \frac{11}{12}$		
72			$0\ \frac{1}{8}\ \frac{1}{3}\ \frac{3}{8}\ \frac{1}{2}\ \frac{5}{8}\ \frac{2}{3}\ \frac{7}{8}$		
73			$0\ \frac{1}{7}\ \frac{2}{7}\ \frac{3}{7}\ \frac{1}{2}\ \frac{4}{7}\ \frac{5}{7}\ \frac{6}{7}$		
74			$0\ \frac{1}{9}\ \frac{2}{9}\ \frac{4}{9}\ \frac{1}{2}\ \frac{5}{9}\ \frac{7}{9}\ \frac{8}{9}$		
75		$\frac{1}{24}\ \frac{5}{24}\ \frac{7}{24}\ \frac{11}{24}\ \frac{13}{24}\ \frac{17}{24}\ \frac{19}{24}\ \frac{23}{24}$;	$0\ \frac{1}{5}\ \frac{1}{4}\ \frac{2}{5}\ \frac{1}{2}\ \frac{3}{5}\ \frac{3}{4}\ \frac{4}{5}$		
76			$0\ \frac{1}{7}\ \frac{2}{7}\ \frac{3}{7}\ \frac{1}{2}\ \frac{4}{7}\ \frac{5}{7}\ \frac{6}{7}$		
77			$0\ \frac{1}{9}\ \frac{2}{9}\ \frac{4}{9}\ \frac{1}{2}\ \frac{5}{9}\ \frac{7}{9}\ \frac{8}{9}$		

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