On dualizable DG categories.

Convention: unless stated otherwise, we live in $DGCat_{cont}$ (i.e., DG categories are presentable and functors are continuous).

Theorem 1. Let $\mathcal{D} \in DGCat_{cont}$. The following are equivalent:

(i) $\mathcal{D}$ is dualizable.

(ii) The functor

$$C \mapsto Funct_{cont}(\mathcal{D}, C), \quad C \in DGCat_{cont}$$

commutes with colimits in $DGCat_{cont}$. (N. Rozenblyum suggests to peraphrize this by saying that C is an "enriched completely compact" object of $DGCat_{cont}$; here "completely" indicates that the colimits are not necessarily filtered and "enriched" indicates that the colimit of $Funct_{cont}(\mathcal{D}, C_i)$ is computed in $DGCat_{cont}$).

(iii) For any localization functor $\pi: C_1 \to C_2$ the corresponding functor $Funct_{cont}(\mathcal{D}, C_1) \to Funct_{cont}(\mathcal{D}, C_2)$ is also a localization functor.

(iv) For any localization functor $\pi: C_1 \to C_2$ the corresponding functor $Funct_{cont}(\mathcal{D}, C_1) \to Funct_{cont}(\mathcal{D}, C_2)$ is essentially surjective.

(v) Any localization functor $\pi: C \to \mathcal{D}$ admits a (continuous!) splitting $s: \mathcal{D} \to C$ ("splitting" means that $\pi \circ s \simeq Id_{\mathcal{D}}$).

(vi) $\mathcal{D}$ can be represented as a retract of some compactly generated DG category C ("retract" means that there exist $\mathcal{D} \xrightarrow{s} C \xrightarrow{\pi} \mathcal{D}$ with $\pi \circ s \simeq Id_{\mathcal{D}}$).

Remarks. (a) In the situation of (v) and (vi) the functor $\$: \mathcal{D} \to \mathcal{C}$ does not have to be fully faithful.

(b) Despite the previous remark the word "retract" is used in this context by Lurie in his "Higher Topoi" book.

(c) The proof given below shows that in the situation of (v) the functor $\pi: \mathcal{C} \to \mathcal{D}$ admits a canonical splitting. More precisely, the category of all splittings has a final object.

Now let us prove the theorem.

For the implication (i)$\Rightarrow$(ii), see Lemma 2.1.6(2) of [GL:DG].

The implications (iii)$\Rightarrow$(iv)$\Rightarrow$(v) are clear.

The implication (v)$\Rightarrow$(vi) holds because $\mathcal{D}$ is presentable and therefore is a localization of some compactly generated DG category.

For the implication (vi)$\Rightarrow$(i) see Lemma 4.3.3 of the article "On some finiteness questions..." by Gaitsgory and me.

Finally, the implication (ii)$\Rightarrow$(iii) follows from the equivalence (a)$\Leftrightarrow$(b) in the next lemma.

Lemma 2. Let $\pi: \mathcal{C}_1 \to \mathcal{C}_2$ be a functor in $\mathcal{DG}Cat_{cont}$. The following are equivalent:

(a) π is a localization functor.

(b) $\mathcal{C}_2 = \mathrm{Coker}(\mathcal{A} \xrightarrow{F} \mathcal{C}_1)$ for some $F: \mathcal{A} \to \mathcal{C}_1$.

(b′) $\mathcal{C}_2 = \mathrm{Coker}(\mathcal{A}' \xrightarrow{F'} \mathcal{C}_1)$ for some fully faithful functor $F': \mathcal{A}' \to C_1$.

Here I am using the notation

$$\mathrm{Coker}\,(\mathcal{A} \xrightarrow{F} C_1) := \mathrm{colim}\,(\mathcal{A} \underset{0}{\overset{F}{\rightrightarrows}} C_1).$$

As for the lemma, the equivalence (a)$\Leftrightarrow$(b') is probably standard. Clearly (b')$\Rightarrow$(b). It is also clear that (b)$\Rightarrow$(b') (take $\mathcal{A}' := \overline{\mathrm{Im}}\, F$, where $\overline{\mathrm{Im}}\, F \subset C_1$ is the full subcategory generated by the essential image of F).

For aesthetic reasons, I would introduce the following definition.

Definition. A functor $F: C_1 \to C_2$ in DGCat_{cont} is *locally split* if for any compactly generated (or equivalently, any dualizable) $\mathcal{A} \in \mathrm{DGCat}_{cont}$ the corresponding functor $\mathrm{Funct}_{cont}(\mathcal{A}, C_1) \to \mathrm{Funct}_{cont}(\mathcal{A}, C_2)$ is essentially surjective. As far as I understand, local splitness just means that F has a splitting in DGCat.

E.g., a localization functor is locally split.

Properties (iii)-(v) have variants with the words "localization functor" replaced by "locally split functor". These variants are equivalent to the original properties (iii)-(v) (i.e., to the dualizability of $\mathcal{D}$). This follows, e.g., from the next observation.

Lemma 3. A functor $F: C_1 \to C_2$ in DGCat_{cont} is locally split if and only if there exists a commutative diagram
$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[rr, bend right, "\Phi"'] & C_1 \arrow[r, "F"] & C_2 \end{tikzcd}$$
with Φ being a localization functor.

Proof. The "if" statement follows from the fact that a localization functor is locally split. To prove the "only if" statement, choose a localization functor $\phi: \mathcal{A} \to C_2$ with $\mathcal{A}$ compactly generated; then local splitness of $F: C_1 \to C_2$ implies the existence of a functor $\mathcal{A} \to C_1$ such that the above diagram commutes. ∎