HOPF ALGEBRAS (SPENCER BLOCH)

Lecture 3: $\boxed{10/02/2007}$

Milnor - Moore theorem

A = graded Hopf algebra over a field k

$$P(A) = \{ x \in \bigoplus_{i \geq 1} A_i \mid \Delta(x) = x \otimes 1 + 1 \otimes x \}$$

Then $P(A)$ is a graded (super) Lie algebra with respect to $[a, b] = ab - (-1)^{|a||b|} ba$ if $a, b \in P(A)$ are homogeneous of degrees $|a|$ and $|b|$, respectively.

If L is a graded Lie algebra over k, its universal enveloping algebra $U(L)$ inherits a grading from L.

Caution: If we want the graded Hopf algebra $U(L)$ to be connected, we must require L to be strictly positively graded: $L = \bigoplus_{i \geq 1} L_i$, which, in particular, forces L to be nilpotent (if L is finite dimensional).

Remark: $U(L)$ is, in general, not the same as the universal enveloping algebra of the underlying Lie algebra of L! Namely, it is the quotient of the tensor algebra $T(L)$ by the two-sided ideal generated by all elements of the form

$$x_1 \otimes x_2 - (-1)^{|x_1||x_2|} x_2 \otimes x_1 - [x_1, x_2]$$

for homogeneous elements $x_1, x_2 \in L$.

Example: If $L = L_1$ (i.e., L is concentrated in degree 1), then L is abelian, but $\mathcal{U}(L)$ is the alternating algebra $\Lambda(L_1)$ on the underlying vector space of L_1.

Similarly, the symmetric algebra of L is defined as the quotient

$$S(L) = T(L) \Big/ \left(x_1 \otimes x_2 - (-1)^{|x_1||x_2|} x_2 \otimes x_1 \right)$$

Poincaré - Birkhoff - Witt theorem.

If $\mathrm{char}(k) = 0$, the composition

$$\begin{array}{ccccc} S(L) & \longrightarrow & T(L) & \twoheadrightarrow & \mathcal{U}(L) \\ x_1 \cdot \ldots \cdot x_n & \longmapsto & \frac{1}{n!} \sum\limits_{\sigma \in S_n} (-1)^{\sigma} x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)} & & \end{array}$$

is an isomorphism of coalgebras.

Sketch of the proof (possibly taken from Serre's book "Algèbres de Lie"). Idea:

1. Equip $S(L)$ with an L-module structure
2. check that the induced composition

$$\begin{array}{ccccc} \mathcal{U}(L) & \longrightarrow & \mathrm{End}_k(S(L)) & \longrightarrow & S(L) \\ & & \alpha & \longmapsto & \alpha(1) \end{array}$$

is inverse to the symmetrization map constructed above.

Note that the construction of #1 will not be canonical.

Choose a well ordered basis $\{x_\mu\}$ of L, consisting of homogeneous elements. Then $S(L)$ has a basis consisting of elements of the form

$$x_M = x_{\mu_1} x_{\mu_2} \cdots x_{\mu_k}, \quad \text{where } k \geq 0$$
$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k,$$

and there are no repetitions of indices μ such that the corresponding x_μ has odd degree

Now we define an action of L on $S(L)$ inductively:

$$x_\mu * x_M = \begin{cases} x_{\mu M} & \text{if } \mu < \mu_1, \text{ or } \mu = \mu_1 \text{ and } |x_{\mu_1}| \text{ is even.} \\ \frac{1}{2}[x_\mu, x_\mu] * x_N & \text{if } \mu = \mu_1 \text{ and } |x_{\mu_1}| \text{ is odd} \\ [x_\mu, x_{\mu_1}] * x_N + (-1)^{|x_{\mu_1}||x_\mu|} x_{\mu_1} * (x_\mu * x_N) & \text{if } \mu_1 < \mu \end{cases}$$

Here, $M = (\mu_1, \mu_2, \ldots, \mu_k)$
and $N = (\mu_2, \ldots, \mu_k)$.

Note that $x_\mu * x_N$ is defined by induction on k, and $x_{\mu_1} * (x_\mu * x_N)$ is defined by induction on μ, because $\mu_1 < \mu$. (This is why we need the basis to be well ordered.)

This defines an action map

$$* : L \times S(L) \longrightarrow S(L)$$

Now the proof of #2 is straightforward, and is left to the reader as an exercise.

<u>Corollary</u>. If $\mathrm{char}(k)=0$, then

$$P(\mathcal{U}(L)) = L.$$

<u>Theorem</u> (Milnor–Moore). Let k be a field of characteristic 0, and let A be a connected graded cocommutative Hopf algebra over k (with the usual assumption that $\dim_k A_n < \infty$ for all n).

Then the natural homomorphism

$$\mathcal{U}(P(A)) \longrightarrow A$$

is an isomorphism.

<u>Proof</u>. Consider $A^* = \bigoplus_{n=0}^{\infty} A_n^*$, which is a commutative connected graded Hopf algebra over k. It was proved in the previous lecture that the composition

$$P(A^*) \hookrightarrow I(A^*) \longrightarrow I(A^*)/I(A^*)^2 = Q(A^*)$$

is injective. Dualizing, we find that

$$P(A) \longrightarrow Q(A) \text{ is surjective.}$$

This easily implies that $P(A)$ generates A as an algebra. Hence the induced homomorphism

$$\mathcal{U}(P(A)) \xrightarrow{\varphi} A \quad \text{is surjective.}$$

Let us write $J = \mathrm{Ker}(\varphi)$. It is a graded two-sided ideal in $\mathcal{U}(P(A))$. Moreover, since $\mathcal{U}(P(A))_1 = P(A)_1$, we have $J_1 = 0$.

Let us show that $J_n = 0$ for all $n \in \mathbb{N}$, by induction on n. First, it is easy to check that φ is a homomorphism of coalgebras. If $J \neq 0$, choose the minimal $n \in \mathbb{N}$ for which $J_n \neq 0$, and pick $x \in J_n$, $x \neq 0$.

We have
$$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_{i=1}^{n-1} x_i' \otimes x_i'',$$
where $x_i' \in U(P(A))_i$ and $x_i'' \in U(P(A))_{n-i}$.

Now $(\varphi \otimes \varphi)(\Delta(x)) = 0$, which (by our choice of n) forces $\Delta(x) = x \otimes 1 + 1 \otimes x$. By the previous corollary, this forces $x \in P(A)$, which is a contradiction. //.

Next complex of ideas to be discussed

- Bar complex
- Chen theory
- Sullivan theory

The bar complex has something to do with path spaces.

Let us write $I = [0,1]$

If X is any topological space, we will write

$$X^I = P(X) = \{\text{continuous maps } [0,1] \to X\}$$
with the natural topology

$= $ the path space of X

Note that we have a natural map

$$\varphi : X^I \longrightarrow X \times X$$
$$\phi \longmapsto (\phi(0), \phi(1))$$

<u>Problem:</u> We have no canonical procedure for concatenating paths. For instance, if we put ${}_aP_b = \varphi^{-1}(a,b)$, we could try to define

$${}_aP_b \times {}_bP_c \longrightarrow {}_aP_c$$

by
$$(\phi, \psi) \longmapsto \left(t \longmapsto \begin{cases} \phi(2t), & 0 \le t \le 1/2 \\ \psi(2t-1), & 1/2 \le t \le 1 \end{cases} \right)$$

However, this would not be associative on the nose.

There is a better version of this construction, due to Moore:

$$P^M(X) = \{ (\phi, l) \mid \phi : [0, +\infty) \longrightarrow X \text{ is continuous, and } l \in [0, +\infty) \text{ is such that } \phi(t) = \phi(l) \text{ for all } t \ge l \}$$

Then we define

$$\varphi : P^M(X) \longrightarrow X \times X,$$
$$(\phi, l) \longmapsto (\phi(0), \phi(l)),$$

and ${}_aP^M_b = \varphi^{-1}(a,b)$, as before. Here we do get an associative multiplication, defined as follows:

$$ {}_aP^M_b(X) \times {}_bP^M_c(X) \longrightarrow {}_aP^M_c(X) $$

$$ ((\varphi_1, l_1), (\varphi_2, l_2)) \longmapsto (\varphi, l_1 + l_2), $$

where

$$ \varphi(t) = \begin{cases} \varphi_1(t), & 0 \le t \le l_1 \\ \varphi_2(t - l_1), & l_1 \le t \le l_1 + l_2 \\ \varphi_2(l_2), & t \ge l_1 + l_2 \end{cases} $$

We have an obvious inclusion

$$ \begin{aligned} X &\hookrightarrow P(X) \\ x &\longmapsto (\text{constant path } t \mapsto x) \end{aligned} $$

which is a homotopy retract.

Given $x \in X$, we write $\Omega(X)_x$ for the loop space of X at x, i.e., $\Omega(X)_x = {}_xP_x(X)$. We also have an obvious version $\Omega^M(X)_x = {}_xP^M_x(X)$, which is an honest topological group.

Note that $H^*(\Omega(X)_x)$ is a Hopf algebra. In fact, for us, it will turn out that loop spaces are some sort of "machines for producing Hopf algebras".

Cosimplicial description of the path space

Let us recall that we have the category Δ whose objects are the linearly ordered sets $[n] = \{0, 1, \ldots, n\}$ for $n = 0, 1, 2, \ldots$, and whose morphisms are the (non-strictly) increasing maps.

Recall that a simplicial set is a functor

$$\Delta^{op} \longrightarrow (\text{Sets}) = \text{the category of sets}$$

We will write

$$I[1] : \Delta^{op} \longrightarrow (\text{Sets})$$

for the functor represented by $[1] \in \Delta$.

Explicitly, note that we have

$$\mathrm{Hom}_{\Delta}([n], [1]) = \{\varphi_i^n\}_{0 \le i \le n+1},$$

where $\varphi_i^n : \{0, 1, \ldots, n\} \longrightarrow \{0, 1\}$ is defined by

$$\varphi_i^n(j) = \begin{cases} 0, & j \le n-i \\ 1, & j > n-i \end{cases}$$

Now if X is any topological space, we obtain a cosimplicial topological space $X^{I[1]}$, defined by $X^{I[1]}([n]) = X^{\mathrm{Hom}_{\Delta}([n],[1])}$

As a topological space, $X_n^{I[1]}$ is just X^{n+2}.

Next time we will work out the cosimplicial structure explicitly.