IDEMPOTENTS IN MONOIDAL CATEGORIES

MITYA BOYARCHENKO AND VLADIMIR DRINFELD

Contents

Introduction

This paper is devoted to an abstract study of idempotents in monoidal categories, building a foundation for the definition of character sheaves for unipotent groups [BD06].

Let us fix a monoidal category $\mathcal{M}$ with unit object $\mathbb{1}$. We define a *closed* (resp., *open*) *idempotent* in $\mathcal{M}$ as a morphism $\mathbb{1} \xrightarrow{\pi} e$ (resp., $e \xrightarrow{\pi} \mathbb{1}$) in $\mathcal{M}$ which becomes an isomorphism after tensoring with e either on the left or on the right. This definition is motivated by the following example: if $\mathcal{M}$ is the category of sheaves of complex vector spaces on a topological space X with the usual tensor product, and $\xi : Y \hookrightarrow X$ is the inclusion of a closed (resp., open) subspace, then the natural arrow $\mathbb{C}_X \longrightarrow \xi_! \mathbb{C}_Y$ (resp., $\xi_! \mathbb{C}_Y \longrightarrow \mathbb{C}_X$), obtained by adjunction from the natural isomorphism $\mathbb{C}_X\big|_Y \cong \mathbb{C}_Y$, is a closed (resp., open) idempotent in the category $\mathcal{M}$. The definition of a *locally closed idempotent*, modelling the sheaf $\xi_! \mathbb{C}_Y$ in the case where ξ is the inclusion of a locally closed subspace, can be obtained as a mixture of the first two definitions; it is studied in some detail in Section 5.

In view of the applications we have in mind, most of our paper is devoted to the study of closed idempotents. (This reflects the fact that coadjoint orbits for a unipotent algebraic group are closed.) Besides, every result about closed idempotents can be formally translated into a result about open idempotents, since if $\mathcal{M}$ is a monoidal category, then its opposite category $\mathcal{M}^{op}$, equipped with the same monoidal structure $\otimes$, is also a monoidal category,

Both authors were supported by NSF grant DMS-0401164.
Address: Department of Mathematics, University of Chicago, Chicago, IL 60637.
E-mail: `mitya@math.uchicago.edu` (M.B.), `drinfeld@math.uchicago.edu` (V.D.).

and an arrow π in $\mathcal{M}$ is a closed idempotent if and only if the corresponding arrow in $\mathcal{M}^{op}$ is an open idempotent.

In Section 1 we study morphisms between closed idempotents, two of the main results being that a closed idempotent $\mathbb{1} \xrightarrow{\pi} e$ is determined by its codomain e up to unique isomorphism, and if $\mathbb{1} \xrightarrow{\pi} e$ and $\mathbb{1} \xrightarrow{\pi'} e'$ are two closed idempotents in $\mathcal{M}$, then there exists at most one morphism $f : e \longrightarrow e'$ in $\mathcal{M}$ such that $f \circ \pi = \pi'$. This leads to a natural definition of a partial order on the set of isomorphism classes of closed idempotents in $\mathcal{M}$. Section 2 begins with the important observation that the notion of a closed idempotent is equivalent to the notion of an *idempotent monoid*, i.e., a monoid whose multiplication map is an isomorphism. This allows us to relate our notion of a closed idempotent to the notion of an *idempotent functor* studied by Adams [Ad73] and to the equivalent notion of an *idempotent monad* studied by Deleanu-Frei-Hilton [DFH1, DFH2].

Let A be an associative unital ring. We say that two idempotents $e_1, e_2 \in A$ are *equivalent* if $e_1 = e_2 e_1$ and $e_2 = e_1 e_2$. There is a bijection between the set of equivalence classes of idempotents in A and the set of right ideals $I \subseteq A$ that are direct summands of A as right A-modules, given by $e \mapsto eA$. Moreover, if $e \in A$ is an idempotent, then eAe is a subring of A which has e as the multiplicative identity, and we have a canonical isomorphism of rings $eAe \xrightarrow{\simeq} \mathrm{End}_{\mathrm{mod}-A}(eA)$. The categorification of these statements, where A is replaced by a monoidal category $\mathcal{M}$ and e is replaced by a closed idempotent $\mathbb{1} \xrightarrow{\pi} e$, is explained in Section 3; in particular, the subring eAe is replaced by the strictly full subcategory $e\mathcal{M}e$ of $\mathcal{M}$ consisting of all objects X of $\mathcal{M}$ such that $e \otimes X \otimes e \cong X$.

With the notation of the previous paragraph, if A is moreover commutative, then the map $A \longrightarrow eAe$ given by $a \mapsto eae$ is a ring homomorphism. The categorification of this result is proved in Section 4, namely, we show that if $\mathcal{M}$ is braided, then the functor $X \mapsto e \otimes X \otimes e$ can be upgraded to a (strong) braided monoidal functor $\mathcal{M} \longrightarrow e\mathcal{M}e$. We also study the properties of closed idempotents from the point of view of the braiding; it turns out that if $\mathbb{1} \longrightarrow e$ is a closed idempotent in $\mathcal{M}$, then the braiding automorphism of $e \otimes e$ is the identity, and, moreover, the square of the braiding is the identity on $X \otimes e$ for each $X \in \mathcal{M}$.

In Section 5 we give two definitions of a locally closed idempotent in a monoidal category, and prove that they are equivalent in a triangulated tensor category.

In conclusion, we would like to mention that even though this article contains many new results, it is written more in the style of an expository paper rather than that of a research paper. In particular, it is essentially self-contained, and includes recollections of some standard material, as well as precise definitions and statements of results taken from works by other authors. The rest of the article can be read independently from this introduction.

Warning. This is a first and very preliminary draft of our paper. While the main results are formulated in detail, many of the proofs are missing. On the other hand, Section 1, on which the definition of character sheaves given in [BD06] is based, is complete.

1. Basic properties of closed idempotents

1.1. Let $\mathcal{M}$ be a monoidal category with unit object $\mathbb{1}$, and let $\mathbb{1} \xrightarrow{\pi} e$ be a morphism in $\mathcal{M}$. We define two full subcategories of $\mathcal{M}$ as follows:

$$_{\pi}\mathcal{M} = \big\{ m \in \mathcal{M} \,\big|\, \pi \otimes \mathrm{id}_m : \mathbb{1} \otimes m \longrightarrow e \otimes m \text{ is an isomorphism} \big\}$$

and

$$\mathcal{M}_{\pi} = \big\{ m \in \mathcal{M} \,\big|\, \mathrm{id}_m \otimes \pi : m \otimes \mathbb{1} \longrightarrow m \otimes e \text{ is an isomorphism} \big\}.$$

Definition 1.1. We say that π is a *closed idempotent* in $\mathcal{M}$ if $e \in_{\pi} \mathcal{M} \cap \mathcal{M}_{\pi}$. We say that an *object* e of $\mathcal{M}$ is a *closed idempotent* if there exists a morphism $\pi : \mathbb{1} \to e$ which is a closed idempotent.

The second part of the definition may seem like a bad idea; however, later on we will see that if $\mathbb{1} \xrightarrow{\pi} e$ and $\mathbb{1} \xrightarrow{\pi'} e$ are two closed idempotents with the same codomain e, then there exists a unique morphism $\sigma : e \to e$ such that $\pi' = \sigma \circ \pi$ (of course, σ is then necessarily an isomorphism, since the roles of π and π' can be interchanged).

1.2. The definition above is motivated in part by the following example. Let X be a topological space, let $\mathcal{M}$ be the category of sheaves of $\mathbb{C}$-vector spaces on X with the usual tensor product, and let $Y \overset{i}{\hookrightarrow} X$ denote the inclusion of a closed subspace. If $\mathbb{C}_X$ and $\mathbb{C}_Y$ denote the constant sheaves with stalks $\mathbb{C}$ on X and Y, respectively, then $\mathbb{C}_X$ is a unit object of $\mathcal{M}$, and the natural isomorphism $i^*\mathbb{C}_X \xrightarrow{\simeq} \mathbb{C}_Y$ gives, by adjunction, a morphism $\mathbb{C}_X \longrightarrow i_*\mathbb{C}_Y$, which is easily seen to be a closed idempotent in the category $\mathcal{M}$. This example also explains the terminology.

In Section 5 we will introduce two other types of idempotents ("open" and "locally closed"); the definition will be such that an arrow $\mathbb{1} \xrightarrow{\pi} e$ can only denote a closed idempotent. In any case, from now on until the rest of this section, the word "idempotent" will mean "closed idempotent". Also, for the time being, we will only work with idempotents as arrows $\mathbb{1} \to e$ satisfying a certain property, as opposed to idempotents as objects of $\mathcal{M}$.

1.3. Before we proceed with a basic study of idempotents, we need to make an important technical remark. Note that the definition of an idempotent $\mathbb{1} \xrightarrow{\pi} e$, as well as of the subcategories $_{\pi}\mathcal{M}$ and $\mathcal{M}_{\pi}$, makes no use of the associativity constraint on $\mathcal{M}$. Likewise, most of our results (at least the ones in this section) do not involve the associativity constraint in their statements. On the other hand, their proofs, of course, make constant use of associativity, in one form or another. Fortunately, it is known that every monoidal category is equivalent to a *strictly associative* one (see Theorem 3.6 below). Thus, for the most part of these notes, we will only deal with strictly associative monoidal categories, with the understanding that all our results that do not involve associativity in their statements carry over automatically to the general case without any modifications.

On the other hand, we do not assume that the unit $\mathbb{1}$ is strict, i.e., we prefer to keep track of the left and right unit isomorphisms. There are two reasons for doing this. First, since the unit object plays such an important role in our definitions, keeping track of it as explicitly as possible will help us avoid needless confusion. Second, even if we start with a strictly associative *and strictly unital* monoidal category $\mathcal{M}$, certain "Hecke subcategories" of $\mathcal{M}$ (to be defined in Section 3) associated to idempotents of $\mathcal{M}$ will not be strictly unital in general. Thus our approach makes the situation look "more symmetric."

By contrast, keeping track of all the associativity isomorphisms would only make the proofs of our results much more complicated than the ones given below and obscure our understanding of the subject.

1.4. For future reference, we will now define precisely the type of categories we will be working with. This avoids all possible confusion as to "how strict" our monoidal structures are, and also makes this text essentially self-contained. Other basic definitions (monoidal functors, braidings, etc.) will be recalled when they are needed. As a general reference we have used Chapters VII and XI of [McL98].

Definition 1.2. A *strictly associative monoidal category* is a 5-tuple $\mathcal{M} = (\mathcal{M}, \otimes, \mathbb{1}, \lambda, \rho)$ consisting of a category $\mathcal{M}$, a bifunctor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ which is strictly associative in the obvious sense (both on objects and on morphisms), a distinguished object $\mathbb{1} \in \mathcal{M}$ (called the *unit object*), and two collections of isomorphisms,

$$\lambda_m : \mathbb{1} \otimes m \xrightarrow{\simeq} m \qquad \text{and} \qquad \rho_m : m \otimes \mathbb{1} \xrightarrow{\simeq} m,$$

for all objects $m \in \mathcal{M}$, both of which are functorial with respect to m, these data being subject to the following conditions:

(i) the two isomorphisms

$$\mathrm{id}_a \otimes \lambda_c : a \otimes \mathbb{1} \otimes c \xrightarrow{\simeq} a \otimes c \quad \text{and} \quad \rho_a \otimes \mathrm{id}_c : a \otimes \mathbb{1} \otimes c \xrightarrow{\simeq} a \otimes c$$

agree for all $a, c \in \mathcal{M}$, and

(ii) we have

$$\lambda_{\mathbb{1}} = \rho_{\mathbb{1}} : \mathbb{1} \otimes \mathbb{1} \xrightarrow{\simeq} \mathbb{1}.$$

These axioms imply also:

(iii) $\lambda_{b \otimes c} = \lambda_b \otimes \mathrm{id}_c : \mathbb{1} \otimes b \otimes c \xrightarrow{\simeq} b \otimes c$ for all $b, c \in \mathcal{M}$ and

(iv) $\rho_{a \otimes b} = \mathrm{id}_a \otimes \rho_b : a \otimes b \otimes \mathbb{1} \xrightarrow{\simeq} a \otimes b$ for all $a, b \in \mathcal{M}$.

We refer the reader to op. cit., §VII.1 for more details.

1.5. Until the end of this section we fix, once and for all, a strictly associative monoidal category $\mathcal{M}$. We begin our analysis of idempotents with two simple lemmas.

Lemma 1.3. *Let $\mathbb{1} \xrightarrow{\pi} e$ be an idempotent in $\mathcal{M}$. The following conditions are equivalent for an object m of $\mathcal{M}$:*

(*i*) *$m \in_\pi \mathcal{M}$ (resp., $m \in \mathcal{M}_\pi$);*

(*ii*) *$m \cong e \otimes m$ (resp., $m \cong m \otimes e$);*
(*iii*) *$m \cong e \otimes x$ for some $x \in \mathcal{M}$ (resp., $m \cong x \otimes e$ for some $x \in \mathcal{M}$).*

The proof is too trivial to write down, and the lemma will be used implicitly from now on without further mention. It shows that if π is an idempotent, then the subcategories ${}_\pi\mathcal{M}$ and $\mathcal{M}_\pi$ depend only on the object e and not on the arrow π, and hence the following notation, which we will use below, is unambiguous:

$$ {}_\pi\mathcal{M} =: e\mathcal{M}, \qquad \mathcal{M}_\pi =: \mathcal{M}e, \qquad {}_\pi\mathcal{M} \cap \mathcal{M}_\pi =: e\mathcal{M}e. $$

Lemma 1.4. *Let $\mathbb{1} \xrightarrow{\pi} e$ be an idempotent in $\mathcal{M}$, and let $x \xrightarrow{\alpha} y \xrightarrow{\beta} x$ be morphisms in $\mathcal{M}$ such that $\beta \circ \alpha = \mathrm{id}_x$. If $y \in e\mathcal{M}$ (resp., $y \in \mathcal{M}e$), then $x \in e\mathcal{M}$ (resp., $x \in \mathcal{M}e$).*

This lemma implies, for example, that if $\mathcal{M}$ is an additive monoidal category which is Karoubi complete, then the categories $e\mathcal{M}$, $\mathcal{M}e$ and $e\mathcal{M}e$ obtained from it are also Karoubi complete.

Proof. It is enough to consider the case where $y \in e\mathcal{M}$. We have a commutative diagram

$$\begin{tikzcd}
\mathbb{1} \otimes x \arrow[r, "\mathrm{id}_{\mathbb{1}} \otimes \alpha"] \arrow[d, "\pi \otimes \mathrm{id}_x"'] & \mathbb{1} \otimes y \arrow[r, "\mathrm{id}_{\mathbb{1}} \otimes \beta"] \arrow[d, "\pi \otimes \mathrm{id}_y"'] & \mathbb{1} \otimes x \arrow[d, "\pi \otimes \mathrm{id}_x"'] \\
e \otimes x \arrow[r, "\mathrm{id}_e \otimes \alpha"] & e \otimes y \arrow[r, "\mathrm{id}_e \otimes \beta"] & e \otimes x
\end{tikzcd}$$

This implies that the arrow $\pi \otimes \mathrm{id}_x$ is a retract of $\pi \otimes \mathrm{id}_y$ (in the obvious sense). It is easy to check that in an arbitrary category, the retract of an isomorphism is an isomorphism, which proves the lemma. □

1.6. We now begin proving a sequence of results on the uniqueness of certain morphisms between idempotents.

Lemma 1.5. *Let $\mathbb{1} \xrightarrow{\pi} e$ be an idempotent in $\mathcal{M}$, let $m \in \mathcal{M}$ be arbitrary, and let $x \in \mathcal{M}e$. If $f, g : m \otimes e \to x$ are two morphisms such that*

$$ f \circ (\mathrm{id}_m \otimes \pi) = g \circ (\mathrm{id}_m \otimes \pi) : m \otimes \mathbb{1} \to x, $$

then $f = g$.

Thus the arrow $\mathrm{id}_m \otimes \pi : m \otimes \mathbb{1} \longrightarrow m \otimes e$ enjoys a certain weakening of the property of being a categorical epimorphism. We leave it to the reader to formulate and prove the dual version of this lemma, where $x \in e\mathcal{M}$ and $\mathrm{id}_m \otimes \pi$ is replaced by $\pi \otimes \mathrm{id}_m$.

Proof. The assumption implies that

$$ (f \otimes \mathrm{id}_e) \circ (\mathrm{id}_m \otimes \pi \otimes \mathrm{id}_e) = (g \otimes \mathrm{id}_e) \circ (\mathrm{id}_m \otimes \pi \otimes \mathrm{id}_e) : m \otimes \mathbb{1} \otimes e \longrightarrow x \otimes e. $$

Since $\pi \otimes \mathrm{id}_e$, and hence also $\mathrm{id}_m \otimes \pi \otimes \mathrm{id}_e$, is an isomorphism, it follows that

$$f \otimes \mathrm{id}_e = g \otimes \mathrm{id}_e : m \otimes e \otimes e \longrightarrow x \otimes e.$$

On the other hand, since $x \in \mathcal{M}e$, the commutative diagram

$$\begin{tikzcd}
m \otimes e \otimes \mathbb{1} \arrow[r, "f \otimes \mathrm{id}_{\mathbb{1}}"] \arrow[d, "\mathrm{id}_m \otimes \mathrm{id}_e \otimes \pi"'] & x \otimes \mathbb{1} \arrow[d, "\mathrm{id}_x \otimes \pi"] \\
m \otimes e \otimes e \arrow[r, "f \otimes \mathrm{id}_e"] & x \otimes e
\end{tikzcd}$$

shows that $f \otimes \mathrm{id}_{\mathbb{1}}$, and hence f, is determined by $f \otimes \mathrm{id}_e$:

$$f \otimes \mathrm{id}_{\mathbb{1}} = (\mathrm{id}_x \otimes \pi)^{-1} \circ (f \otimes \mathrm{id}_e) \circ (\mathrm{id}_{m \otimes e} \otimes \pi).$$

Since the roles of f and g can be interchanged, it follows that $f = g$. □

Corollary 1.6. *If* $\mathbb{1} \xrightarrow{\pi} e$ *is an idempotent in* $\mathcal{M}$ *and* $f : e \to e$ *is a morphism such that* $f \circ \pi = \pi$, *then* $f = \mathrm{id}_e$. (*In particular, an idempotent has no nontrivial automorphisms.*)

Proof. Apply the lemma to the special case $m = \mathrm{id}_{\mathbb{1}}$ (identifying $\mathbb{1} \otimes e$ with e) and $x = e$, with f being the given morphism and $g = \mathrm{id}_e$. □

1.7. The following construction will be needed for proving several important results below. Consider two arrows, $\mathbb{1} \xrightarrow{\pi} e$ and $\mathbb{1} \xrightarrow{\pi'} e'$, in $\mathcal{M}$, which are not necessarily idempotents (just arbitrary morphisms). We define a new arrow, denoted $\mathbb{1} \xrightarrow{\pi\pi'} e \otimes e'$, as

$$\pi\pi' := (\pi \otimes \pi') \circ \lambda_{\mathbb{1}}^{-1} = (\pi \otimes \pi') \circ \rho_{\mathbb{1}}^{-1} \quad : \quad \mathbb{1} \longrightarrow \mathbb{1} \otimes \mathbb{1} \longrightarrow e \otimes e'.$$

The commutative diagram

$$\begin{tikzcd}
\mathbb{1} \otimes \mathbb{1} \arrow[rr, "\pi \otimes \pi'"] \arrow[dr, "\mathrm{id}_{\mathbb{1}} \otimes \pi'"] \arrow[dd, "\lambda_{\mathbb{1}}"'] & & e \otimes e' \\
& \mathbb{1} \otimes e' \arrow[ur, "\pi \otimes \mathrm{id}_{e'}"'] \arrow[dd, "\lambda_{e'}"] & \\
\mathbb{1} \arrow[dr, "\pi'"'] & & \\
& e' &
\end{tikzcd}$$

leads to a commutative diagram

$$\begin{tikzcd} \mathbb{1} \arrow[rr, "\pi\pi'"] \arrow[dr, "\pi'"'] & & e \otimes e' \\ & e' \arrow[ur, "(\pi\otimes \mathrm{id}_{e'})\circ\lambda_{e'}^{-1}"'] & \end{tikzcd} \tag{1.1}$$

By symmetry, we also have a commutative diagram

$$\begin{tikzcd} \mathbb{1} \arrow[rr, "\pi\pi'"] \arrow[dr, "\pi"'] & & e \otimes e' \\ & e \arrow[ur, "(\mathrm{id}_e \otimes\pi')\circ\rho_e^{-1}"'] & \end{tikzcd} \tag{1.2}$$

We now use this construction to prove a few other uniqueness results.

Lemma 1.7. *If $\mathbb{1} \xrightarrow{\pi} e$ is an idempotent in $\mathcal{M}$, then so is $\mathbb{1} \xrightarrow{\pi^2} e \otimes e$.*

Proof. We must show that

$$\mathbb{1} \otimes \mathbb{1} \otimes e \otimes e \xrightarrow{\pi\otimes\pi\otimes\mathrm{id}_{e\otimes e}} e \otimes e \otimes e \otimes e$$

and

$$e \otimes e \otimes \mathbb{1} \otimes \mathbb{1} \xrightarrow{\mathrm{id}_{e\otimes e} \otimes\pi\otimes\pi} e \otimes e \otimes e \otimes e$$

are isomorphisms. But

$$\pi \otimes \pi \otimes \mathrm{id}_{e\otimes e} = \big[(\pi \otimes \mathrm{id}_e) \otimes \mathrm{id}_e \otimes \mathrm{id}_e\big] \circ \big[\mathrm{id}_{\mathbb{1}} \otimes(\pi \otimes \mathrm{id}_e) \otimes \mathrm{id}_e\big]$$

and

$$\mathrm{id}_{e\otimes e} \otimes\pi \otimes \pi = \big[\mathrm{id}_e \otimes(\mathrm{id}_e \otimes\pi) \otimes \mathrm{id}_{\mathbb{1}}\big] \circ \big[\mathrm{id}_e \otimes \mathrm{id}_e \otimes(\mathrm{id}_e \otimes\pi)\big],$$

which proves the lemma. □

Proposition 1.8. *If $\mathbb{1} \xrightarrow{\pi} e$ is an idempotent in $\mathcal{M}$, then the following diagram commutes:*

$$\begin{tikzcd} & \mathbb{1} \otimes e \arrow[dr, "\pi\otimes\mathrm{id}_e"] & \\ e \arrow[ur, "\lambda_e^{-1}"] \arrow[dr, "\rho_e^{-1}"'] & & e \otimes e \\ & e \otimes \mathbb{1} \arrow[ur, "\mathrm{id}_e \otimes\pi"'] & \end{tikzcd} \tag{1.3}$$

Proof. We apply the construction above to $e' = e$, $\pi' = \pi$. By assumption, both $\pi \otimes \mathrm{id}_e$ and $\mathrm{id}_e \otimes\pi$ are isomorphisms; thus (1.1) and (1.2) together imply that

$$\big[(\mathrm{id}_e \otimes\pi) \circ \rho_e^{-1}\big]^{-1} \circ \big[(\pi \otimes \mathrm{id}_e) \circ \lambda_e^{-1}\big] \circ \pi = \pi.$$

In view of Corollary 1.6, this proves the proposition. □

1.8. We are now ready to prove that an idempotent $\mathbb{1} \xrightarrow{\pi} e$ is determined by its codomain up to unique isomorphism. We begin with

Proposition 1.9. *If $\mathbb{1} \xrightarrow{\pi} e$ is an idempotent in $\mathcal{M}$, then the map*

$$\pi^* : \mathrm{Hom}(e,e) \longrightarrow \mathrm{Hom}(\mathbb{1},e), \qquad f \longmapsto f \circ \pi,$$

is bijective.

Proof. Injectivity of π^* is immediate from Lemma 1.5. To prove that π^* is surjective, consider an arbitrary morphism $\pi' : \mathbb{1} \to e$. We apply the construction of §1.7 to $e' = e$ and the two morphisms π and π'. Even though the map $\mathrm{id}_e \otimes \pi'$ may not be an isomorphism, the map $\pi \otimes \mathrm{id}_e$ still is, so we obtain

$$\pi' = \left[(\pi \otimes \mathrm{id}_e) \circ \lambda_e^{-1}\right]^{-1} \circ \left[(\mathrm{id}_e \otimes \pi') \circ \rho_e^{-1}\right] \circ \pi,$$

proving the surjectivity of π^*. □

Corollary 1.10. *If $\mathbb{1} \xrightarrow{\pi} e$ and $\mathbb{1} \xrightarrow{\pi'} e$ are two idempotents in $\mathcal{M}$ with the same codomain e, then there exists a unique morphism $\sigma : e \to e$ such that $\pi' = \sigma \circ \pi$, and σ is an isomorphism.*

Proof. The first statement is a special case of Proposition 1.9. The second statement follows since the roles of π and π' can be interchanged. □

1.9. We conclude the section with a study of morphisms between idempotents with *different* codomains, culminating in a definition of a partial order on the set of isomorphism classes of idempotents. Let us consider a commutative diagram in $\mathcal{M}$,

$$\begin{tikzcd} e \arrow[rr, "f"] & & e' \\ & \mathbb{1} \arrow[ul, "\pi"] \arrow[ur, "\pi'"'] & \end{tikzcd} \tag{1.4}$$

Lemma 1.11. *If π and π' are idempotents, then $e' \in e\mathcal{M}e$.*

Proof. Tensoring the diagram (1.4) with e' on the left, we obtain

$$\begin{tikzcd} e' \otimes e \arrow[rr, "\mathrm{id}_{e'} \otimes f"] & & e' \otimes e' \\ & e' \otimes \mathbb{1} \arrow[ul, "\mathrm{id}_{e'} \otimes \pi"] \arrow[ur, "\mathrm{id}_{e'} \otimes \pi'"'] & \end{tikzcd}$$

Since $\mathrm{id}_{e'} \otimes \pi'$ is an isomorphism, we can define morphisms

$$\alpha = (\mathrm{id}_{e'} \otimes \pi) \circ \rho_{e'}^{-1} \ : \ e' \longrightarrow e' \otimes \mathbb{1} \longrightarrow e' \otimes e$$

and

$$\beta = \rho_{e'} \circ (\mathrm{id}_{e'} \otimes \pi')^{-1} \circ (\mathrm{id}'_e \otimes f) \ : \ e' \otimes e \longrightarrow e' \otimes e' \longrightarrow e' \otimes \mathbb{1} \longrightarrow e'$$

with the property that $\beta \circ \alpha = \mathrm{id}_{e'}$. Since $e' \otimes e \in \mathcal{M}e$, Lemma 1.4 implies that $e' \in \mathcal{M}e$. By symmetry, we also obtain $e' \in e\mathcal{M}$, which proves the desired result. ☐

Proposition 1.12. *In the same situation, if π and π' are idempotents and* (1.4) *commutes, then f is the only morphism $e \to e'$ making that diagram commute. Furthermore, the map*

$$f^* : \mathrm{Hom}(e', e') \longrightarrow \mathrm{Hom}(e, e')$$

is bijective.

Proof. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(e', e') \arrow[rr, "f^*"] \arrow[dr, "\pi'^*"'] & & \mathrm{Hom}(e, e') \arrow[dl, "\pi^*"] \\
& \mathrm{Hom}(\mathbb{1}, e') &
\end{tikzcd}$$

In this diagram, the map π'^* is bijective by Proposition 1.9, and the map π^* is injective by Lemmas 1.11 and 1.5. This implies that both f^* and π^* are bijective. The first statement of the proposition also follows from the injectivity of π^*. ☐

One of the main results of this section is the following

Theorem 1.13. *Let e and e' be closed idempotents in $\mathcal{M}$. The following conditions are all equivalent to one another:*

- (a) *$e' \in e\mathcal{M}e$;*
- (a_1) *$e' \in e\mathcal{M}$;*
- (a_2) *$e' \in \mathcal{M}e$;*
- (b) *there exists a morphism $f : e \to e'$ such that $\mathrm{id}_{e'} \otimes f : e' \otimes e \longrightarrow e' \otimes e'$ and $f \otimes \mathrm{id}_{e'} : e \otimes e' \longrightarrow e' \otimes e'$ are isomorphisms;*
- (b_1) *there exists a morphism $f : e \to e'$ such that $\mathrm{id}_{e'} \otimes f : e' \otimes e \longrightarrow e' \otimes e'$ is an isomorphism;*
- (b_2) *there exists a morphism $f : e \to e'$ such that $f \otimes \mathrm{id}_{e'} : e \otimes e' \longrightarrow e' \otimes e'$ is an isomorphism;*
- (c) *for every choice of closed idempotents $\mathbb{1} \xrightarrow{\pi} e$ and $\mathbb{1} \xrightarrow{\pi'} e'$, there exists a morphism $f : e \to e'$ with $\pi' = f \circ \pi$;*
- (d) *there exist closed idempotents $\mathbb{1} \xrightarrow{\pi} e$, $\mathbb{1} \xrightarrow{\pi'} e'$ and a morphism $f : e \to e'$ such that $\pi' = f \circ \pi$.*

Proof. It is clear that (a) implies (a_1) and (a_2). We claim that either (a_1) or (a_2) implies (b). If either (a_1) or (a_2) holds, then either $e \otimes e' \cong e'$ or $e' \otimes e \cong e'$, whence $e' \otimes e \otimes e' \cong e'$

(because e' is also an idempotent). Choose an isomorphism $\gamma : e' \otimes e \otimes e' \xrightarrow{\simeq} e'$. It is then clear that the composition

$$f \ : \ e \xrightarrow{\simeq} \mathbb{1} \otimes e \otimes \mathbb{1} \xrightarrow{\pi' \otimes \mathrm{id}_e \otimes \pi'} e' \otimes e \otimes e' \xrightarrow{\gamma} e'$$

satisfies the condition of (b). Again, (b) trivially implies both (b_1) and (b_2).

Suppose that (b_1) holds, and let $f : e \to e'$ be a morphism such that $\mathrm{id}_{e'} \otimes f : e' \otimes e \longrightarrow e' \otimes e'$ is an isomorphism. Consider the morphism $f \circ \pi : \mathbb{1} \to e'$. By Proposition 1.9, there exists a (unique) morphism $\alpha : e' \to e'$ with $f \circ \pi = \alpha \circ \pi'$. Moreover, since $e' \otimes e \cong e' \otimes e' \cong e'$, we see that $e' \in \mathcal{M}e$; thus $\mathrm{id}_{e'} \otimes \pi : e' \otimes \mathbb{1} \longrightarrow e' \otimes e$ is an isomorphism. But now

$$(\mathrm{id}_{e'} \otimes f) \circ (\mathrm{id}_{e'} \otimes \pi) = (\mathrm{id}_{e'} \otimes \alpha) \circ (\mathrm{id}_{e'} \otimes \pi')$$

implies that $\mathrm{id}_{e'} \otimes \alpha : e' \otimes e' \longrightarrow e' \otimes e'$ is an isomorphism. The usual argument now implies that α is an isomorphism, since

$$(\pi' \otimes \mathrm{id}_{e'}) \circ (\mathrm{id}_{\mathbb{1}} \otimes \alpha) = (\mathrm{id}_{e'} \otimes \alpha) \circ (\pi' \otimes \mathrm{id}_{e'}).$$

So we see that $\pi' = \alpha^{-1} \circ f \circ \pi$. Thus we have proved that (b_1) implies (c). By symmetry, (b_2) also implies (c). Finally, the fact that (c) implies (d) is a tautology, and the fact that (d) implies (a) is the content of Lemma 1.11. □

1.10. **Partial order.** If $\mathbb{1} \xrightarrow{\pi} e$ and $\mathbb{1} \xrightarrow{\pi'} e'$ are idempotents in a monoidal category $\mathcal{M}$, we will write $e' \leq e$, or $\pi' \leq \pi$, or $(e', \pi') \leq (e, \pi)$, whenever the equivalent conditions of Theorem 1.13 hold. Note that by the theorem, the condition $e' \leq e$ depends only on e and e' and not on the choice of π and π'; thus we obtain a well defined relation on the set of isomorphism classes of closed idempotents in $\mathcal{M}$.

Proposition 1.14. *This relation is a partial order, i.e., if e, e' and e'' are idempotents in $\mathcal{M}$, we have: $e \leq e$; if $e' \leq e$ and $e \leq e'$, then $e \cong e'$; and if $e'' \leq e'$ and $e' \leq e$, then $e'' \leq e$.*

This proposition follows trivially from our previous results.

1.11. **Another rigidity result.** The following result can be thought of as a converse to Corollary 1.10.

Proposition 1.15. *Let $\mathbb{1} \xrightarrow{\pi} e$ be a closed idempotent in $\mathcal{M}$. If there exist morphisms $\pi' : \mathbb{1} \longrightarrow e$ and $f : e \to e$ in $\mathcal{M}$ satisfying $\pi = f \circ \pi'$, then f is an isomorphism (and hence π' is also a closed idempotent in $\mathcal{M}$).*

Proof. By Proposition 1.9, there exists a (unique) morphism $g : e \longrightarrow e$ such that $\pi' = g \circ \pi$. In particular, $f \circ g \circ \pi = \pi$, whence $f \circ g = \mathrm{id}_e$ by Corollary 1.6. It remains to show that $g \circ f = \mathrm{id}_e$. Now by Proposition 3.1 below, f and g are morphisms in the monoidal category $e\mathcal{M}e$, and e is a unit object of $e\mathcal{M}e$, so our result follows from the observation that the monoid of endomorphisms of the unit object in any monoidal category is commutative. Indeed, if $\mathcal{N}$ is a monoidal category with unit object $\mathbb{1}_{\mathcal{N}}$, the functor $X \mapsto \mathbb{1}_{\mathcal{N}} \otimes X$ is isomorphic to the identity functor $\mathrm{Id}_{\mathcal{N}}$, whence $\mathrm{End}(\mathbb{1}_{\mathcal{N}})$ embeds as a submonoid of $\mathrm{End}(\mathrm{Id}_{\mathcal{N}})$, the latter being the Bernstein center of $\mathcal{N}$, which is well known to be commutative. □

2. Idempotents and idempotent monads

2.1. In this section, unlike the previous one, the associativity constraints will play a role even in the statements of our results; thus we prefer to work with general monoidal categories. The precise definition is given in [McL98], §VII.1, but we remark that Definition 1.2 has to be modified as follows. First, we do not require $\otimes$ to be strictly associative; instead, we need to specify the extra data of a collection α of trifunctorial isomorphisms

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\simeq} X \otimes (Y \otimes Z)$$

for all objects X,Y,Z of $\mathcal{M}$, called the *associativity constraint*, satisfying the "pentagon axiom," stating that for any four objects X, Y, Z, W of $\mathcal{M}$, the diagram formed by the five possible objects of $\mathcal{M}$ obtained by grammatically correct insertions of parentheses in the expression $X \otimes Y \otimes Z \otimes W$, and by the five isomorphisms between these objects arising from the associativity constraint, is a commutative pentagon. The only other change we need to make is that in condition (i) and in properties (iii) and (iv) of Definition 1.2, the equality of two possible isomorphisms is replaced by the commutativity of the triangle formed by these isomorphisms and the appropriate associativity constraint.

As remarked before, the definition of a closed idempotent, as well as all the results of Section 1, remain unchanged in this setup.

2.2. Let us recall that a *monoid* in a monoidal category $\mathcal{M}$ is a triple (G, μ, η) consisting of an object G of $\mathcal{M}$, a morphism $\mu : G \otimes G \to G$, and a morphism $\eta : \mathbb{1} \to G$ such that the following diagrams commute:

$$\begin{tikzcd}
(G \otimes G) \otimes G \arrow[rr, "\alpha_{G,G,G}"] \arrow[d, "\mu\otimes \mathrm{id}_G"'] & & G \otimes (G \otimes G) \arrow[d, "\mathrm{id}_G \otimes \mu"] \\
G \otimes G \arrow[dr, "\mu"] & & G \otimes G \arrow[dl, "\mu"'] \\
& G &
\end{tikzcd} \tag{2.1}$$

(this states that μ is associative, i.e., makes G into a "semigroup"), and

$$\begin{tikzcd}
& G \otimes G \arrow[dd, "\mu"] & \\
\mathbb{1} \otimes G \arrow[ur, "\eta\otimes \mathrm{id}_G"] \arrow[dr, "\lambda_G"'] & & G \otimes \mathbb{1} \arrow[ul, "\mathrm{id}_G \otimes \eta"'] \arrow[dl, "\rho_G"] \\
& G &
\end{tikzcd} \tag{2.2}$$

(this states that η is both a left and a right unit for the multiplication μ).

Definition 2.1. An *idempotent monoid* in a monoidal category $\mathcal{M}$ is a monoid (G, μ, η) in $\mathcal{M}$ such that μ is an isomorphism.

If (G, μ, η) is an idempotent monoid in $\mathcal{M}$, it is clear that $\mathbb{1} \xrightarrow{\eta} G$ is a closed idempotent in $\mathcal{M}$, since the commutativity of (2.2) implies that both $\eta \otimes \mathrm{id}_G$ and $\mathrm{id}_G \otimes \eta$ are isomorphisms. The converse is slightly less obvious, but is also true; more precisely:

Lemma 2.2. *If $\mathbb{1} \xrightarrow{\pi} e$ is a closed idempotent in $\mathcal{M}$, then there exists a unique morphism $e \otimes e \xrightarrow{\mu} e$ such that (e, μ, π) is a monoid; moreover, μ is an isomorphism.*

Proof. The commutativity of (1.3) and the fact that $\mathrm{id}_e \otimes \pi$ and $\pi \otimes \mathrm{id}_e$ are isomorphisms implies that there is a unique arrow $e \otimes e \xrightarrow{\mu} e$ such that the diagram (2.2), with G replaced by e, commutes; and, moreover, this μ is an isomorphism. Thus we only need to check that μ is associative. This follows from the commutativity of the diagram

$$\begin{tikzcd}
(e \otimes e) \otimes e \arrow[rr, "\alpha_{e,e,e}"] \arrow[d, "(\mathrm{id}_e \otimes \pi)^{-1} \otimes \mathrm{id}_e"'] & & e \otimes (e \otimes e) \arrow[d, "\mathrm{id}_e \otimes (\pi \otimes \mathrm{id}_e)^{-1}"] \\
(e \otimes \mathbb{1}) \otimes e \arrow[rr, "\alpha_{e,\mathbb{1},e}"] \arrow[d, "\rho_e \otimes \mathrm{id}_e"'] & & e \otimes (\mathbb{1} \otimes e) \arrow[d, "\mathrm{id}_e \otimes \lambda_e"] \\
e \otimes e \arrow[rr, equal] \arrow[dr, "\mu"'] & & e \otimes e \arrow[dl, "\mu"] \\
& e &
\end{tikzcd}$$

Indeed, the commutativity of the top square follows from the naturality of α, the commutativity of the middle square is one of the axioms of a monoidal category, and the commutativity of the lower triangle is obvious. Since $\mu = \rho_e \circ (\mathrm{id}_e \otimes \pi)^{-1} = \lambda_e \circ (\pi \otimes \mathrm{id}_e)^{-1}$ by construction, we see that μ is associative. $\square$

2.3. We can now relate the theory of idempotents developed above to the theory of idempotent functors and idempotent monads, studied in the works of Adams [Ad73] and Deleanu-Frei-Hilton [DFH1, DFH2].

Let us first recall that setup considered by Adams. Let $\mathcal{C}$ be a category. Adams defines an *idempotent functor* on $\mathcal{C}$ as a pair (E, η) consisting of a functor $E : \mathcal{C} \longrightarrow \mathcal{C}$ and a natural transformation $\eta : \mathrm{Id}_{\mathcal{C}} \longrightarrow E$ satisfying two axioms:

Axiom 1. For every object X of $\mathcal{C}$, we have

$$\eta_{E(X)} = E(\eta_X) : E(X) \longrightarrow E^2(X).$$

Axiom 2. For every $X \in \mathcal{C}$, the map $\eta_{E(X)} : E(X) \longrightarrow E^2(X)$ is an isomorphism.

We observe that the category $\mathrm{Funct}(\mathcal{C}, \mathcal{C})$ of all functors $\mathcal{C} \to \mathcal{C}$ is in fact a strictly associative and strictly unital monoidal category, the monoidal functor being given by composition of endofunctors. Conversely, every monoidal category acts on itself via tensoring on the left, which realizes $\mathcal{M}$ as a monoidal subcategory of $\mathrm{Funct}(\mathcal{M}, \mathcal{M})$. [In fact, $\mathcal{M}$ is equivalent to the category of endofunctors of $\mathcal{M}$ which "commute with the right action of $\mathcal{M}$" in the appropriate sense; the details are explained in §3.3.] Thus the language of endofunctors is more or less equivalent to the language we have been using in the previous section.

In particular, an idempotent functor on a category $\mathcal{C}$ in the sense of Adams is the same as a closed idempotent in the monoidal category $\mathrm{Funct}(\mathcal{C}, \mathcal{C})$ in the sense of Definition 1.1; this follows from Proposition 1.8. Moreover, in view of Lemma 2.2, we see that an idempotent functor on $\mathcal{C}$ is more or less the same thing as an *idempotent monad* on $\mathcal{C}$ (which, by definition, is just an idempotent monoid in the category $\mathrm{Funct}(\mathcal{C}, \mathcal{C})$)—a fact which was asserted without proof by Adams [Ad73].

Remark 2.3. From the theory developed in the previous section (particularly Proposition 1.8) it follows that in the definition of an idempotent functor (E, η) it suffices to require the a priori weaker condition that both $\eta_{E(X)} : E(X) \longrightarrow E^2(X)$ and $E(\eta_X) : E(X) \longrightarrow E^2(X)$ are isomorphisms for every object X of $\mathcal{C}$; the equality $\eta_{E(X)} = E(\eta_X)$ is then automatic. For the reader's convenience we now give a different (and independent) proof of the same fact, namely, one which is written purely within the framework of functors, objects and morphisms and does not involve the language of monoidal categories.

Suppose that $\eta_{E(X)}$ and $E(\eta_X)$ are isomorphisms for every $X \in \mathcal{C}$, and fix $X \in \mathcal{C}$. The diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "\eta_X"'] & E(X) \arrow[d, "\eta_{E(X)}"] \\ E(X) \arrow[r, "E(\eta_X)"'] & E^2(X) \end{tikzcd}$$

commutes because η is a morphism of functors. Applying E to this diagram and remembering that $E(\eta_X)$ is an isomorphism, we obtain that

$$E^2(\eta_X) = E(\eta_{E(X)}) : E^2(X) \longrightarrow E^3(X).$$

The diagram

$$\begin{tikzcd} E(X) \arrow[r, "\eta_{E(X)}"] \arrow[d, "\eta_{E(X)}"'] & E^2(X) \arrow[d, "\eta_{E^2(X)}"] \\ E^2(X) \arrow[r, "E(\eta_{E(X)})"'] & E^3(X) \end{tikzcd}$$

commutes for the same reason as the one above; since $\eta_{E(X)}$ is an isomorphism, we obtain that

$$E(\eta_{E(X)}) = \eta_{E^2(X)} : E^2(X) \longrightarrow E^3(X).$$

Finally, the diagram

$$\begin{tikzcd} E(X) \arrow[r, "E(\eta_X)"] \arrow[d, "\eta_{E(X)}"'] & E^2(X) \arrow[d, "\eta_{E^2(X)}"] \\ E^2(X) \arrow[r, "E^2(\eta_X)"'] & E^3(X) \end{tikzcd}$$

also commutes for the same reason. But $\eta_{E^2(X)} = E^2(\eta_X)$ by the previous two observations. Thus $E(\eta_X) = \eta_{E(X)}$, as desired.

Remark 2.4. On the other hand, it is *not* enough to require only one of the morphisms $\eta_{E(X)}$ and $E(\eta_X)$ to be an isomorphism. As an example, let $\mathcal{C}$ be a category with one object. Then the data of $\mathcal{C}$ is equivalent to the data of a monoid G (in the category of sets). Moreover, a functor $E : \mathcal{C} \to \mathcal{C}$ is the same as a homomorphism $e : G \to G$ of monoids, and a natural transformation $\eta : \mathrm{Id}_{\mathcal{C}} \to E$ is the same as an element $\eta \in G$ with the property

$$e(g) \cdot \eta = \eta \cdot g \qquad \forall\, g \in G.$$

However, it is trivial to find an example where these conditions are satisfied and where $e(\eta)$ is invertible, but η is not. For instance, we can take e to be the constant map, defined by $e(g) = 1$ for all $g \in G$. Then the condition above reduces to $\eta \cdot g = \eta$. Now we can take G to be the monoid of all maps of sets $\{0,1\} \to \{0,1\}$ (with respect to the operation of composition), and η to be the constant map $0 \mapsto 0$, $1 \mapsto 0$.

3. Hecke subcategories

3.1. Let $\mathcal{M} = (\mathcal{M}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category, and let $\mathbb{1} \xrightarrow{\pi} e$ be a closed idempotent in $\mathcal{M}$. In Section 1 we have introduced the full subcategory $e\mathcal{M}e \subseteq \mathcal{M}$ consisting of all objects X of $\mathcal{M}$ such that $X \cong e \otimes X \otimes e$. Recall that a *strictly full subcategory* of a category $\mathcal{C}$ is a full subcategory $\mathcal{D} \subseteq \mathcal{C}$ such that if $X \in \mathcal{C}$ is isomorphic to an object of $\mathcal{D}$, then $X \in \mathcal{D}$. It is clear that $e\mathcal{M}e$ is a strictly full subcategory of $\mathcal{M}$ which is closed under the tensor product. By abuse of notation, we will denote by $\otimes$ the induced bifunctor on $e\mathcal{M}e$, and by α the induced associativity constraint on $e\mathcal{M}e$. Moreover, for every object X of $e\mathcal{M}e$ we define isomorphisms

$$\lambda_X^\pi = \lambda_X \circ (\pi \otimes \mathrm{id}_X)^{-1} \;:\; e \otimes X \xrightarrow{\simeq} \mathbb{1} \otimes X \xrightarrow{\simeq} X$$

and

$$\rho_X^\pi = \rho_X \circ (\mathrm{id}_X \otimes \pi)^{-1} \;:\; X \otimes e \xrightarrow{\simeq} X \otimes \mathbb{1} \xrightarrow{\simeq} X.$$

Proposition 3.1. *The category* $e\mathcal{M}e = \big(e\mathcal{M}e, \otimes, e, \alpha, \lambda^\pi, \rho^\pi\big)$ *is monoidal.*

Proof. The fact that the induced associativity constraint α on $e\mathcal{M}e$ satisfies the pentagon axiom is automatic. Thus we need to verify the following two conditions:

(i) for all $a, c \in e\mathcal{M}e$, the diagram

$$\begin{tikzcd}[column sep=small] (a \otimes e) \otimes c \arrow[rr, "\alpha_{a,e,c}"] \arrow[dr, "\rho_a^\pi \otimes \mathrm{id}_c"'] & & a \otimes (e \otimes c) \arrow[dl, "\mathrm{id}_a \otimes \lambda_c^\pi"] \\ & a \otimes c & \end{tikzcd} \tag{3.1}$$

commutes, and

(ii) we have $\lambda_e^\pi = \rho_e^\pi : e \otimes e \longrightarrow e$.

Now property (ii) is the content of Proposition 1.8. To prove (i), we expand the diagram (3.1) into

$$\begin{tikzcd}[column sep=small] (a \otimes e) \otimes c \arrow[rr, "\alpha_{a,e,c}"] \arrow[d, "(\mathrm{id}_a \otimes \pi)^{-1} \otimes \mathrm{id}_c"'] & & a \otimes (e \otimes c) \arrow[d, "\mathrm{id}_a \otimes (\pi \otimes \mathrm{id}_c)^{-1}"] \\ (a \otimes \mathbb{1}) \otimes c \arrow[rr, "\alpha_{a,\mathbb{1},c}"] \arrow[dr, "\rho_a \otimes \mathrm{id}_c"'] & & a \otimes (\mathbb{1} \otimes c) \arrow[dl, "\mathrm{id}_a \otimes \lambda_c"] \\ & a \otimes c & \end{tikzcd}$$

The commutativity of the top square follows from the naturality of α, and the commutativity of the bottom triangle is one of the axioms for the monoidal category $\mathcal{M}$. □

Remark 3.2. In view of Corollary 1.10, the category $e\mathcal{M}e$ depends only on the idempotent e and not on the arrow π, up to equivalence.

We call $e\mathcal{M}e$ the *Hecke subcategory* of $\mathcal{M}$ corresponding to the idempotent e, by analogy with the term *Hecke subalgebra*, used for the subring eAe in a ring A associated to an idempotent e.

3.2. **An example.** In this subsection we show that a certain naive version of Proposition 3.1 is false. Namely, there exists a monoidal category $\mathcal{M}$ and an object $e \in \mathcal{M}$, which is a *weak idempotent* in the sense that $e \otimes e \cong e$, such that the subcategory $e\mathcal{M}e \subset \mathcal{M}$ is not monoidal with respect to the induced convolution functor.

In other words, even though we have $e \otimes X \cong X \cong X \otimes e$ for all $X \in e\mathcal{M}e$, the functor $X \mapsto e \otimes X$ is *not* an autoequivalence of the category $e\mathcal{M}e$.

Example 3.3. Let $\mathbb{A}^1$ denote the affine line over an algebraically closed field k, and let $\mathcal{M}$ be the bounded derived category $D^b_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$ of constructible ℓ-adic complexes on $\mathbb{A}^1$, equipped with the usual tensor product. Here ℓ is a prime invertible in k and $\overline{\mathbb{Q}}_\ell$ is a fixed algebraic closure of the field $\mathbb{Q}_\ell$ of ℓ-adic numbers.

Put $e = j_!\overline{\mathbb{Q}}_\ell \oplus i_!\overline{\mathbb{Q}}_\ell$, where $j : \mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathbb{A}^1$ is the open embedding and $i : \{0\} \hookrightarrow \mathbb{A}^1$ is the corresponding closed embedding. Then $e \otimes e \cong e$, but the functor $X \mapsto e \otimes X$ is not an autoequivalence of the category $e\mathcal{M}e = e\mathcal{M}$, which follows from the lemma below

Lemma 3.4. *Let $\mathcal{M}$ be a triangulated monoidal category, let $e_1, e_2 \in \mathcal{M}$ be weak idempotents such that $e_1 \otimes e_2 = 0 = e_2 \otimes e_1$, and suppose that there exists a nonzero morphism $e_1 \to e_2[k]$ for some $k \in \mathbb{Z}$. If $e = e_1 \oplus e_2$, then e is a weak idempotent with the property that the functor $X \mapsto e \otimes X$ on the category $e\mathcal{M}e$ is not faithful.*

Proof. Since $e_1 \otimes e_2 = 0 = e_2 \otimes e_1$, and since e_1 and e_2 are weak idempotents, it follows that e is also a weak idempotent, and the objects e_1 and $e_2[k]$ lie in $e\mathcal{M}e$. Any nonzero morphism $e_1 \to e_2[k]$ is annihilated by each of the functors $X \mapsto e_1 \otimes X$ and $X \mapsto e_2 \otimes X$, whence it is also annihilated by the functor $X \mapsto e \otimes X$. $\square$

Note that in the example above, the objects $e_1 = i_!\overline{\mathbb{Q}}_\ell$ and $e_2 = j_!\overline{\mathbb{Q}}_\ell$ are weak idempotents in $\mathcal{M}$, and we have (writing $\mathbb{D}$ for the Verdier duality functor)

$$\begin{aligned} \operatorname{Hom}(e_1, e_2) &= \operatorname{Hom}(i_!\overline{\mathbb{Q}}_\ell, j_!\overline{\mathbb{Q}}_\ell) \cong \operatorname{Hom}(\mathbb{D}i_*\mathbb{D}\overline{\mathbb{Q}}_\ell, \mathbb{D}j_*\mathbb{D}\overline{\mathbb{Q}}_\ell) \\ &\cong \operatorname{Hom}(j_*\mathbb{D}\overline{\mathbb{Q}}_\ell, i_*\mathbb{D}\overline{\mathbb{Q}}_\ell) \cong \operatorname{Hom}(j_*\overline{\mathbb{Q}}_\ell[2](1), i_*\overline{\mathbb{Q}}_\ell) \\ &\cong \operatorname{Hom}(i^*j_*\overline{\mathbb{Q}}_\ell[2](1), \overline{\mathbb{Q}}_\ell) \cong \overline{\mathbb{Q}}_\ell[-2](-1). \end{aligned}$$

In our situation, Tate twists can be (noncanonically) trivialized, so we see that there exists a nonzero morphism $e_1 \to e_2[2]$ in $\mathcal{M}$, whence the lemma can be applied.

3.3. We now review the construction of a strictly associative and strictly unital monoidal category equivalent to a given monoidal category, due to J. Bernstein; it was explained to us by D. Kazhdan. For future purposes, we slightly generalize our definition as follows. Consider a monoidal category $\mathcal{M} = (\mathcal{M}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ and a strictly full subcategory $\mathcal{M}' \subseteq \mathcal{M}$ with the property that $\mathcal{M}'\mathcal{M} \subseteq \mathcal{M}'$, i.e., if $X \in \mathcal{M}'$ and $Y \in \mathcal{M}$, then $X \otimes Y \in \mathcal{M}'$. We will call such an $\mathcal{M}'$ a *right tensor ideal* in $\mathcal{M}$.

We define a category $\mathcal{E}nd_{\mathcal{M}}(\mathcal{M}')$ as follows. Its objects are pairs (F, R) consisting of a functor $F : \mathcal{M}' \longrightarrow \mathcal{M}'$ and a bifunctorial collection of isomorphisms

$$R_{X,Y} : F(X) \otimes Y \xrightarrow{\simeq} F(X \otimes Y) \qquad \forall\, X \in \mathcal{M}',\ Y \in \mathcal{M}$$

satisfying an analogue of the pentagon axiom, namely, the diagram

$$\begin{tikzcd}
(F(X)\otimes Y)\otimes Z \arrow[rr, "\alpha"] \arrow[d, "R_{X,Y}\otimes \mathrm{id}_Z"'] & & F(X)\otimes (Y\otimes Z) \arrow[d, "R_{X,Y\otimes Z}"] \\
F(X\otimes Y)\otimes Z \arrow[dr, "R_{X\otimes Y,Z}"'] & & F(X\otimes (Y\otimes Z)) \\
& F((X\otimes Y)\otimes Z) \arrow[ur, "F(\alpha)"'] &
\end{tikzcd}$$

commutes. Morphisms $(F,R)\to(F',R')$ are defined as natural transformations $F\to F'$ that are compatible with R and R' in the obvious sense. Moreover, $\mathcal{E}nd_{\mathcal{M}}(\mathcal{M}')$ has a strictly associative and strictly unital monoidal structure given by composition of functors; more precisely, $(F,R)\otimes(F',R')=(F\circ F',RR')$, where, by definition,

$$(RR')_{X,Y}=F(R'_{X,Y})\circ R_{F'(X),Y} \;:\; FF'(X)\otimes Y \xrightarrow{\simeq} F(F'(X)\otimes Y) \xrightarrow{\simeq} FF'(X\otimes Y)$$

Remark 3.5. One can generalize the setup above by defining the category of functors between two right module categories over $\mathcal{M}$. However, we do not need the formalism of module categories in these notes, and thus we will not explain this generalization.

Bernstein's observation can be stated as follows:

Theorem 3.6. *The category $\mathcal{E}nd_{\mathcal{M}}(\mathcal{M})$ is equivalent to $\mathcal{M}$ as a monoidal category.*

This result is a special case of Theorem 3.7 proved below.

3.4. In order to proceed we need the notion of a monoidal functor between two monoidal categories. We review the definition following [McL98], §XI.2. If $\mathcal{M}=(\mathcal{M},\otimes,\mathbb{1},\alpha,\lambda,\rho)$ and $\mathcal{M}'=(\mathcal{M}',\otimes',\mathbb{1}',\alpha',\lambda',\rho')$ are monoidal categories, a (*strong*) *monoidal functor* $F:\mathcal{M}\longrightarrow\mathcal{M}'$ consists of the following data:

(i) a functor $F:\mathcal{M}\longrightarrow\mathcal{M}'$ between the underlying categories;
(ii) a collection of bifunctorial isomorphisms
$$F_2(a,b):F(a)\otimes' F(b)\xrightarrow{\simeq}F(a\otimes b)$$
for all objects a,b of $\mathcal{M}$;
(iii) an isomorphism
$$F_0:\mathbb{1}'\xrightarrow{\simeq}F(\mathbb{1}).$$

These data are required to satisfy the obvious conditions of compatibility with the associativity constraints and the left and right unit isomorphisms (loc. cit.). The adjective "strong" refers to the fact that in the definition of a plain monoidal functor one only requires F_0 and all the $F_2(a,b)$ to be morphisms in $\mathcal{M}'$ (as opposed to isomorphisms). However, in this paper we only use the notion of a strong monoidal functor, and thus the adjective "strong" will be omitted from now on.

The notions of a morphism of monoidal functors and the composition of monoidal functors are defined in the obvious way. A useful fact, which can be checked trivially using the definition above, is that if $F : \mathcal{M} \longrightarrow \mathcal{M}'$ is a monoidal functor which is an equivalence of the underlying categories, then it is automatically an equivalence of monoidal categories, in the sense that if $G : \mathcal{M}' \longrightarrow \mathcal{M}$ is quasi-inverse to F, then G can be equipped with the structure of a monoidal functor such that $F \circ G$ and $G \circ F$ are isomorphic to the identity functors of $\mathcal{M}'$ and $\mathcal{M}$, respectively, as monoidal functors.

3.5. Let $\mathcal{M}$ be a monoidal category and $\mathbb{1} \xrightarrow{\pi} e$ a closed idempotent in $\mathcal{M}$. It is clear that $e\mathcal{M}$ is a right tensor ideal in $\mathcal{M}$. Moreover, there is a natural monoidal functor

$$\Phi \ : \ e\mathcal{M}e \longrightarrow \mathcal{E}nd_{\mathcal{M}}(e\mathcal{M}) \tag{3.2}$$

taking every object m of $e\mathcal{M}e$ to the pair $(F^m, R^m) = (m \otimes \cdot, \alpha_{m,\cdot,\cdot})$, i.e.,

$$F^m(X) = m \otimes X, \qquad R^m_{X,Y} = \alpha_{m,X,Y} \ : \ (m \otimes X) \otimes Y \xrightarrow{\simeq} m \otimes (X \otimes Y).$$

The monoidal structure on Φ is defined by $\Phi_2(a,b) = \alpha^{-1}_{a,b,\cdot}$ and $\Phi_0 = (\lambda^\pi_\cdot)^{-1}$, i.e.,

$$\Phi_2(a,b) : a \otimes (b \otimes X) \xrightarrow{\alpha^{-1}_{a,b,X}} (a \otimes b) \otimes X \quad \text{and} \quad \Phi_0 : X \xrightarrow{(\lambda^\pi_X)^{-1}} e \otimes X.$$

Theorem 3.7. *With the notation and assumptions as above, the monoidal functor* (Φ, Φ_2, Φ_0) *is an equivalence of monoidal categories.*

In the special case $(e, \pi) = (\mathbb{1}, \mathrm{id}_{\mathbb{1}})$, we recover Theorem 3.6.

Proof. As remarked above, it is enough to show that Φ is an equivalence of the underlying categories. We construct an explicit quasi-inverse of Φ. Let us define a functor

$$\Psi \ : \ \mathcal{E}nd_{\mathcal{M}}(e\mathcal{M}) \longrightarrow e\mathcal{M}e$$

by $\Psi(F,R) = F(e)$. The functorial collection of isomorphisms

$$\rho^\pi_m : m \otimes e \xrightarrow{\simeq} m, \qquad m \in e\mathcal{M}e,$$

constructed above yields an isomorphism of functors

$$\rho^\pi : \Psi \circ \Phi \xrightarrow{\simeq} \mathrm{Id}_{e\mathcal{M}e}\,.$$

It remains to construct an isomorphism

$$\Phi \circ \Psi \xrightarrow{\simeq} \mathrm{Id}_{\mathcal{E}nd_{\mathcal{M}}(e\mathcal{M})}\,.$$

For each pair $(F,R) \in \mathcal{E}nd_{\mathcal{M}}(e\mathcal{M})$ we have, by construction,

$$\Phi(\Psi(F,R)) = \big(F^{F(e)}, R^{F(e)}\big),$$

and we need to produce a functorial isomorphism between (F,R) and $\big(F^{F(e)}, R^{F(e)}\big)$ in order to complete the proof of the theorem. For each $m \in e\mathcal{M}$, we have a functorial isomorphism

$$\theta_m : F(e) \otimes m \xrightarrow{\simeq} F(m)$$

given by the composition

$$F(\lambda_m) \circ F\big((\pi \otimes \mathrm{id}_m)^{-1}\big) \circ R_{e,m} \ : \ F(e) \otimes m \xrightarrow{\simeq} F(e \otimes m) \xrightarrow{\simeq} F(\mathbb{1} \otimes m) \xrightarrow{\simeq} F(m).$$

The collection of isomorphisms θ_m defines an isomorphism of functors

$$\theta \ : \ F^{F(e)} \xrightarrow{\simeq} F,$$

which itself depends functorially on the pair (F, R). Hence it remains to check that θ is compatible with R and $R^{F(e)} = \alpha_{F(e),\cdot,\cdot}$ in the obvious sense. This is equivalent to proving the commutativity of the diagram

$$\begin{tikzcd}
F^{F(e)}(X) \otimes Y \arrow[r, "R^{F(e)}_{X,Y}"] \arrow[d, "\theta_X \otimes \mathrm{id}_Y"'] & F^{F(e)}(X \otimes Y) \arrow[d, "\theta_{X \otimes Y}"] \\
F(X) \otimes Y \arrow[r, "R_{X,Y}"'] & F(X \otimes Y)
\end{tikzcd}$$

To show that it commutes, we expand it as follows:

$$\begin{tikzcd}
\big(F(e) \otimes X\big) \otimes Y \arrow[rr] \arrow[d] & & F(e) \otimes (X \otimes Y) \arrow[d] \\
F(e \otimes X) \otimes Y \arrow[dr] \arrow[dd] & & F\big(e \otimes (X \otimes Y)\big) \arrow[dl] \arrow[dd] \\
& F\big((e \otimes X) \otimes Y\big) \arrow[dd] & \\
F(\mathbb{1} \otimes X) \otimes Y \arrow[dr] \arrow[dd] & & F\big(\mathbb{1} \otimes (X \otimes Y)\big) \arrow[dl] \arrow[dd] \\
& F\big((\mathbb{1} \otimes X) \otimes Y\big) \arrow[dr] & \\
F(X) \otimes Y \arrow[rr] & & F(X \otimes Y)
\end{tikzcd}$$

All arrows in this diagram are the obvious ones. It is easy to check that the diagram above commutes using the pentagon axiom for (F, R), the naturality of α, and the naturality of R. The (straightforward) details are left to the reader. $\square$

3.6. Let $\mathcal{C}$ be a category and $\mathcal{D} \subseteq \mathcal{C}$ a subcategory. We recall ([McL98], §IV.3) that $\mathcal{D}$ is said to be *reflective* (resp., *coreflective*) if the inclusion functor $\mathcal{D} \hookrightarrow \mathcal{C}$ admits a left (resp., right) adjoint.

Lemma 3.8. *If e is a closed idempotent in a monoidal category $\mathcal{M}$, then the strictly full subcategories $e\mathcal{M}$, $\mathcal{M}e$ and $e\mathcal{M}e$ of $\mathcal{M}$ are reflective.*

The following result is a partial converse to this lemma:

Theorem 3.9. *Assume that $\mathcal{M}$ is a monoidal category which is left closed, i.e., for any pair of objects X, Y of $\mathcal{M}$, there exists an object $\underline{\mathrm{Hom}}^{\ell}(X,Y) \in \mathcal{M}$ and a functorial isomorphism*

$$\mathrm{Hom}_{\mathcal{M}}(Z \otimes X, Y) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{M}}(Z, \underline{\mathrm{Hom}}^{\ell}(X,Y))$$

for all objects Z of $\mathcal{M}$. Then the map $e \mapsto e\mathcal{M}$ provides a bijection between

(i) *the set of isomorphism classes of closed idempotents in $\mathcal{M}$, and*
(ii) *the set of reflective right tensor ideals $\mathcal{M}' \subseteq \mathcal{M}$ which have the property that if $Y \in \mathcal{M}'$ and $X \in \mathcal{M}$, then $\underline{\mathrm{Hom}}^{\ell}(X,Y) \in \mathcal{M}'$.*

This theorem has an obvious analogue for right closed monoidal categories and left tensor ideals. We also note that if $\mathcal{M}$ is *left rigid* (this property is stronger than being left closed), then the second condition in (ii) holds automatically for right tensor ideals, since in this case we have a natural isomorphism $\underline{\mathrm{Hom}}^{\ell}(X,Y) \xrightarrow{\simeq} Y \otimes X^*$ for any two objects $X, Y \in \mathcal{M}$.

4. Idempotents in braided monoidal categories

4.1. As in Section 1, we begin by giving a precise definition of the type of categories we work with. See [McL98], §XI.1 for more details.

Definition 4.1. A *strictly associative braided monoidal category*, to be abbreviated BMC, is a 6-tuple $\mathcal{M} = (\mathcal{M}, \otimes, \mathbb{1}, \lambda, \rho, \beta)$, where $(\mathcal{M}, \otimes, \mathbb{1}, \lambda, \rho)$ is a strictly associative monoidal category in the sense of Definition 1.2 and β is a bifunctorial collection of isomorphisms

$$\beta_{a,b} : a \otimes b \xrightarrow{\simeq} b \otimes a \qquad \forall\, a, b \in \mathcal{M},$$

called the *braiding* of $\mathcal{M}$, satisfying the axioms

(i) for every $a \in \mathcal{M}$, the triangle

$$\begin{tikzcd} a \otimes \mathbb{1} \arrow[rr, "\beta_{a,\mathbb{1}}"] \arrow[dr, "\rho_a"'] & & \mathbb{1} \otimes a \arrow[dl, "\lambda_a"] \\ & a & \end{tikzcd}$$

commutes, so that, in particular, $\beta_{\mathbb{1},\mathbb{1}} = \mathrm{id}_{\mathbb{1}\otimes\mathbb{1}}$; and

(ii) the following two diagrams, which result from the "hexagon axiom" for a usual braided monoidal category, but which reduce to triangles due to the strict associativity assumption, commute for all $a, b, c \in \mathcal{M}$:

$$\begin{tikzcd} & a \otimes b \otimes c \arrow[dl, "\mathrm{id}_a \otimes \beta_{b,c}"'] \arrow[dr, "\beta_{a\otimes b,c}"] & \\ a \otimes c \otimes b \arrow[rr, "\beta_{a,c}\otimes \mathrm{id}_b"'] & & c \otimes a \otimes b \end{tikzcd} \tag{4.1}$$

and

$$\begin{tikzcd} & a \otimes b \otimes c \arrow[dl, "\beta_{a,b}\otimes \mathrm{id}_c"'] \arrow[dr, "\beta_{a,b\otimes c}"] & \\ a \otimes c \otimes b \arrow[rr, "\mathrm{id}_b \otimes \beta_{a,c}"'] & & c \otimes a \otimes b \end{tikzcd} \tag{4.2}$$

4.2. From now on until the end of this section, we fix a BMC $\mathcal{M}$ and study closed idempotents in $\mathcal{M}$.

Proposition 4.2. *If e is a closed idempotent in $\mathcal{M}$, then*
(*a*) $\beta_{e,e} = \mathrm{id}_{e\otimes e}$, *and*
(*b*) *for every object $m \in \mathcal{M}$, we have $\beta_{m,e} \circ \beta_{e,m} = \mathrm{id}_{e\otimes m}$.*

Theorem 4.3. *If e is a closed idempotent in $\mathcal{M}$, the natural functor*

$$\mathcal{M} \longrightarrow e\mathcal{M}e, \qquad X \mapsto e \otimes X \otimes e$$

can be upgraded to a braided monoidal functor.

5. Idempotents in triangulated monoidal categories

5.1. **Open idempotents.** As mentioned in the introduction, there is a notion dual to that of a closed idempotent:

Definition 5.1. An *open idempotent* in a monoidal category $\mathcal{M}$ is an arrow $e \xrightarrow{\pi} \mathbb{1}$ in $\mathcal{M}$ such that the two induced arrows,

$$e \otimes e \xrightarrow{\mathrm{id}_e \otimes \pi} e \otimes \mathbb{1} \qquad \text{and} \qquad e \otimes e \xrightarrow{\pi \otimes \mathrm{id}_e} \mathbb{1} \otimes e,$$

are isomorphisms. An object e of $\mathcal{M}$ is an *open idempotent* if there exists an arrow $e \xrightarrow{\pi} \mathbb{1}$ which is an open idempotent.

It is clear that if $\mathcal{M}$ is a monoidal category, then the opposite category $\mathcal{M}^{op}$ inherits a natural monoidal structure, and we have an obvious bijection between open idempotents in $\mathcal{M}$ and closed idempotents in $\mathcal{M}^{op}$. Thus the results of Sections 1, 3 and 4 have obvious analogues for open idempotents whose formulation is left to the reader. In particular, if e is an open idempotent in $\mathcal{M}$, the strictly full subcategories $e\mathcal{M}$, $\mathcal{M}e$ and $e\mathcal{M}e$ of $\mathcal{M}$ are defined in the same way as for closed idempotents. More interesting is the fact that if $\mathcal{M}$ is a *triangulated* monoidal category, then there is a bijection between the set of isomorphism classes of closed idempotents in $\mathcal{M}$ and the set of isomorphism classes of open idempotents in *the same* category $\mathcal{M}$, see Proposition 5.3 below.

5.2. **Locally closed idempotents.** To motivate our definition of a locally closed idempotent in a triangulated monoidal category, let us first consider the abelian situation. Let X be a topological space and $\mathcal{M}$ the category of sheaves of complex vector spaces on X with the usual tensor product. If $\xi : Y \hookrightarrow X$ is the inclusion of a locally closed subspace, we can consider the constant sheaf $\mathbb{C}_Y$ with stalks $\mathbb{C}$ on Y and form $\xi_! \mathbb{C}_Y \in \mathcal{M}$. To study the properties of this sheaf, let us write $Y = U \cap Z$, where U is open in X and Z is closed in X. We assume moreover that $X = U \cup Z$; this is always possible to achieve, and under this condition each of U, Z determines the other one uniquely. We have a diagram of inclusions

$$\begin{tikzcd}
& X & \\
U \arrow[ur, hook, "j"] & & Z \arrow[ul, hook', "i"'] \\
& Y \arrow[ul, hook', "i'"] \arrow[ur, hook, "j'"'] \arrow[uu, "\xi"] &
\end{tikzcd}$$

which by adjunction induces a diagram

$$\begin{tikzcd}
j_! \mathbb{C}_U \arrow[r, hook] \arrow[d] & \mathbb{C}_X \arrow[d] \\
\xi_! \mathbb{C}_Y \arrow[r, hook] & i_! \mathbb{C}_Z
\end{tikzcd}$$

that is both cartesian and cocartesian (this uses the assumption that $X = U \cup Z$). Moreover, the left (resp., right) vertical arrow is a closed idempotent in the category of sheaves of complex vector spaces on X (resp., on U), and the top (resp., bottom) horizontal arrow is an open idempotent in the category of sheaves of complex vector spaces on X (resp., on Z). This motivates two possible definitions of a locally closed idempotent in an arbitrary monoidal category:

Definition 5.2. A *locally closed idempotents* in a monoidal category $\mathcal{M}$ is either

(i) an open idempotent $u \longrightarrow e$ in the category $e\mathcal{M}e$ corresponding to a closed idempotent $\mathbb{1} \longrightarrow e$ in $\mathcal{M}$, or
(ii) a closed idempotent $v \longrightarrow u$ in the category $v\mathcal{M}v$ corresponding to an open idempotent $v \longrightarrow \mathbb{1}$ in $\mathcal{M}$.

In a triangulated monoidal category these two definitions turn out to be equivalent, see Theorem 5.4 below. In the rest of the section we explain the definition of a triangulated monoidal category and prove the two results we have mentioned.

5.3. **Main results.**

Proposition 5.3. *Let $\mathcal{M}$ be a triangulated monoidal category. If $e \xrightarrow{\pi} \mathbb{1}$ is an open idempotent in $\mathcal{M}$ and*

$$e \xrightarrow{\pi} \mathbb{1} \xrightarrow{\pi'} e' \longrightarrow e[1]$$

is a distinguished triangle obtained from π, then $\mathbb{1} \xrightarrow{\pi'} e'$ is a closed idempotent in $\mathcal{M}$. Moreover, the map $\pi \mapsto \pi'$ defined this way yields a bijection between the set of isomorphism classes of open idempotents in $\mathcal{M}$ and that of closed idempotents in $\mathcal{M}$.

Theorem 5.4. *Let $\mathcal{M}$ be a triangulated monoidal category. An object of $\mathcal{M}$ is a locally closed idempotent in the sense of Definition 5.2(i) if and only if it is a locally closed idempotent in the sense of Definition 5.2(ii).*

References

[Ad73] J.F. Adams, *Idempotent functors in homotopy theory*, in: "Manifolds—Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973)," pp. 247–253. Univ. Tokyo Press, Tokyo, 1975.

[BD06] M. Boyarchenko and V. Drinfeld, *A motivated introduction to character sheaves and the orbit method for unipotent groups in positive characteristic*, Preprint, September 2006, arXiv: `math.RT/0609769`

[DFH1] A. Deleanu, A. Frei and P. Hilton, *Idempotent triples and completion*, Math. Z. **143** (1975), 91–104.

[DFH2] A. Deleanu, A. Frei and P. Hilton, *Generalized Adams completion*, Cahiers Topologie Géom. Différentielle **15** (1974), 61–82.

[McL98] S. MacLane, "Categories for the working mathematician", 2nd ed. Graduate Texts in Mathematics, **5**. Springer-Verlag, New York, 1998.