Centralizers of semisimple elements of a simply connected semisimple group $G$.

Let $k$ be an algebraically closed field, and $G$ a connected semisimple group over $k$. The centralizer of $g \in G$ will be denoted by $Z(g)$.

**Reductivity of centralizers**

**Fact:** if $g \in G$ is semisimple then $Z(g)$ is reductive.

Note that if $T \subset G$ is a maximal torus containing $g$ then $T$ is also a maximal torus of $Z(g)$. So the rank of $Z(g)$ equals that of $G$.

**Connectedness of centralizers.**

**Fact:** if $G$ is simply connected and $g \in G$ is semisimple then $Z(g)$ is connected.

**Remarks.** (i) The simply-connectedness of $G$ is essential: take $G=\text{PGl}(2)$, $g=(1\ 0)$. (iii) The semisimplicity of $g$ is also essential. In fact, it is known that if $G$ is not isomorphic to a product $\prod_{i=1}^{m} \text{Sl}(n_i)$ then $G$ has a unipotent element whose centralizer is disconnected.
Centralizers and Levi subgroups.

Any Levi subgroup $L \leq G$ can be represented as $Z(g)$ for some semisimple $g \in G$ (take $g$ to be a generic element of the center of $L$).

If $G = SL(n)$ then the centralizer of any semisimple $g \in G$ is a Levi subgroup. This is not true for a general simply connected semisimple group $G$:

Assume that $\text{char } k \neq 2$. Let $m, n > 0$

Example 1. $G = Sp(2m+2n)$, $g = \begin{pmatrix} 4m & 0 \\ 0 & -12n \end{pmatrix}$

Then $Z(g) = Sp(2m) \times Sp(2n) \subset Sp(2m+2n)$.

This is not a Levi subgroup. $m, n > 0$.

Example 2. Let $\text{char } k \neq 2$ and $G = Spin(m+n)$.

Assume that $n$ is even, then $SO(n) \vartriangleleft 1_n$.

Let $\bar{g} = \begin{pmatrix} 1m & 0 \\ 0 & -1n \end{pmatrix} \in SO(m+n)$ and let $g \in Spin(m+n)$ be a preimage of $\bar{g}$.

One has $Z(\bar{g}) = \{(A, B) \in O(m) \times O(n) \mid \det A = \det B\}$, so $Z(\bar{g})$ is disconnected. But $Z(g)$ has to be connected, so $Z(g)$ equals the preimage of $SO(m) \times SO(n)$ in $Spin(m+n)$. This is not a Levi subgroup unless $m = 2$ or $n = 2$. 
Centralizers can be semisimple.

A Levi subgroup of $G$ cannot be semisimple unless it equals $G$. On the other hand, in Example 1 the centralizer $Z(g)$ is semisimple. This is also true in Example 2 unless $m=2$ or $n=2$.

Remarks. (i) Note that if $Z(g)$ is semisimple then $Z(g)$ is not contained in any parabolic $P \leq G$ different from $G$.

(ii) Schur’s lemma says that if $G = \text{SL}(n)$ and $P \leq G$ is a subgroup whose centralizer is bigger than the center of $G$ then $P$ is contained in a parabolic $P \leq G$, $P \neq G$. The above Remark (i) combined with Examples 1–2 show that this is not true for a general simply connected semisimple group $G$.

It may happen that $[Z(g),Z(g)]$ is not simply connected.

If $L$ is a Levi subgroup of a simply connected semisimple group $G$ then $[L,L]$ is simply connected. On the other hand,
in Example 2 with \( m, n \neq 2 \) the group \( [Z(g), Z(g)] = Z(g) \) is not simply connected.

Semisimple elements \( \xi \in G \) such that \( Z(g) \) is semisimple.

From now on we assume that \( G \) is simply connected and almost simple. We also assume that \( \text{char } k = 0 \) (otherwise the statements would become slightly more complicated).

Let \( A(G) \) be the set of conjugacy classes of semisimple elements \( \xi \in G \) such that \( Z(g) \) is semisimple. Let \( B(G) \) be the set of vertices of the extended Dynkin diagram of \( G \) (so \( \text{Card } B(G) = i + 1 \)). In 1969 Victor Kac constructed a bijection \( B(G) \rightarrow A(G) \).

Notation: \( \alpha_1, \ldots, \alpha_r \) are the simple roots; \( \check{\omega}_1, \ldots, \check{\omega}_r \) are the fundamental coweights, i.e., \( (\check{\omega}_i, \omega_j) = \delta_{ij} \); \( \alpha_{\text{max}} = \sum_{i=1}^{r} \max \alpha_i \) is the maximal root. We choose an isomorphism \( \varepsilon: Q/Z \rightarrow \{ \text{roots of unity in } k^* \} \).
Construction of the map $B(G) \rightarrow A(G)$. The vertex of the extended Dynkin diagram corresponding to $\alpha_{\text{max}}$ goes to $1 \in G$. The vertex corresponding to $\alpha_i$ goes to $\tilde{g}_i = \tilde{\omega}_i \left( e \left( \frac{1}{m_i} \right) \right)$, where $\tilde{\omega}_i$ is considered as a morphism $\mathbb{G}_m \rightarrow \{\text{maximal torus}\}$.

Exercise. $Z(g_i)$ is semisimple. The Dynkin diagram of $Z(g_i)$ is obtained from the extended Dynkin diagram of $G$ by removing its $i$-th vertex.

Theorem. The above map $B(G) \rightarrow A(G)$ is bijective.

This theorem was proved by V. Kac as a part of his classification of elements of finite order in $G$ up to conjugacy; see §3.6 of ch. 3 of the book: V.V. Gorbatevich, A.L. Onishchik, and E.B. Vinberg, Structure of Lie groups and Lie algebras. Lie groups and Lie algebras, III. Encyclopaedia of Mathematical Sciences, 41. Springer-Verlag, Berlin, 1994.

Consistency check. Let $G = \text{SL}(n)$. Then $Z(g)$ is semi-simple only if $g \in \{\text{center of } G\}$. So Card $A(G) = n$. On the other hand, the extended Dynkin diagram has $n$ vertices.
Surprise no. 1. For each $n \in \mathbb{Z}$ one has the map $\alpha_n: A(G) \to A(G)$ defined by $\alpha_n(g) = g^n$. It is hard to imagine the corresponding map $B(G) \to B(G)$.

Surprise no. 2. Instead of considering $\tilde{\omega}_i(e(\tfrac{1}{m_i}))$ one could consider $g = \omega_i(e(\alpha)), \alpha \in \mathbb{Q}/\mathbb{Z}$. It is easy to show that if the denominator of $\alpha$ is $\leq m_i$ then $Z(g)$ is semisimple, so $g$ defines an element of $A(G)$. It is hard to see why it comes from an element of $B(G)$.

Nevertheless, the theorem is true!

Sketch of the proof. Note that $g$ belongs to the center of $Z(g)$, so if $Z(g)$ is semisimple then $g$ has finite order. We have

$$\{\text{Elements of finite order in } G \text{ conjugation} = \{\text{Elements of finite order in } T\}/W =$$

$$= (C \otimes \mathbb{Q}/\mathbb{Z})/W,$$

where $C$ is the group generated by the coroots $\xi_i$ (we have used the isomorphism $e: \mathbb{Q}/\mathbb{Z} \cong \{\text{roots of unity in } \mathbb{R}\}$). Now

$$(C \otimes \mathbb{Q}/\mathbb{Z})/W = (C \otimes \mathbb{Q})/W_{aff},$$

where $W_{aff}$ is the affine Weyl group. The theory of $W_{aff}$ identifies $(C \otimes \mathbb{Q})/W_{aff}$ with the simplex

$$\Delta := \{x \in C \otimes \mathbb{Q} \mid (x, x_{\max}) \leq 1, (x, x_i) \geq 0 \text{ for } 1 \leq i \leq r\}.$$  

Finally, one checks that if $x \in \Delta$ and $g_x$ is the corresponding element of $G$ then $Z(g_x)$ is semisimple if and only if $x$ is a vertex of $\Delta$. 