Today char $k=0$ (or char k positive and big enough). If you prefer, k is algebraically closed

G reductive (connected), $\mathfrak{g} = \mathrm{Lie}(G)$

$$
\begin{tikzcd}
\mathfrak{g} \arrow[r, "\chi"] & c := \mathfrak{t}/W \\
\text{open } \mathfrak{g}^{reg} \arrow[u, phantom, "\cup"] & c^{reg,ss} \text{ open} \arrow[u, phantom, "\cup"] \\
\text{open } \mathfrak{g}^{reg,ss} \arrow[u, phantom, "\cup"] \arrow[ur] &
\end{tikzcd}
$$

$c = \{$"characteristic polynomials"$\}$

Letters c, χ stand for "characteristic"

G acts transitively on each fiber of $\mathfrak{g}^{reg} \xrightarrow{\chi} c$

$$\mathfrak{g}^{reg,ss} = \chi^{-1}(c^{reg,ss}).$$

How big are $c \setminus c^{reg,ss}$, $\mathfrak{g} \setminus \mathfrak{g}^{reg,ss}$, $\mathfrak{g} \setminus \mathfrak{g}^{reg}$?

Discriminant divisor

(Codimension $3 \geq 2$).

Centralizers and regular centralizers

$x \in \mathfrak{g} \Rightarrow I_x := \{ g \in G \mid g x g^{-1} = x \}$ - affine alg. group scheme

As x varies, get an affine group $I \to \mathfrak{g}$.

It is not flat: $\dim I_x$ can jump.

Now consider $I^{reg} := I|_{\mathfrak{g}^{reg}}$.

By definition, $x \in \mathfrak{g}^{reg} \Leftrightarrow \dim I_x = r$ (minimal value).

So I^{reg} is flat over $\mathfrak{g}$. Since char $k=0$ each I_x is smooth, so I^{reg} is smooth over $\mathfrak{g}$.

Fact: $x \in \mathfrak{g}^{reg} \Rightarrow I_x$ is abelian.

<u>Corollary</u>. $x \in \mathfrak{g}^{reg} \Rightarrow I_x$ depends only on $\chi(x)$.

More precisely: if $x, x' \in \mathfrak{g}^{reg}$ and $\chi(x') = \chi(x)$ then $x' = g x g^{-1}$. A choice of g defines an isomorphism $\varphi_g : I_x \xrightarrow{\sim} I_{x'}$. I_x abelian $\Rightarrow \varphi_g$ doesn't depend on g.

<u>Def.</u> Let $y \in c$. Then $J_y := I_x$, where $x \in \mathfrak{g}^{reg}$, $\chi(x) = y$.

As $y \in \mathfrak{c}$ varies we get a smooth commutative group scheme J over $\mathfrak{c}$ of relative dimension r.

<u>Def.</u> J is the <u>scheme of regular centralizers</u>.

We have a group scheme $\chi^* J$ over $\mathfrak{g}$. (It will also be called the scheme of regular centralizers.)

<u>Example.</u> $G = GL(n)$, $\mathfrak{g} = \mathfrak{gl}(n)$, $\mathfrak{c} = \{$monic polynomials of degree $n\}$.

$f \in k[x]$ monic, $\deg f = n \Rightarrow J_f = (k[x]/(f))^\times$

($(k[x]/(f))^\times$ is viewed as an algebraic group).

By def., $\chi^* J|_{\mathfrak{g}^{reg}} = I^{reg}$.

<u>Prop.</u> The isomorphism $\chi^* J|_{\mathfrak{g}^{reg}} \xrightarrow{\sim} I^{reg}$ uniquely extends to a homomorphism $\chi^* J \to I$.

<u>Example.</u> $G = GL(n)$, $A \in \mathfrak{gl}(n)$, $f \in k[x]$ the charact. polynomial of A,

$$J_A := (k[x]/(f))^*, \quad I_A := \text{centralizer of } A \text{ in } GL(n)$$

$$\begin{array}{ccc} \Psi & & \Psi \\ p & \longmapsto & p(A) \end{array}$$

(A not regular $\Rightarrow$ neither mono nor epi).

<u>Proof.</u> $I' := \chi^* J$, $I'^{\,reg} := I'|_{\mathfrak{g}^{reg}}$

Extension problem:
$$\begin{array}{ccc} I'^{\,reg} & = & I^{reg} \\ \cap & & \cap \\ I' & \dashrightarrow & I \end{array}$$

$I' \to \mathfrak{g}$ smooth, so I' is smooth

$\operatorname{codim}(I' \setminus I'^{\,reg}) \geq 2$

I is affine.

$\forall$ morphism $I'^{\,reg} \to \{$affine scheme$\}$ extends to I' in a unique way. (The extension is a morphism of group schemes over $\mathfrak{g}$ because this is true over $\mathfrak{g}^{reg} \subset \mathfrak{g}$). ■