A formulation of the fundamental lemma for $SL(n)$ in concrete terms.

Notation: F a non-archimedean local field, $O_F \subset F$ the ring of integers, $m_F \subset O_F$ the maximal ideal, $q = |O_F/m|$. Let $E \supset F$ be an unramified extension of degree n and $O_E \subset E$ the ring of integers.

Now let $\gamma \in O_E$ be such that $F[\gamma] = E$. Then the ring $R := O_F[\gamma]$ is a subgroup of finite index in O_E. For each $s \in \mathbb{Z}$ let $N_{\gamma,F}(s)$ denote the number of R-submodules $M \subset E$ such that the number

$$\mathrm{size}(M) := \mathrm{length}_O(M/M\cap R) - \mathrm{length}_O(R/M\cap R)$$

equals s. (Note that if $\mathrm{size}(M) \neq \pm\infty$ then M is finitely generated and nonzero). Since $\mathrm{size}(m_O M) = \mathrm{size}(M) - n$ the number $N_{\gamma,F}(s)$ depends only on the image of s in $\mathbb{Z}/n\mathbb{Z}$. The fundamental lemma says **something** about the Fourier transform of the function $N_{\gamma,F} : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}$.

More precisely, for each $\zeta \in \mathbb{C}$ such that $\zeta^n = 1$ define

$$u_{\gamma,F}(\zeta) := \frac{1}{(O_E : R)} \sum_{s \in \mathbb{Z}/n\mathbb{Z}} \zeta^s N_{\gamma,F}(s),$$

where $(O_E : R)$ is the index of the subgroup $R \subset O_E$.

As far as I understand, the Fundamental Lemma for $SL(n)$ says that if ζ is a primitive root of degree $m|n$ then

$$u_{\gamma, F}(\zeta) = u_{\gamma, F_m}(1), \qquad (*)$$

where $F_m < E$ is the unique subfield containing F with $[F_m : F] = m$ and the definition of u_{γ, F_m} is similar to that of $u_{\gamma, F}$ (consider E as an extension of F_m of degree $\frac{n}{m}$ rather than an extension of F of degree n, consider $O_{F_m}[\gamma]$ instead of $R := O_F[\gamma]$, etc.). E.g., if $\zeta = 1$ then the Fundamental Lemma is a tautology, and if ζ is a primitive n-th root of 1 it says that $u_{\gamma, E}(\zeta) = 1$.

<u>Exercise 1.</u> Prove (or disprove!) that the Fundamental Lemma indeed **implies** formula $(*)$. and moreover, in the case that $G = SL(n)$ and

the endoscopy group H equals $\mathrm{Ker}(E^\times \xrightarrow{N} F^\times)$ the Fundamental Lemma is equivalent to formula $(*)$ for $m=n$. (However, if
$H=\mathrm{Ker}\,(GL(\frac{n}{m}, F_m) \xrightarrow{\det} F_m^\times \xrightarrow{N} F^\times)$, $m|n$, $m\neq n$
then the Fundamental Lemma is stronger than $(*)$).

As a first step, prove that the normalized absolute value of the discriminant of γ equals $(O_E : R)^{-2}$.

Exercise 2. Prove the Fundamental Lemma for $SL(2)$. Hint: show that if $(O_E : R) = q^k$ then $u_{\gamma,F}(-1) = a_k - a_{k-1} + a_{k-2} - \ldots + (-1)^k a_0$, where $a_0 = 1$ and $a_i = q^{i-1}(q+1)$ for $i>0$. In fact, a_i is the number of vertices of the Bruhat-Tits tree at distance i from a fixed vertex).

Historical remark. The fundamental lemma for $SL(3)$ was proved by Kottwitz. The fundamental lemma for $SL(n)$ was proved by Kazhdan and Waldspurger.

Remark. Equality $(*)$ on p. 2 implies that the function $N_{\gamma,F} : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}$ defined on p. 1 satisfies $N_{\gamma,F}(as) = N_{\gamma,F}(s)$ for all $a \in \mathbb{Z}$ coprime to n. This looks mysterious! Anyway, I do not know how to prove this without $(*)$.