Tate-Nakayama duality

Let $F$ be a local non-archimedean field. Let $T$ be a torus over $F$ and $X^*(T)$ its group of characters. $\text{Gal}(\overline{F}/F)$ acts on $X^*(T)$. The obvious pairing $T(\overline{F}) \times X^*(T) \rightarrow \overline{F}^\times$ induces a pairing $H^i(F, T) \times H^{2-i}(F, X^*(T)) \rightarrow H^2(F, G_m) = Br(F) = \mathbb{Q}/\mathbb{Z}$.

Theorem. (i) This pairing is nondegenerate.
(ii) $H^4(F, T)$ and $H^4(F, X^*(T))$ are finite.

What about $H^0$ and $H^2$? (It is known that $F$ has cohomological dimension 2). Notation: $A^i := H^i(F, T)$, $B^{2-i} := H^{2-i}(F, X^*(T))$.

Example: $T = G_m$

$A^i := H^i(F, G_m)$
$B^i := H^{2-i}(F, \mathbb{Z})$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$F^\times$</th>
<th>$\text{Hom}<em>{\text{cont}}(\text{Gal}(F/F), \mathbb{Q}/\mathbb{Z}) = \text{Hom}</em>{\text{cont}}(F^\times, \mathbb{Q}/\mathbb{Z})$</th>
</tr>
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<tbody>
<tr>
<td>$i = 0$</td>
<td>$F^\times$</td>
<td>$\text{Hom}<em>{\text{cont}}(\text{Gal}(F/F), \mathbb{Q}/\mathbb{Z}) = \text{Hom}</em>{\text{cont}}(F^\times, \mathbb{Q}/\mathbb{Z})$</td>
</tr>
<tr>
<td>$i = 1$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$i = 2$</td>
<td>$\mathbb{Q}/\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
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Claim. The pairing $A^i \times B^{2-i} \rightarrow \mathbb{Q}/\mathbb{Z}$ induces isomorphisms $A^2 \cong \text{Hom}(B^0, \mathbb{Q}/\mathbb{Z})$ and $B^2 \cong \text{Hom}_{\text{cont}}(A^0, \mathbb{Q}/\mathbb{Z})$, where $A^0 = T(F)$ is equipped with the topology that comes from the topology on $F$. As said before, $A^4$ and $B^4$ are finite and dual to each other.

Proofs can be found in
S. Shatz, Profinite groups, arithmetic, and geometry, ch. VI, §5.

At least, the statement about $H^4(F, T)$ is an exercise (once you believe that $Br(F) = \mathbb{Q}/\mathbb{Z}$).

First, forget about the local field $F$.

Let $k$ be any algebraically closed field of characteristic 0, let $T$ be a torus over $k$ and $X_*(T) = \text{Hom}(\hat{G}_m, T)$ the group of cocharacters.

Kottwitz' lemma. $\text{Hom}(H^4(\Gamma, X_*(T)), \mathbb{Q}/\mathbb{Z}) = \text{Hom}(\pi_0(\hat{T}^\Gamma), k^\times)$.

Tate twist

Reformulation. $\pi_0(\hat{T}^\Gamma) = H^4(\Gamma, X_*(T))$ (1)

(most people take $k = \mathbb{C}$ and skip the Tate twist).

Combining the lemma with Tate-Nakayama duality Kottwitz immediately gets:

Corollary. Let $F$ be a local nonarchimedean field and $T$ a torus over $F$, let $\hat{T}$ be the dual torus over $k$, i.e., $X_*(\hat{T}) = X_*(T)$. Let $\hat{T}^\Gamma$ be the invariants of $\Gamma = \text{Gal}(\bar{F}/F)$ acting on $\hat{T}$. Then $H^4(F, T) = \text{Hom}(\pi_0(\hat{T}^\Gamma), k^\times)$.

(Note that $k$ does not appear in the l.h.s. of this equality while in the r.h.s. it appears twice because $\hat{T}$ depends on $k$).

Proof of Kottwitz' lemma for $k = \mathbb{C}$. We have to prove that $\pi_0(\hat{T}^\Gamma) = H^4(\Gamma, X_*(T))$. One has $\pi_0(\hat{T}^\Gamma) = \text{Coker}(\text{Lie} T^\Gamma \to \text{Lie} \hat{T}^\Gamma) = (X_*(T) \otimes \mathbb{C})^\Gamma$, $T^\Gamma = (X_*(T) \otimes \mathbb{C}/\mathbb{Z})^\Gamma$. Now use the exact sequence $(X_*(T) \otimes \mathbb{C})^\Gamma \to (X_*(T) \otimes \mathbb{C}/\mathbb{Z})^\Gamma \to}$
\( \to H^1(\Gamma, X^*_\chi(\mathcal{T})) \to H^1(\Gamma, X^*_\chi(\mathcal{T}) \otimes \mathbb{Q}) = 0 \)

(we have used the vanishing of \( H^i \) of a profinite group acting on a \( \mathbb{Q} \)-vector space, \( i > 0 \)).

A variant of the proof which makes sense for any \( q \).

Notation: \( A^*_q = X^*_\chi(\mathcal{T})^q \).

Then \( X^*_\chi(\mathcal{T}^q) = A^*_q \), \( \text{Hom}(\mathcal{X}_0(\mathcal{T}^q), \mathbb{Q}^*) = (A^*_q)_{\text{tors}} \).

On the other hand, let \( A^*_q = \text{Hom}(\mathcal{A}, \mathbb{Z}) = X^*_\chi(\mathcal{T}) \), then \( H^1(\Gamma, A^*_q) = \ker (H^1(\Gamma, A^*_q) \to H^1(\Gamma, A^*_q \otimes \mathbb{Q})) = \text{Coker} (((A^*_q \otimes \mathbb{Q})^p \to (A^*_q \otimes \mathbb{Q}/\mathbb{Z})^p)). \)

But \( (A^*_q \otimes \mathbb{Q})^p = \text{Hom}(A^*_q, \mathbb{Q}) \) and \( (A^*_q \otimes \mathbb{Q}/\mathbb{Z})^p = \text{Hom}(A^*_q, \mathbb{Q}/\mathbb{Z}) \), so \( H^1(\Gamma, A^*_q) = \text{Ext}(A^*_q, \mathbb{Z}) = \text{Hom}((A^*_q)_{\text{tors}}, \mathbb{Q}/\mathbb{Z}). \)

Remark. The last part of the proof was devoted to showing that \( (H^1(\Gamma, A^*_q))_{\text{tors}} = \text{Hom}((A^*_q)_{\text{tors}}, \mathbb{Q}/\mathbb{Z}). \) In fact, this is a particular case of the universal coefficients formula.