CHAPTER 4

Global Theory: Chiral Homology

Чиновник умирает, и ордена его остаются на лице земли.

Козьма Прутков, "Мысли и Афоризмы", 1860 †

4.1. The cookware

This section collects some utensils and implements needed for the construction of chiral homology. A sensible reader should skip it, returning to the material when necessary.

Here is the inventory together with a brief comment on the employment mode.

(i) In 4.1.1–4.1.2 we consider homotopy direct limits of complexes and discuss their multiplicative properties. The material will be used in 4.2 where the de Rham complex of a $\mathcal{D}$-module on Ran's space $\mathcal{R}(X)$ is defined as the homotopy direct limit of de Rham complexes of the corresponding $\mathcal{D}$-modules on all the X^n's with respect to the family of all diagonal embeddings. See [**BK**], [**Se**].

(ii) In 4.1.3 and 4.1.4 we discuss the notion of the Dolbeault algebra which is an algebraic version of the $\bar\partial$-resolution. Dolbeault resolutions are functorial and have nice multiplicative properties, so they are very convenient for computing the global de Rham cohomology of $\mathcal{D}$-modules on $\mathcal{R}(X)$, in particular, the chiral homology of chiral algebras. An important example of Dolbeault algebras comes from the Thom-Sullivan construction; see [**HS**] §4.

(iii) In 4.1.5 we recall the definitions of semi-free DG modules, semi-free commutative DG algebras, and the cotangent complex following [**H**] (see also [**Dr1**] and [**KrM**]). The original construction of the cotangent complex, due to Grothendieck [**Gr1**], [**Il**], was performed in the setting of simplicial algebras. We follow the DG setting of [**H**] (using the notation ${}^L\Omega_F$ instead of the standard $\mathbb{L}_{F/k}$).

(iv) Sections 4.1.6 and 4.1.7 deal with Batalin-Vilkovisky algebras and the corresponding homotopy categories. BV algebras are "quantum deformations" of odd Poisson (alias braid, alias Gerstenhaber) algebras. We will see in 4.3.1 that chiral chain complexes of, respectively, commutative $\mathcal{D}_X$-algebras, coisson algebras, and general chiral algebras are naturally commutative DG algebras, odd Poisson algebras, and BV algebras. For a more lively account, see, e.g., [**Ge**] and [**Schw**].

(v) A typical example of BV algebra is the Chevalley homology complex of a Lie algebra; from the BV viewpoint it is the BV envelope of the Lie algebra (see 4.1.8). In 4.1.9–4.1.10 we explain what BV envelopes of Lie algebroids are. To

† "A civil servant dies, and regalia of his stay on the face of the earth." Koz'ma Proutkoff, "Thoughts and Aphorisms", 1860.

define it, an extra structure – that of the BV extension – is needed. The situation is parallel to that of chiral envelopes from 3.9; we will see in 4.8 that the chiral homology functor transforms chiral envelopes to BV envelopes. The homotopy properties of this construction are discussed in 4.1.11 and 4.1.12; in 4.1.13 we prove a technical statement (to be used in 4.1.18) asserting that while considering BV Lie R-algebroids from the homotopy point of view, one can change R at will by any cofibrant representative in its homotopy class.

If R is a plain commutative algebra (i.e., a DG algebra sitting in degree 0) and $\mathcal{L}$ a plain Lie R-algebroid, then BV extensions of $\mathcal{L}$ are the same as *right* $\mathcal{L}$-module structures on R. Then the BV envelope is the de Rham-Chevalley complex for this $\mathcal{L}$-module (see 2.9.1); as a mere graded algebra it equals $\mathrm{Sym}(\mathcal{L}[1])$. They were considered in [**Kos**] (the case of the tangent algebroid), [**Hue**], [**X**], and (under the name of Calabi-Yau or vertex 0-algebroid structures) in §11 of [**GMS2**].

(vi) In 4.1.14 and 4.1.15 we consider homotopy unital commutative algebras and BV algebras and show that the corresponding homotopy categories are the same as the homotopy categories of the corresponding strictly unital objects. We need this material since the chiral chain complexes of unital chiral algebras are naturally homotopy unital BV algebras (not the strict ones); see 4.3.4.

(vii) In 4.1.16–4.1.18 we discuss perfect complexes, perfect commutative DG algebras, and perfect BV algebras. Perfect commutative algebras are immediate counterparts in the homotopy DG world of the usual smooth algebras, and perfect BV algebras correspond to de Rham complexes for a flat connection on the line bundle ω^{-1}. The cohomology of a perfect BV algebra is finite-dimensional. We show in 4.6.9 that the chiral homology of a (very) smooth commutative $\mathcal{D}_X$-algebra is a perfect commutative algebra, and that of a cdo is a perfect BV algebra (see 4.8.5).

We refer to [**CFK**] for a geometric discussion of commutative DG algebras supported in non-positive degrees. The perfectness property for commutative DG algebras should be compared with a much stronger (and deeper) condition on a DG algebra F – perfectness of F as an $F \otimes F$-bimodule – introduced by M. Kontsevich (the latter property makes sense for arbitrary associative DG algebras).

4.1.1. Homotopy direct limits. Below $\mathcal{A}$ is a category which we tacitly assume to be closed under direct sums of sufficiently high cardinality.

(i) Let B be a simplicial set. So for every $n \geq 0$ we have a set B_n and for every monotonous map of intervals $\alpha : [0, \ldots, m] \to [0, \ldots, n]$ we have the corresponding map $\partial_\alpha : B_n \to B_m$ compatible with the composition of the α's.

Denote by $\mathcal{C}(B, \mathcal{A})$ the category of homology type coefficient systems on B with coefficients in $\mathcal{A}$. Thus $F \in \mathcal{C}(B, \mathcal{A})$ is a rule that assigns to every simplex $b \in B_n$ an object $F_b \in \mathcal{A}$ and to every α as above a morphism $\partial_\alpha = \partial_\alpha^F : F_b \to F_{\partial_\alpha(b)}$ compatible with the composition of the α's.

Denote by $\mathcal{C}_s(\mathcal{A})$ the category of simplicial objects in $\mathcal{A}$. We have a functor

$$C_s : \mathcal{C}(B, \mathcal{A}) \to \mathcal{C}_s(\mathcal{A}) \tag{4.1.1.1}$$

where $C_s(F)_n := \bigoplus_{b \in B_n} F_b$ and the structure maps $\partial_\alpha : C_s(F)_n \to C_s(F)_m$ are direct sums of the corresponding morphisms ∂_α^F.

(ii) Let $\mathcal{P}$ be a small category. It yields a simplicial set $B\mathcal{P}$ (the classifying simplicial set; see [**Se**] or [**Q3**]); an n-simplex $\tilde{p} \in B\mathcal{P}_n$ is a diagram $(p_0 \to \cdots \to p_n)$

in $\mathcal{P}$. Consider the category $\mathcal{F}un(\mathcal{P},\mathcal{A})$ of functors $F:\mathcal{P}\to\mathcal{A}$, $p\mapsto F_p$. There is a fully faithful embedding

$$\mathcal{F}un(\mathcal{P},\mathcal{A})\hookrightarrow\mathcal{C}(B\mathcal{P},\mathcal{A}) \tag{4.1.1.2}$$

which identifies a functor F with the system $F_{(p_0\to\cdots\to p_n)}=F_{p_0}$.

(iii) Suppose that $\mathcal{A}$ is a k-category. Let $C(\mathcal{A})$ be the category of complexes in $\mathcal{A}$. By Dold-Puppe one has the fully faithful embedding

$$\mathcal{N}orm : C_s(\mathcal{A})\hookrightarrow C(\mathcal{A}) \tag{4.1.1.3}$$

which identifies $C_s(\mathcal{A})$ with the full subcategory of complexes having degrees ≤ 0.

(iv) Suppose that $\mathcal{A}$ is a DG (super) k-category.[1] We have the functor $\operatorname{tot}: C(\mathcal{A})\to\mathcal{A}$ which sends a complex $C=(C^{\cdot},d)$ to an object $\operatorname{tot}C\in\mathcal{A}$ such that $\operatorname{tot}C=\oplus C^n[-n]$ as a plain object without differential; the structure differential is the sum of d and the structure differentials of $C^n[-n]$. Set

$$\mathrm{Norm}:=\operatorname{tot}\mathcal{N}orm : C_s(\mathcal{A})\to\mathcal{A} \tag{4.1.1.4}$$

For $F\in\mathcal{C}(B,\mathcal{A})$ as in (i) set $C(B,F):=\mathrm{Norm}\,C_s(B,F)$. For a functor $F:\mathcal{P}\to\mathcal{A}$ as in (ii) set $C(\mathcal{P},F):=C(B\mathcal{P},F)$; this is the *homotopy direct* $\mathcal{P}$*-limit* of F.

Notice that the DG structure on $\mathcal{A}$ yields DG structures on all the above categories, and (4.1.1.1)–(4.1.1.4) are DG functors.

Remark. Suppose that $\mathcal{A}=C(\mathcal{B})$ for an abelian category $\mathcal{B}$. Let $\varinjlim F\in C(\mathcal{B})$ be the plain direct $\mathcal{P}$-limit of F. There is an obvious canonical morphism of complexes $C(\mathcal{P},F)\to\varinjlim F$. If F takes values in $\mathcal{B}\subset\mathcal{A}$, then the complex $C(\mathcal{P},F)$ has degrees ≤ 0 and the above morphism yields an isomorphism $H^0C(\mathcal{P},F)\xrightarrow{\sim}\varinjlim F$.

Exercise. Let $S\in\mathcal{P}$ be an object such that the group $G:=\operatorname{Aut}P$ acts freely on every set $\operatorname{Hom}(S,T)$, $T\in\mathcal{P}$. Let $A\in\mathcal{A}$ be an object equipped with an $\operatorname{Aut}S$-action, and let $F=\operatorname{Ind}A$ be the corresponding induced functor, $F(T):=A[\operatorname{Hom}(S,T)]_G$. Then the complex $C(\mathcal{P},F)$ computes the homology of G with coefficients in A. In particular, if G is finite and we are dealing with k-categories where k is a field of characteristic 0, then the map $C(\mathcal{P},F)\to\varinjlim F=A_G$ is a quasi-isomorphism.

4.1.2. Operations. From now on $\mathcal{A}$ is a pseudo-tensor category; we assume that direct sums in $\mathcal{A}$ are compatible with operations.[2]

(i) Consider the category of k-modules. This is a tensor category, so both categories of simplicial k-modules and k-complexes are tensor categories. $\mathcal{N}orm$ is *not* a tensor functor. However for any finite set of simplicial k-modules N_i there is a canonical functorial morphism of complexes

$$c=c_{\{N_i\}}:\otimes\mathcal{N}orm(N_i)\to\mathcal{N}orm(\otimes N_i) \tag{4.1.2.1}$$

[1] We always assume $\mathcal{A}$ to be pretriangulated (the cones of morphisms are well defined, see [**BoKa**] or [**Dr1**]) and closed under appropriate direct limits.

[2] I.e., $P_I(\{\underset{\alpha\in A_i}{\oplus}F_{\alpha i}\},G)=\prod P_I(\{F_{\alpha i}\},G)$.

which equals $id_{\otimes N_i}$ if the N_i are constant simplicial k-modules;[3] such a c is unique. According to 1.1.6(ii), this makes $\mathcal{N}orm$ a pseudo-tensor functor.[4]

Now if $\mathcal{C}$ is any pseudo-tensor simplicial k-category, we define its *normalization* as a pseudo-tensor DG k-category $\mathcal{N}orm\,\mathcal{C}$ having the same objects as $\mathcal{C}$ and operations $P^{\mathcal{N}orm\,\mathcal{C}} := \mathcal{N}orm\,P^{\mathcal{C}}$. The composition of operation is defined using the above c.

Remarks. (a) The plain k-categories corresponding to $\mathcal{C}$ and $\mathcal{N}orm\,\mathcal{C}$ coincide.
(b) Morphisms (4.1.2.1) are homotopy equivalences of complexes.

(ii) $C_s(\mathcal{A})$ is a simplicial pseudo-tensor catregory. Namely, for $F_i, G \in C_s(\mathcal{A})$ the simplicial set $P_I(\{F_i\}, G)_s$ is defined as follows. Consider $P^a_b := P_I(\{F_{i\,a}\}, G_b)$; they form a cosimplicial-simplicial set P. Our $P_I(\{F_i\}, G)_s$ is the corresponding total simplicial set $\mathcal{T}ot\,P$ (for the definition of $\mathcal{T}ot$ see, e.g., [**BK**]).

Remark. If $\mathcal{A}$ is a tensor category, then $C_s(\mathcal{A})$ has an obvious structure of a tensor simplicial category. The above pseudo-tensor simplicial structure comes from this tensor simplicial structure.

(iii) If $\mathcal{A}$ is a pseudo-tensor k-category, then $C(\mathcal{A})$ is a pseudo-tensor DG k-category. The normalization functor extends in the obvious way to a pseudo-tensor DG functor

$$\mathcal{N}orm : \mathcal{N}orm C_s(\mathcal{A}) \to C(\mathcal{A}). \tag{4.1.2.2}$$

(iv) Assume that $\mathcal{A}$ is a pseudo-tensor DG (super) k-category. Then the categories in (4.1.2.2) are bi-DG categories and $\mathcal{N}orm$ is a pseudo-tensor bi-DG functor. Let us consider them as DG categories (with respect to the total grading and differential).

The functor tot: $C(\mathcal{A}) \to \mathcal{A}$ is a pseudo-tensor DG functor in the obvious way. We get a pseudo-tensor DG functor

$$\mathrm{Norm} := \mathrm{tot}\,\mathcal{N}orm : C_s(\mathcal{A}) \to \mathcal{A}. \tag{4.1.2.3}$$

4.1.3. Dolbeault algebras. Below "scheme" means "k-scheme" where k is our base field of characteristic 0.

Definition. Let X be a scheme. A *Dolbeault $\mathcal{O}_X$-algebra* is a commutative unital DG $\mathcal{O}_X$-algebra $\mathcal{Q}$, quasi-coherent as an $\mathcal{O}_X$-module, such that:
(a) The structure morphism $\mathcal{O}_X \xrightarrow{\alpha} \mathcal{Q}$ is a quasi-isomorphism.
(b) $\mathcal{Q}$ is homotopically $\mathcal{O}_X$-flat (see 2.1.1).
(c) Spec $\mathcal{Q}^0$ is an affine scheme.
By (c) for each $\mathcal{O}_X$-quasi-coherent $\mathcal{Q}$-module N one has $\Gamma(X,N) \xrightarrow{\sim} R\Gamma(X,N)$.
A *Dolbeault $\mathcal{D}_X$-algebra* is a DG $\mathcal{D}_X$-algebra which is a Dolbeault $\mathcal{O}_X$-algebra.

Lemma. *If X is separated and quasi-compact, then it admits a Dolbeault $\mathcal{O}_X$-algebra. If, in addition, X is smooth, then it admits a Dolbeault $\mathcal{D}_X$-algebra.*

Proof. We present two constructions; for the second one X has to be quasi-projective. In both situations the Dolbeault algebras we define satisfy $\mathcal{Q}^{<0} = 0$.
(i) Let us begin with the Thom-Sullivan construction (see [**HS**] §4).

[3] Which amounts to the fact that the $\mathcal{N}orm(N_i)$ are complexes supported in degree 0.
[4] The associativity diagram from 1.1.6(ii) for $\tau = \mathcal{N}orm$, $\nu = c$ is commutative due to the uniqueness property of c.

For a finite set I we denote by Δ_I the subscheme of $\mathbb{A}^I$ defined by the equation $\Sigma t_i = 1$. Let $\Omega_I := \Gamma(\Delta_I, DR_{\Delta_I})$ be the de Rham algebra of Δ_I; this is a resolution of k. For any $I' \subset I$ we have an evident embedding $\Delta_{I'} \subset \Delta_I$, hence a projection $\Omega_I \to \Omega_{I'}$.

Choose a finite affine covering $\{U_s\}$, $s \in S$, of X. For a non-empty subset $I \subset S$ we have an affine embedding $j_I : U_I := \bigcap\limits_{i \in I} U_i \hookrightarrow X$. Our $\mathcal{Q}$ is a subalgebra of the DG algebra $\prod\limits_I j_{I*}\mathcal{O}_{U_I} \otimes \Omega_I$ which consists of those collections (f_I) that for every $I' \subset I$ the images of $f_{I'}$, f_I under the maps $j_{I'*}\mathcal{O}_{U_{I'}} \otimes \Omega_{I'} \to j_{I*}\mathcal{O}_{U_I} \otimes \Omega_{I'} \leftarrow j_{I*}\mathcal{O}_{U_I} \otimes \Omega_I$ coincide.

It follows easily from [**HS**] 4.1 that $\mathcal{Q}$ is a Dolbeault $\mathcal{O}_X$-algebra. If X is smooth, then it is a $\mathcal{D}_X$-algebra in the evident way.

Exercise. If X is a projective curve, then the $\mathcal{O}_X$-algebra $\mathcal{Q}^0$ is *not* finitely generated.

(ii) Here is another construction. We suppose that X is quasi-projective. The Dolbeault algebras we will construct have the property that each $\mathcal{Q}^i$ is a locally free $\mathcal{O}_X$-module.

Let $\pi : Y \to X$ be a Jouanolou map; i.e., Y is a torsor over X with respect to an action of some vector bundle such that Y is an affine scheme.[5] Consider now the relative de Rham complex $DR(Y/X)$. It is clear that $\mathcal{Q} := \pi_* DR_{Y/X}$ is a Dolbeault $\mathcal{O}_X$-algebra.

Suppose X is smooth. Then the jet algebra $\mathcal{J}\mathcal{Q} = DR(\mathcal{J}\mathcal{Q}^0/\mathcal{O}_X)$[6] (see 2.3.2) is a Dolbeault $\mathcal{D}_X$-algebra. Indeed, it obviously satisfies conditions (a) and (b), and condition (c) follows since the morphism $\operatorname{Spec} \mathcal{J}\mathcal{Q}^0 = \mathcal{J}Y \to Y$ is affine. □

In the next lemma (parallel to the lemma from 2.2.10) we consider the category of all Dolbeault algebras in either $\mathcal{O}_X$- or $\mathcal{D}_X$-setting. Since the tensor product of two Dolbeault algebras is obviously a Dolbeault algebra, this is a tensor category.

Lemma. *Every functor from the category of Dolbeault algebras to a groupoid is isomorphic to a trivial functor.*

Proof. Denote our functor by $\bar{\ }$.

(i) Let $\zeta, \eta : \mathcal{Q}_1 \to \mathcal{Q}_2$ be any two morphisms. Let us show that $\bar{\zeta} = \bar{\eta}$. Consider the morphisms $\chi, \chi' : \mathcal{Q}_1 \to \mathcal{Q}_1^{\otimes 2}$, $\mu : \mathcal{Q}_1^{\otimes 2} \to \mathcal{Q}_1$, $\xi : \mathcal{Q}_1^{\otimes 2} \to \mathcal{Q}_2$ where $\chi(a) = a \otimes 1$, $\chi'(a) = 1 \otimes a$, $\mu(a \otimes b) = ab$, $\xi(a \otimes b) = \zeta(a)\eta(b)$. Since $\mu\chi = \mu\chi'$, one has $\bar{\chi} = \bar{\chi}'$. Since $\xi\chi = \zeta$, $\xi\chi' = \eta$, we have $\bar{\zeta} = \bar{\eta}$.

(ii) For any finite collection $\{\mathcal{Q}_\alpha\}$ of Dolbeault algebras one can find a Dolbeault algebra $\mathcal{Q}$ such that for every α there is a morphism $\mathcal{Q}_\alpha \to \mathcal{Q}$. Indeed, one can take $\mathcal{Q} = \otimes \mathcal{Q}_\alpha$ and the standard morphisms.

(iii) To finish the proof, let us show that for a pair $\mathcal{Q}_0$, $\mathcal{Q}_n$ and a chain of morphisms $\mathcal{Q}_0 \xleftarrow{\chi_1} \mathcal{Q}'_1 \xrightarrow{\psi_1} \mathcal{Q}_1 \xleftarrow{\chi_2} \cdots \xleftarrow{\chi_n} \mathcal{Q}'_n \xrightarrow{\psi_n} \mathcal{Q}_n$ the composition $\bar{\psi}_n \bar{\chi}_n^{-1} \cdots \bar{\psi}_1 \bar{\chi}_1^{-1}$:

[5]Recall that one constructs Y as follows. Choose an open embedding $X \subset \bar{X}$ such that $\bar{X}$ is a projective variety. Blowing $\bar{X} \smallsetminus X$ up if necessary, we can assume that $\bar{X} \smallsetminus X \subset \bar{X}$ is locally defined by one equation. Choose an embedding $\bar{X} \hookrightarrow \mathbb{P}^n$. Let $V \subset \mathbb{P}^{n*} \times \mathbb{P}^n$ be the complement to the incidence divisor. Set $Y := X \underset{\mathbb{P}^n}{\times} V$.

[6]The fact that functors DR and $\mathcal{J}$ commute follows from their universal properties. Precisely, for an $\mathcal{O}_X$-algebra B both $DR(\mathcal{J}B)$ and $\mathcal{J}DR(B)$ represent the same functor $F \mapsto \operatorname{Hom}_{\mathcal{O}_X}(B, F^0)$ on the category of commutative DG $\mathcal{D}_X$-algebras.

$\bar{\mathcal{Q}}_0 \to \bar{\mathcal{Q}}_n$ does not depend on the chain. To see this, choose $\phi_i : \mathcal{Q}_i \to Q$ as in (ii). Since $\bar{\phi}_{i-1}\bar{\chi}_i = \bar{\phi}_i\bar{\psi}_i$ by (i), our composition equals $\bar{\phi}_n^{-1}\bar{\phi}_0$; q.e.d. $\square$

4.1.4. Dolbeault resolutions. Let $\mathcal{Q}$ be a Dolbeault $\mathcal{D}_X$-algebra. For a $\mathcal{D}_X$-complex the morphism $\alpha_M : M \to M \otimes \mathcal{Q}$ is called a Dolbeault resolution of M. It is a quasi-isomorphism by (a), (b) from the definition in 4.1.3, and by (c) the morphisms $\Gamma(X, M \otimes \mathcal{Q}) \to R\Gamma(X, M \otimes \mathcal{Q})$, $\Gamma_{DR}(X, M \otimes \mathcal{Q}) \to R\Gamma_{DR}(X, M \otimes \mathcal{Q})$ are quasi-isomorphisms. Thus we have canonical quasi-isomorphisms

$$\Gamma(X, M \otimes \mathcal{Q}) \xrightarrow{\sim} R\Gamma(X, M), \quad \Gamma_{DR}(X, M \otimes \mathcal{Q}) \xrightarrow{\sim} R\Gamma_{DR}(X, M). \tag{4.1.4.1}$$

REMARKS. (i) Another way to compute $R\Gamma_{DR}(X, M)$ is to use a smaller complex $\Gamma_{DR}(Y, \pi^* M)$ where $\pi : Y \to X$ is a Jouanolou map. We will use Dolbeault $\mathcal{D}$-resolutions for the computations of the de Rham homology of $\mathcal{D}$-modules on $\mathcal{R}(X)$ (see 4.2); it is not clear if one can do this using the above type of complexes.

(ii) Let $i : X \hookrightarrow Z$ be a closed embedding of smooth varieties. The for $\mathcal{Q}$, M as above one has $\Gamma_{DR}(Z, i_*(M \otimes \mathcal{Q})) \xrightarrow{\sim} R\Gamma_{DR}(Z, i_* M)$.[7]

(iii) According to Remark (i) in 1.4.6 the functor $C\mathcal{M}(X) \to C\mathcal{M}(X, \mathcal{Q})$, $M \mapsto M \otimes \mathcal{Q}$, is a compound tensor DG functor (here C denotes the category of complexes or DG modules). The "forgetting of the $\mathcal{Q}$-action" functor $C\mathcal{M}(X, \mathcal{Q}) \to C\mathcal{M}(X)$ is a $*$ pseudo-tensor functor. So $\alpha_M : M \to M \otimes \mathcal{Q}$ is a morphism of $*$ pseudo-tensor functors. Thus for $\varphi \in P_I^*(\{M_i\}, N)$ its action on $R\Gamma_{DR}$ coincides with the morphism of complexes $\underset{I}{\boxtimes}\Gamma_{DR}(X, M_i \otimes \mathcal{Q}) \to \Gamma_{DR}(X^I, \Delta_*^{(I)} N \otimes \mathcal{Q})$.

LEMMA. *Suppose X is a curve and $N \in \mathcal{M}(X)$ is such that $H^1(X, N) = 0$. Then $H^{\geq 1}(X, h(N)) = 0$. In particular, if $\mathcal{Q}$ is a Dolbeault $\mathcal{D}_X$-algebra, then for any $M \in C\mathcal{M}(X)$ one has $\Gamma(X, h(M \otimes \mathcal{Q})) \xrightarrow{\sim} R\Gamma(X, h(M \otimes \mathcal{Q}))$. So, if M is homotopically quasi-induced (see 2.1.11), then $\Gamma(X, h(M \otimes \mathcal{Q}))$ computes $R\Gamma_{DR}(X, M)$.*

Proof. Set $K := \mathrm{Ker}(N \twoheadrightarrow h(N))$. Then $H^i(X, h(N)) = H^{i+1}(X, K)$ for $i \geq 1$, and the latter groups vanish since $\dim X = 1$. $\square$

Sometimes it is convenient to deal with non-quasi-coherent versions of Dolbeault algebras. Precisely, suppose we have a commutative unital DG $\mathcal{O}_X$-algebra $\mathcal{Q}$ (which we do not assume to be $\mathcal{O}_X$-quasi-coherent). We say that $\mathcal{Q}$ is a *Dolbeault-style $\mathcal{O}_X$-algebra* if for every quasi-coherent $\mathcal{O}_X$-module M the obvious morphisms $\Gamma(X, M \otimes \mathcal{Q}) \to R\Gamma(X, M \otimes \mathcal{Q})$, $R\Gamma(X, M) \to R\Gamma(X, M \otimes \mathcal{Q})$ are quasi-isomorphisms. For a smooth X, a *Dolbeault-style $\mathcal{D}_X$-algebra* is a Dolbeault-style $\mathcal{O}_X$-algebra equipped with a flat connection. The protoexample is the classical Dolbeault algebra: assuming that $k = \mathbb{C}$, X is smooth and proper, we take for $\mathcal{Q}(U)$, where U/X is étale, the sections of the $\bar{\partial}$-resolution $\Omega_U^{0,0} \xrightarrow{\bar{\partial}} \Omega_U^{0,1} \xrightarrow{\bar{\partial}} \cdots$ of the algebra of holomorphic functions. Quasi-isomorphisms (4.1.4.1) are valid for any Dolbeault-style $\mathcal{D}_X$-algebra $\mathcal{Q}$.

4.1.5. A reminder on semi-free objects and the cotangent complex. We follow the super conventions of 1.1.16, as always dropping the adjective "super"; the base field k is of characteristic 0. Commutative and associative DG k-algebras are called simply commutative algebras; the corresponding category is

[7] To see this, notice that the terms of the complex $DR(i_*(M \otimes \mathcal{Q}))$ admit an increasing bounded below filtration whose successive quotients are $\mathcal{Q}^0$-modules; hence their higher cohomology vanishes.

denoted by $\mathcal{C}om$. The subcategory of unital algebras is denoted by $\mathcal{C}omu$. These are tensor categories.

For all details for this section we refer to, e.g., [**H**] or [**Dr1**].

Suppose we have $F \in \mathcal{C}omu$. A (DG) F-module P is said to be *semi-free* if it admits a filtration $P_0 \subset P_1 \subset \cdots$ by DG submodules such that $\bigcup P_i = P$ and each $\mathrm{gr}_i P$ is a free F-module.[8] Equivalently, P is semi-free if and only if it admits a system of *semi-free generators*, i.e., a base $\{e_i\}$ of P considered as a mere graded F-module, with index set I equipped with a projection $I \to \mathbb{Z}_{\geq 0}$, $i \mapsto n(i)$, such that for every $i \in I$ the element $d(e_i) \in P$ is a linear combination of e_j's with $n(j) < n(i)$ (so $d(e_i) = 0$ if $n(i) = 0$). Homotopically projective (= K-projective in the sense of [**Sp**]) F-modules are the same as direct summands of semi-free F-modules. Every F-module admits a semi-free left resolution.

An algebra $F \in \mathcal{C}omu$ is said to be *semi-free* if it admits a system of *semi-free generators*, i.e., a system of elements $\{f_i\} \subset F$ indexed by a set I equipped with a projection $I \to \mathbb{Z}_{\geq 0}$, $i \mapsto n(i)$, such that for every $i \in I$ the element $d(f_i)$ belongs to the subalgebra generated by f_j, $n(j) < n(i)$, and the $\{f_i\}$ freely generate F as a mere graded commutative algebra. Any commutative algebra admits a semi-free resolution.

The category $\mathcal{C}omu$ of commutative unital DG algebras is naturally a closed model category (with quasi-isomorphisms as weak equivalences, surjective morphisms as fibrations). Its cofibrant objects are the same as retracts of semi-free algebras.

For $F \in \mathcal{C}omu$ its *cotangent complex* (or the *cotangent F-module*) ${}^L\Omega_F$ is an object of the derived category $D(F)$ defined as follows. The cotangent module is equipped with a canonical morphism ${}^L\Omega_F \to \Omega_F$. If F is semi-free, then this morphism is an isomorphism in $D(F)$. Otherwise one chooses a semi-free resolution of F and defines ${}^L\Omega_F$ as the image of $\Omega_{\tilde{F}}$ under the canonical equivalence of categories $D(\tilde{F}) \xrightarrow{\sim} D(F)$; one checks that this definition does not depend on the choice of $\tilde{F}$; i.e., for different $\tilde{F}$ the corresponding objects are canonically identified.

4.1.6. Batalin-Vilkovisky algebras. The *Batalin-Vilkovisky operad* BV is a DG k-operad which coincides with the 1-Poisson (alias braid, alias Gerstenhaber) operad as a mere $\mathbb{Z}$-graded super operad. So it is generated by binary operations $\cdot$ of degree 0 (the product) and $\{\ \}$ of degree 1 (the 1-Poisson bracket) that satisfy the usual relations. The differential is determined by the property $d(\cdot) = \{\ \}$. Therefore for a complex C a BV algebra structure on C is a 1-Poisson structure on C considered as a mere graded vector space (see 1.4.18) such that $d(\cdot_C) := d_C \cdot_C - \cdot_C\, d_{C\otimes C} = \{\ \}_C$.

Remark. So $\{\ \}_C$ measures the extent to which d_C is *not* a derivation of $\cdot_C$. In fact, d_C is a second order differential operator with respect to the commutative algebra structure; its symbol equals $\{\ \}$.

Notice that the action of the Lie algebra $L := C[-1]$ on C extends canonically to an action of the contractible Lie algebra $L_\dagger$ (see 1.1.16). Namely, the component $L[1] = C \subset L_\dagger$ acts on C as $\cdot$.

For a BV algebra C a structure of a BV C-module on a complex M amounts to a BV algebra structure on $C \oplus M$ such that $C \leftrightarrows C \oplus M$ are morphisms of BV algebras and $\cdot$, hence $\{\ \}$, vanishes on $M \subset C \oplus M$. For $m \in M$ its *centralizer*

[8]I.e., is isomorphic to a direct sum of modules isomorphic to $F[n]$, $n \in \mathbb{Z}$.

$\mathcal{C}ent(m)$ consists of all $c \in C$ such that $\{c, m\} = 0$; m is *C-central* if $\mathcal{C}ent(m) = C$. Notice that C-central elements form a subcomplex $M_c \subset M$, and the product map defines a morphism of complexes $\cdot : C \otimes M_c \to M$.

The quotient operad of BV modulo the relation $\{\ \} = 0$ is the unit operad $\mathcal{C}om$. Therefore a commutative (DG) algebra is the same as a *commutative* BV algebra, i.e., a BV algebra C with $\{\ \}_C = 0$.

Suppose we have a $k[t]$-flat family of BV algebras C_t such that C_0 is commutative. Then C_0 acquires a 1-Poisson bracket $\{\ \}$, $\{a_0, b_0\} := (t^{-1}\{a_t, b_t\}_{C_t}) \bmod t$. We refer to C_t as *BV quantization* of the 1-Poisson algebra $(C_0, \{\ \})$.

Example. Let C be any BV algebra. Consider C_t which coincides with $C[t]$ as a mere commutative graded algebra and has differential $d_{C_t} := td_C$ (so $\{\ \}_{C_t} = t\{\ \}_C$). Then C_t is a BV quantization of C considered as a 1-Poisson algebra with zero differential.

Notice that the more non-degenerate $\{\ \}$ on $H^{\cdot} C_0$ is, the fewer cohomology classes of C_0 survive the quantization.

The BV operad is acyclic: one has $H^{\cdot}(\mathrm{BV}_n) = 0$ for $n > 0$. So the homotopy category of BV algebras coincides with that $D(k)$. An interesting homotopy BV theory arises in a *filtered* setting. Namely, BV is naturally a DG *filtered* operad: the (increasing) filtration is the stupid one $BV^{\geq -n}$. Notice that $\mathrm{gr}\,\mathrm{BV}$ equals the 1-Poisson operad (the differential is trivial). A *filtered* BV algebra is a complex C equipped with a BV algebra structure and an increasing filtration which is compatible with the BV algebra structure (i.e., the products $\mathrm{BV}_n \otimes C^{\otimes n} \to C$ are compatible with the filtrations). This amounts to the property that the filtration on C is compatible with the product $\cdot$ and with the differential, and the induced product on $\mathrm{gr}_. C$ is compatible with the differential. So $\mathrm{gr}_. C$ is a 1-Poisson DG algebra (and $C_t := \oplus C_i$ is its BV quantization).

We denote be $\mathcal{BV}$ the category of filtered BV algebras C such that $C_{-1} = 0$, $\bigcup C_n = C$. Let $\bar{\mathcal{BV}} \subset \mathcal{BV}$ be the full subcategory of those C for which $C_0 = 0$. Notice that for any $C \in \mathcal{BV}$ the odd Poisson bracket on C_0 vanishes, C_0 is a commutative DG algebra, and $C_1[-1]$ is a Lie DG algebra with respect to $\{\ \}$.

Remark. The embedding $\bar{\mathcal{BV}} \hookrightarrow \mathcal{BV}$ admits a right adjoint $\mathcal{BV} \to \bar{\mathcal{BV}}$ which assigns to $C_. \in \mathcal{BV}$ the same C with a new filtration which is the old C_i for $i > 0$, and the new C_0 equals 0.

The BV operad is augmented in the obvious way. So we have the notion of a *unital* BV algebra. Explicitly, a BV algebra C is unital if it has a unit $1 \in C^0$ with respect to $\cdot$ such that $d(1) = 0$ (then 1 lies in the $\{\ \}$-center of C). In the filtered setting we assume that $1 \in C_0$. The subcategory of unital filtered algebras in $\mathcal{BV}$ is denoted by $\mathcal{BV}_u$. The embedding $\mathcal{BV}_u \to \mathcal{BV}$ admits an obvious left adjoint (adding the unit) $\mathcal{BV} \to \mathcal{BV}_u$.

The DG operad BV has a canonical coproduct $\delta : \mathrm{BV} \to \mathrm{BV} \otimes \mathrm{BV}$, $\delta(\cdot) = \cdot \otimes \cdot$, so we know what the tensor product of the BV algebras is (this is the usual tensor product of 1-Poisson algebras with the obvious differential). The tensor product of BV compatible filtrations is BV compatible, so we know what the tensor product of filtered BV algebras is. The tensor product of unital algebras is obviously unital.

4.1.7. Proposition. *$\mathcal{BV}$, $\bar{\mathcal{BV}}$, and $\mathcal{BV}_u$ are closed model categories with weak equivalences being filtered quasi-isomorphisms and fibrations those morphisms f for which* $\mathrm{gr}\, f$ *is surjective.*

Sketch of a proof. The reference [**H**], strictly speaking, does not cover our filtered setting, but the arguments easily adopt to it. Here are the needed changes; we consider the case of $\mathcal{BV}$. The general Theorem 2.2.1 of [**H**] remains valid if we replace the category of complexes $C(k)$ by the DG category of filtered complexes $C.$ such that $C_{-1}=0$, $\bigcup C_i = C$. In Axiom (H1) from [**H**] 2.2 we assume that M is contractible as a filtered complex, and in the definition of the CMC structure on $\mathcal{C}$ one takes for weak equivalences filtered quasi-isomorphisms, for fibrations those f for which $\mathrm{gr}(f^\natural)$ is surjective. The only modifications in the proof are that in the definition of standard cofibration [**H**] 2.2.3(i) one takes for M any filtered complex with $\mathrm{gr}(d)=0$, and in that of the standard acyclic cofibration [**H**] 2.2.3(ii) a contractible filtered complex. The proof that $\mathcal{BV}$ fits into this framework coincides with that of Theorem 4.1.1 of [**H**]. □

Below, the homotopy category of a closed model category $\mathcal{C}$ is denoted by $\mathcal{H}o\mathcal{C}$.

The embedding $\bar{\mathcal{BV}} \hookrightarrow \mathcal{BV}$ identifies $\mathcal{H}o\bar{\mathcal{BV}}$ with the full subcategory of $\mathcal{H}o\mathcal{BV}$ that consists of BV algebras C such that C_0 is acyclic.[9] The forgetting and adding of the unit functors $\mathcal{H}o\mathcal{BV}_u \rightleftarrows \mathcal{H}o\mathcal{BV}$ remain adjoint on the level of homotopy categories.

The notion of a filtered BV algebra makes sense in any abelian tensor k-category $\mathcal{A}$; if $\mathcal{A}$ has a unit object, then we can consider unital filtered BV algebras. The corresponding categories are denoted by $\mathcal{BV}(\mathcal{A})$, $\bar{\mathcal{BV}}(\mathcal{A})$, and $\mathcal{BV}_u(\mathcal{A})$.

Remark. In fact, the notion of a BV algebra makes sense in any DG pseudo-tensor category and that of a filtered BV algebra in a filtered DG pseudo-tensor category. Unital algebras make sense in the augmented setting (see 1.2.8).

4.1.8. BV enveloping algebras. Below we write down several constructions of BV algebras. Let us begin with the BV envelopes of Lie algebras which are the same as (homological) Chevalley complexes.

(a) Let $\mathcal{L}ie$ be the category of Lie DG algebras. The obvious functor $\bar{\mathcal{BV}} \to \mathcal{L}ie$, $C \mapsto C_1[-1]$, admits left adjoint $\bar{C} : \mathcal{L}ie \to \bar{\mathcal{BV}}$. Similarly, we have a pair of adjoint functors $\mathcal{BV}_u \to \mathcal{L}ie$, $C : \mathcal{L}ie \to \mathcal{BV}_u$ where C is the composition of $\bar{C}$ and the adding of the unit.

For $L \in \mathcal{L}ie$ the corresponding $C(L)$ is the Chevalley complex of L, and $\bar{C}(L)$ is the reduced Chevalley complex. As a plain graded commutative algebra, $C(L)$ equals $\mathrm{Sym}(L[1])$, the filtration $C(L)_i$ is $\mathrm{Sym}^{\leq i}(L[1])$, the differential and the 1-Poisson bracket are determined by the condition that the embedding $L = \mathrm{Sym}^1(L[1])[-1] \subset C[-1]$ is a morphism of Lie DG algebras. Similarly, as a plain graded commutative algebra, $\bar{C}(L)$ equals $\mathrm{Sym}^{>0}(L[1])$, etc.

Our functors preserve quasi-isomorphisms so they descend to homotopy categories; we get pairs of adjoint functors $\mathcal{H}o\mathcal{L}ie \leftrightarrows \mathcal{H}o\mathcal{BV}$, $\mathcal{H}o\mathcal{L}ie \leftrightarrows \mathcal{H}o\mathcal{BV}_u$.

Remark. The above definitions make sense in any abelian tensor k-category $\mathcal{A}$ (for the unital setting we have to assume that $\mathcal{A}$ has a unit).

(b) More generally, suppose we have a filtration $L_0 \subset L_1 \subset \cdots$ on L such that $\bigcup L_i = L$, $[L_i, L_j] \subset L_{i+j-1}$ (we call such an $L.$ a *commutative filtration* on L). Then the filtration $C(L).$ on $C(L)$ generated by $L.$ (as on a commutative algebra generated by $L.[1]$) is compatible with the BV structure. Denote by $\mathcal{L}ie.$ the category of Lie DG algebras equipped with a commutative filtration. Then

[9]The inverse functor comes from Remark above.

the functor $\mathcal{L}ie. \to \mathcal{BV}_u$ which assigns to $L.$ its Chevalley complex filtered in the above manner is left adjoint to the functor $\mathcal{BV}_u \to \mathcal{L}ie$ which assigns to a filtered BV algebra $C \in \mathcal{BV}_u$ the filtered Lie DG algebra $C[-1]$. The same is true for a non-unital version.

The filtration on $C(L)$ we considered in (a) corresponds to $L_0 = 0$, $L_1 = L$.

(c) Suppose we have a central extension $L^\flat$ of L by $k[-1]$. Consider the Chevalley complex $C(L^\flat)$ filtered according to the filtration $L_0^\flat := k[-1]$, $L_1^\flat = L^\flat$. Then $C(L^\flat)_0 = \mathrm{Sym}(k) = k[t]$, so $C(L^\flat)$ is a filtered BV $k[t]$-algebra. Set $C(L)^\flat := C(L^\flat)_{t=1} \in \mathcal{BV}_u$; this is the $\flat$-*twisted* Chevalley complex of L. The filtration on $C(L)^\flat$ is called the *standard* filtration.

4.1.9. BV extensions. To define the BV envelope of a Lie *algebroid*, one needs an extra structure of its BV extension. The situation is parallel to that of chiral envelopes of Lie* algebroids considered in see 3.9.

Suppose we have $R \in \mathcal{C}omu$ and a DG Lie R-algebroid $\mathcal{L}$ (see 2.9.1). A *BV extension* of $\mathcal{L}$ is an extension of complexes $0 \to R[-1] \xrightarrow{\iota} \mathcal{L}^\flat \xrightarrow{\pi} \mathcal{L} \to 0$ together with a Lie R-algebroid structure on $\mathcal{L}^\flat$ as on a mere graded module (i.e., with differentials forgotten). The following properties should hold:

(i) π is a morphism of graded Lie R-algebroids, ι a morphism of R-modules, and $\iota(k[-1])$ belongs to the center of $\mathcal{L}^\flat$. The Lie bracket on $\mathcal{L}^\flat$ is compatible with the differential.

(ii) The morphism[10] $\mathcal{L}^\flat \otimes R = R \otimes \mathcal{L}^\flat \xrightarrow{d(\cdot)} \mathcal{L}^\flat[1]$ equals $-\iota$ composed with the structure action of $\mathcal{L}^\flat$ on R.

We call such $(\mathcal{L}, \mathcal{L}^\flat)$ a *BV Lie R-algebroid*, and abbreviate it to $\mathcal{L}^\flat$. The pairs $(R, \mathcal{L})$ and $(R, \mathcal{L}^\flat)$ form the categories $\mathcal{L}ie\mathcal{A}lg$ and $\mathcal{L}ie\mathcal{A}lg^{BV}$. So a morphism $\phi : (R_1, \mathcal{L}_1) \to (R_1, \mathcal{L}_2)$ is a pair $(\phi_R, \phi_\mathcal{L})$ of a morphism $\phi_R : R_1 \to R_2$ in $\mathcal{C}omu$ and a morphism $\phi_\mathcal{L} : \mathcal{L}_1 \to \mathcal{L}_2$ in $\mathcal{L}ie$ which are compatible in the obvious sense, a morphism in $\mathcal{L}ie\mathcal{A}lg^{BV}$ is a triple $\phi^\flat = (\phi_R, \phi_\mathcal{L}, \phi_{\mathcal{L}^\flat})$, etc. For a fixed R the categories of (BV) Lie R-algebroids are denoted by $\mathcal{L}ie\mathcal{A}lg_R$, $\mathcal{L}ie\mathcal{A}lg_R^{BV}$.

EXAMPLE. Let L be a Lie algebra acting on R, $L^\flat$ a central extension of L by $k[-1]$. It yields the L-rigidified Lie R-algebroid $L_R := R \otimes L$ and $L_R^\flat \in \mathcal{P}^c(L_R)$ (see 2.9.1). We also have a BV extension $L_R^{\flat BV} \in \mathcal{P}^{BV}(\mathcal{L})$ which equals $\mathcal{L}_R^\flat$ as a mere graded Lie R-algebroid and whose differential is $r \otimes \ell^\flat \mapsto d_{\mathcal{L}_R^\flat}(r \otimes \ell^\flat) - \iota(\ell(r))$. We refer to $L_R^{\flat BV}$ as the $L^\flat$-*rigidified* BV extension of L_R. If $L^\flat$ is the trivialized extension of L, then $L_R^{\flat BV}$ is called the L-*rigidified* BV extension.

Denote by $\mathcal{P}^c(\mathcal{L})$ the groupoid of Lie R-algebroid extensions $\mathcal{L}^c$ of $\mathcal{L}$ by $R[-1]$ (we assume that $k[-1] \subset R[-1]$ belongs to the center of $\mathcal{L}^c$) and by $\mathcal{P}^{BV}(\mathcal{L})$ the groupoid of BV extensions. The Baer sum defines a Picard groupoid (in fact, a k-vector space groupoid) structure on $\mathcal{P}^c(\mathcal{L})$ and makes $\mathcal{P}^{BV}(\mathcal{L})$, if non-empty, a $\mathcal{P}^c(\mathcal{L})$-torsor.

Let $\mathcal{P}_s^{BV}(\mathcal{L})$ be the set of (isomorphism classes of) pairs $(\mathcal{L}^\flat, s)$ where $s : \mathcal{L} \to \mathcal{L}^\flat$ is an R-linear section of π compatible with the grading, but not necessary with the bracket and the differential. Define $\mathcal{P}_s^c(\mathcal{L})$ in a similar way. Then $\mathcal{P}_s^c(\mathcal{L})$ is a vector space, and $\mathcal{P}_s^{BV}(\mathcal{L})$, if non-empty, is a $\mathcal{P}_s^c(\mathcal{L})$-torsor. Let us describe them explicitly.

[10]Here $\cdot$ is the R-action on $\mathcal{L}^\flat$.

For $(\mathcal{L}^\flat, s) \in \mathcal{P}_s^{BV}(\mathcal{L})$ consider a pair (ω^{BV}, μ^{BV}) where $\omega^{BV} : \mathcal{L} \times \mathcal{L} \to R[-1]$, $\mu^{BV} : \mathcal{L} \to R$ are maps $\iota\omega^{BV}(\ell, \ell') := [s(\ell), s(\ell')] - s([\ell, \ell'])$, $\iota\mu^{BV} := [d, s]$. For $(\mathcal{L}^\flat, s) \in \mathcal{P}_s^{BV}(\mathcal{L})$ we have a similarly defined pair (ω^c, μ^c).

Lemma. *Both ω^{BV}, $\omega^c \in \mathrm{Hom}_R(\Lambda^2\mathcal{L}, R[-1])$ are 2-cocycles of $\mathcal{L}$ considered as a mere graded R-algebroid. The map $\mu^c : \mathcal{L} \to R$ is R-linear, and for $r \in R$, $\ell \in \mathcal{L}$ one has $\mu^{BV}(r\ell) = r\mu^{BV}(\ell) - \ell(r)$. Both in BV and in the c setting, one has $[d, \mu] = 0$ and $[d, \omega](\ell, \ell') = \ell(\mu(\ell')) - \ell'(\mu(\ell)) - \mu([\ell, \ell'])$.*

The maps $(\mathcal{L}^\flat, s) \mapsto (\omega^{BV}, \mu^{BV})$, $(\mathcal{L}^c, s) \mapsto (\omega^c, \mu^c)$ are bijections between $\mathcal{P}_s^{BV}(\mathcal{L})$, $\mathcal{P}_s^c(\mathcal{L})$ and the sets of pairs (ω, μ) that satisfy the above conditions. □

Remarks. (i) Suppose that $s : \mathcal{L} \to \mathcal{L}^\flat$ commutes with the bracket; i.e., it is a morphism of graded Lie R-algebroids. This amounts to the vanishing of ω^{BV}, and the equation on μ^{BV} just means that the formula $r \cdot \ell := -\ell(r) + r\mu^{BV}(\ell)$ is a *right* $\mathcal{L}$-module structure on the R-module R. Therefore one gets a bijection between the set of pairs $(\mathcal{L}^\flat, s)$ as above and the set of *right* $\mathcal{L}$-module structures on R.

(ii) Suppose that R, $\mathcal{L}$ have degree 0 (i.e., R is a plain commutative algebra, $\mathcal{L}$ a plain Lie algebroid). Then any BV extension $\mathcal{L}^\flat$ admits a unique splitting s compatible with the grading $s : \mathcal{L} \xrightarrow{\sim} \mathcal{L}^{\flat 0} \subset \mathcal{L}^\flat$, which is automatically a morphism of graded Lie R-algebroids. We see that BV extensions of $\mathcal{L}$ are the same as *Calabi-Yau*, or *vertex 0-algebroid*, structures on $\mathcal{L}$ from §11 of [**GMS2**].

(iii) For arbitrary $\mathcal{L}$ consider the de Rham-Chevalley DG algebra $\mathcal{C}_R(\mathcal{L})$ (see 2.9.1). Recall that as a mere graded algebra, $\mathcal{C}_R(\mathcal{L})$ equals $\mathrm{Hom}_R(\mathrm{Sym}_R(\mathcal{L}[1]), R)$. Now the conditions on (ω^c, μ^c) from the lemma just mean that it is an even 0-cocycle in $\mathcal{C}_R(\mathcal{L})[1]$, i.e., an odd 1-cocycle in $C_R(\mathcal{L})$.

4.1.10. The BV envelopes of BV algebroids. If C is a unital filtered BV algebra, then $R_C := C_0 = \mathrm{gr}_0 C$ is a commutative unital DG algebra, $\mathcal{L} := \mathrm{gr}_1 C[-1]$ is a Lie DG R_C-algebroid, and $C_1[-1]$ is a BV extension of $\mathcal{L}$. We have defined a functor $\mathcal{BV}_u \to \mathcal{L}ie\mathcal{A}lg^{BV}$. It admits a left adjoint $\mathcal{L}ie\mathcal{A}lg^{BV} \to \mathcal{BV}_u$, $(R, \mathcal{L}^\flat) \mapsto C_{BV}(R, \mathcal{L})^\flat$ (the *BV envelope* of $\mathcal{L}^\flat$). As a plain commutative graded algebra, $C_{BV}(R, \mathcal{L})^\flat$ equals $\mathrm{Sym}_R^\flat(\mathcal{L}[1]) :=$ the quotient of $\mathrm{Sym}_R(\mathcal{L}^\flat[1])$ modulo the relation $1^\flat = 1$ where $1^\flat := \iota(1) \in \mathcal{L}^\flat[1]$. The filtration subspaces $C_{BV}(R, \mathcal{L})_a^\flat$ are images of $\mathrm{Sym}^{\leq a}(\mathcal{L}^\flat)$, and the 1-Poisson bracket and differential are uniquely determined by the condition that $\mathcal{L}^\flat \to C_{BV}(R, \mathcal{L})_1^\flat[-1]$ is a morphism of Lie (DG) algebras. The morphism of Lie algebras $\mathcal{L} \to \mathrm{gr}_1 C_{BV}(R, \mathcal{L})^\flat[-1]$ defines a morphism of 1-Poisson algebras $\mathrm{Sym}_R(\mathcal{L}[1]) \to \mathrm{gr}\, C_{BV}(R, \mathcal{L})^\flat$ which is a quasi-isomorphism if $\mathcal{L}$ is homotopically R-flat.

Example. Suppose that R is a plain smooth algebra, $X := \mathrm{Spec}\, R$. Then a BV extension of Θ_R is the same as a right $\mathcal{D}$-module structure on $\mathcal{O}_X$ (see Remark (ii) in 4.1.9), which is the same as a left $\mathcal{D}$-module structure (= the flat connection) on ω_X^{-1}, and $C_{BV}(R, \Theta_R)^\flat$ is the corresponding de Rham complex. In particular, it has finite-dimensional cohomology.

Consider the de Rham-Chevalley DG algebra $\mathcal{C}_R(\mathcal{L})$ as in Remark (iii) in 4.1.9. Notice that $C_{BV}(R, \mathcal{L})^\flat$, considered as a mere graded module, is naturally a $\mathcal{C}_R(\mathcal{L})$-module.[11] One checks in a moment that this action is compatible with the differentials. Therefore $C_{BV}(R, \mathcal{L})^\flat$ is a DG $\mathcal{C}_R(\mathcal{L})$-module.

[11] An element $\varphi \in \mathcal{C}_R^1(\mathcal{L}) = \mathrm{Hom}_R(\mathcal{L}[1], R)$ acts on $C_{BV}(R, \mathcal{L})^\flat$ as a derivation whose restriction to $C_{BV}(R, \mathcal{L})_1^\flat$ is the composition $\mathcal{L}^\flat \to \mathcal{L} \xrightarrow{\varphi} R$.

REMARKS. (i) Take any element of $\mathcal{P}^c_s(\mathcal{L})$; let $\gamma = (\omega^c, \mu^c)$ be the corresponding odd 1-cocycle in $\mathcal{C}_R(\mathcal{L})$ (see Remark (iii) in 4.1.9). Let $\mathcal{L}^{\flat_\gamma}$ the translation of $\mathcal{L}^\flat$ by the corresponding "classical" extension of $\mathcal{L}$. Then $C_{BV}(R,\mathcal{L})^{\flat_\gamma}$ identifies naturally with $C_{BV}(R\mathcal{L})^\flat$ as a mere graded $\mathcal{C}_R(\mathcal{L})$-module, so that its differential d_γ becomes $d+\gamma$ where d is the differential of $C_{BV}(R\mathcal{L})^\flat$.

(ii) The DG algebra $\mathcal{C}_R(\mathcal{L})$ carries a natural filtration $\mathcal{C}_R(\mathcal{L})^{\geq i}$ such that $\mathrm{gr}_i\mathcal{C}_R(\mathcal{L}) = \mathrm{Hom}_R(\mathrm{Sym}^i(\mathcal{L}[1]), R)$. The cocycles γ as above lie in $\mathcal{C}_R(\mathcal{L})^{\geq 1}$. In fact, any odd 1-cocycle γ of $\mathcal{C}_R(\mathcal{L})^{\geq 1}$ (not necessary coming from $\mathcal{P}^c_s(\mathcal{L})$) defines a twisted differential $d_\gamma := d+\gamma$ on $C_{BV}(R,\mathcal{L})^\flat$. If f is an even element of degree 0 in $\mathcal{C}_R(\mathcal{L})^{\geq 1}$, then $d_{\gamma+df}$ is equal to[12] $(1+f)d_\gamma(1+f)^{-1}$, so the cohomology of $C_{BV}(R,\mathcal{L})^\flat$ with respect to d_γ depends only on the class of γ in $H^1\mathcal{C}_R(\mathcal{L})^{\geq 1}$.

4.1.11. Let us discuss the homotopy aspects of the above construction.

PROPOSITION. (i) *The categories $\mathcal{L}ie\mathcal{A}lg$, $\mathcal{L}ie\mathcal{A}lg^{BV}$ have natural closed model category structures with weak equivalences being those morphisms ϕ, resp. $\phi^\flat$, for which both ϕ_R, $\phi_\mathcal{L}$ are quasi-isomorphisms, and fibrations those morphisms for which both ϕ_R, $\phi_\mathcal{L}$ are surjective.*

(ii) *For a fixed $R \in \mathcal{C}omu$ the categories $\mathcal{L}ie\mathcal{A}lg_R$, $\mathcal{L}ie\mathcal{A}lg_R^{BV}$ are naturally closed model categories with quasi-isomorphisms as weak equivalences and surjective morphisms as fibrations.*

Sketch of a proof. Our situation does not fit into the setting of [**H**] 2.2 directly, but the arguments of loc. cit. can be easily modified to do the job.

(i) Let us replace $C(k)$ by $C(k)\times C(k)$ in the general setting of [**H**] 2.2. With conditions (H0), (H1) in [**H**] 2.2 modified in the obvious way, Theorem 2.2.1 of loc. cit. (together with its proof) remains valid in the present situation.

Consider a functor $\mathcal{L}ie\mathcal{A}lg \to C(k)\times C(k)$ which assigns to $(R,\mathcal{L})$ the same pair considered as mere complexes; there is a similar functor $\mathcal{L}ie\mathcal{A}lg^{BV} \to C(k)\times C(k)$, $(R,\mathcal{L}^\flat)\mapsto (R,\mathcal{L})$. These functors admit left adjoints $F : C(k)\times C(k) \to \mathcal{L}ie\mathcal{A}lg$, $F^{BV} : C(k)\times C(k) \to \mathcal{L}ie\mathcal{A}lg^{BV}$. One has $F(P,Q) = (R,\mathcal{L})$ where $R = \mathrm{Sym}(P\otimes Fr(Q))$, where $Fr(Q)$ is the free Lie algebra generated by Q, and $\mathcal{L} = Fr(Q)_R = R\otimes Fr(Q)$; similarly, $F^{BV}(P,Q) = (R, Fr(Q)^\flat_R)$ where $Fr(Q)^\flat_R$ is the $Fr(Q)$-rigidified BV extension (see 4.1.9).

Our functors satisfy conditions (H0), (H1), and we are done.

(ii) Replace $C(k)$ in [**H**] 2.2 by the category $C(k)_{\Theta_R}$ of pairs (Q,τ) where $Q\in C(k)$, $\tau : Q\to \Theta_R := \mathrm{Der}(R,R)$ is a morphism of complexes. Theorem 2.2.1 from loc. cit. (with conditions (H0), (H1) modified in the evident way) remains valid.

The functors $\mathcal{L}ie\mathcal{A}lg_R \to C(k)_{\Theta_R}$, $\mathcal{L}ie\mathcal{A}lg_R^{BV} \to C(k)_{\Theta_R}$ sending $\mathcal{L}$ or $\mathcal{L}^\flat$ to $(\mathcal{L},\tau_\mathcal{L})$ admit left adjoints F and F^{BV}. Namely, $F(Q,\tau) = Fr(Q)_R$, where the free Lie algebra $Fr(Q)$ acts on R according to τ, and $F^{BV}(Q,\tau)$ is its $Fr(Q)$-rigidified BV-extension.

Our functors satisfy conditions (H0), (H1), and we are done. □

The usual constructions, such as adding a variable to kill a cycle (see [**H**] 2.2.2), work for the above closed model categories. For example, consider the case of $\mathcal{L}ie\mathcal{A}lg$. Let $\mathcal{L}$ be a Lie R-algebroid, and suppose we have a datum (Q,τ_Q,ϵ) where Q is a k-complex with zero differential, $\tau_Q : Q\to\Theta_R$ a morphism of graded

[12] Since $f\in\mathcal{C}_R(\mathcal{L})^{\geq 1}$, the multiplication by $1+f$ is an automorphism of $C_{BV}(R,\mathcal{L})^\flat$.

vector spaces, $\epsilon : Q[-1] \to \mathcal{L}$ a morphism of complexes such that $\tau_{\mathcal{L}}\epsilon = d\tau_Q$. Then we have $(\mathcal{C}one(\epsilon), \tau) \in C(k)_{\Theta_R}$ where $\tau := (\tau_{\mathcal{L}}, \tau_Q)$. Define a new Lie R-algebroid $\mathcal{L}(Q, \tau_Q, \epsilon)$ such that for any $\mathcal{L}' \in \mathcal{L}ie\mathcal{A}lg_R$ a morphism $\mathcal{L}(Q, \tau_Q, \epsilon) \to \mathcal{L}'$ is the same as a morphism $\nu : (\mathcal{C}one(\epsilon), \tau) \to \mathcal{L}'$ in $C(k)_{\Theta_R}$ such that $\nu|_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}'$ is a morphism of Lie R-algebroids. There is an evident morphism $\mathcal{L} \to \mathcal{L}(Q, \tau_Q, \epsilon)$ of Lie R-algebroids; any morphism isomorphic to such an arrow for some datum as above is called an *elementary cofibration.* A *standard cofibration* is a morphism $\mathcal{L} \to \mathcal{L}'$ such that $\mathcal{L}'$ admits a filtration $\mathcal{L}'_0 \subset \mathcal{L}'_1 \subset \cdots$ such that $\bigcup \mathcal{L}'_i = \mathcal{L}'$, $\mathcal{L} \xrightarrow{\sim} \mathcal{L}'_0$, and each $\mathcal{L}'_i \to \mathcal{L}'_{i+1}$ is an elementary cofibration. Each cofibration in $\mathcal{L}ie\mathcal{A}lg_R$ is a retract of a standard one. We say that a Lie R-algebroid $\mathcal{L}$ is *semi-free* if the morphism $0_R \to \mathcal{L}$ is a standard cofibration. Every Lie R-algebroid admits a left semi-free resolution.

REMARK. A morphism of graded vector spaces $\chi : Q \to \mathcal{L}$ yields an identification $\mathcal{L}(Q, \tau_Q, \epsilon) \xrightarrow{\sim} \mathcal{L}(Q, \tau_Q + \tau_{\mathcal{L}}\chi, \epsilon + d\chi)$ coming from the standard isomorphism $\mathcal{C}one(\epsilon) \xrightarrow{\sim} \mathcal{C}one(\epsilon + d\chi)$ defined by χ.

4.1.12. The evident functors $\mathcal{L}ie\mathcal{A}lg^{BV} \to \mathcal{L}ie\mathcal{A}lg \to \mathcal{C}omu$, $\mathcal{L}ie\mathcal{A}lg_R \to \mathcal{L}ie\mathcal{A}lg$, and $\mathcal{L}ie\mathcal{A}lg_R^{BV} \to \mathcal{L}ie\mathcal{A}lg^{BV}$ preserve (co)fibrations and weak equivalences; we have the corresponding functors between the homotopy categories. There is a fully faithful embedding $\mathcal{C}omu \hookrightarrow \mathcal{L}ie\mathcal{A}lg$ left adjoint to the projection $\mathcal{L}ie\mathcal{A}lg \to \mathcal{C}omu$, which assigns to R the trivial R-algebroid 0_R and its lifting $\mathcal{C}omu \to \mathcal{L}ie\mathcal{A}lg^{BV}$, $R \mapsto 0_R^\flat$. They also preserve (co)fibrations and weak equivalences

The functor $\mathcal{L}ie\mathcal{A}lg^{BV} \to \mathcal{BV}$, $(R, \mathcal{L}^\flat) \mapsto C_{BV}(R, \mathcal{L})^\flat$ does not preserve weak equivalences. But its restriction to the subcategory of those $(R, \mathcal{L}^\flat)$ for which $\mathcal{L}$ is homotopically R-flat (which includes cofibrant objects of $\mathcal{L}ie\mathcal{A}lg_R^{BV}$) preserves them, so we have a well-defined functor between the homotopy categories $\mathcal{H}o\mathcal{L}ie\mathcal{A}lg^{BV} \to \mathcal{H}o\mathcal{BV}$, $(R, \mathcal{L}^\flat) \mapsto C^L_{BV}(R, \mathcal{L})^\flat$.

To compute $C^L_{BV}(R, \mathcal{L})^\flat$, one should consider a left resolution $\mathcal{L}^L \to \mathcal{L}$ which is homotopically R-flat as an R-module (for example, one can always find $\mathcal{L}^L$ which is a semi-free R-module). The BV extension $\mathcal{L}^\flat$ defines, by pull-back, a BV extension of $\mathcal{L}^L$. One has $C^L_{BV}(R, \mathcal{L})^\flat = C_{BV}(R, \mathcal{L}^L)^\flat$. Notice that $\operatorname{gr} C^L_{BV}(R, \mathcal{L})^\flat = \operatorname{Sym}^L_R \mathcal{L}$.

4.1.13. The next technical proposition assures that while doing homotopy computations with Lie R-algebroids, or BV Lie R-algebroids, one can replace R by any *cofibrant* algebra of the same homotopy class.

For $R \in \mathcal{C}omu$ denote by $[R] \in \mathcal{H}o\mathcal{C}omu$ our R considered as an object of the homotopy category; the same notation is used for morphisms in $\mathcal{C}omu$.

Let $\mathcal{H}o\mathcal{L}ie\mathcal{A}lg_{[R]}$ be the fiber of $\mathcal{H}o\mathcal{L}ie\mathcal{A}lg$ over $[R]$; i.e., it is the category of pairs $(\mathcal{L}_P, [\phi])$ where $\mathcal{L}_P = (P, \mathcal{L}_P) \in \mathcal{H}o\mathcal{L}ie\mathcal{A}lg$ and $[\phi] : [R] \xrightarrow{\sim} [P]$ an isomorphism in $\mathcal{H}o\mathcal{C}omu$. One has a similar category $\mathcal{H}o\mathcal{L}ie\mathcal{A}lg^{BV}_{[R]}$. There are evident functors

$$\mathcal{H}o\mathcal{L}ie\mathcal{A}lg_R \to \mathcal{H}o\mathcal{L}ie\mathcal{A}lg_{[R]}, \quad \mathcal{H}o\mathcal{L}ie\mathcal{A}lg^{BV}_R \to \mathcal{H}o\mathcal{L}ie\mathcal{A}lg^{BV}_{[R]}. \tag{4.1.13.1}$$

PROPOSITION. *For a cofibrant R these functors are essentially surjective.*

Proof. We will consider the case of Lie algebroids; the BV setting is treated similarly.

Below we denote Lie R-algebroids as $\mathcal{L}_R$, etc.

Surjectivity on objects: (i) Suppose we have $(\mathcal{L}_P, [\varphi]) \in \mathcal{H}o\mathcal{L}ie\mathcal{A}lg_{[R]}$. We want to define weak equivalences $\mathcal{L}_R \leftarrow \mathcal{L}_F \rightarrow \mathcal{L}_P$ in $\mathcal{L}ie\mathcal{A}lg$ such that the composition $R \leftarrow F \rightarrow P$ is of homotopy class $[\varphi]$. We can assume that $\mathcal{L}_P$ is a semi-free Lie P-algebroid. We move step-by-step by elementary layers of $\mathcal{L}_P$. The steps are constructed as follows.

(ii) Suppose we have a weak equivalence $\nu : \mathcal{L}_S \rightarrow \mathcal{L}_T$ in $\mathcal{L}ie\mathcal{A}lg$ such that $S \in \mathcal{C}omu$ is cofibrant and an elementary cofibration $i_T : \mathcal{L}_T \rightarrow \mathcal{L}'_T$ in $\mathcal{L}ie\mathcal{A}lg_T$. We will construct an elementary cofibration $i_S : \mathcal{L}_S \rightarrow \mathcal{L}'_S$ in $\mathcal{L}ie\mathcal{A}lg_S$ homotopy equivalent to i_T. More precisely, we construct morphisms $\mathcal{L}'_S \xleftarrow{\pi} \mathcal{L}_U \xrightarrow{\mu} \mathcal{L}'_T$ and $j : \mathcal{L}_S \rightarrow \mathcal{L}_U$ in $\mathcal{L}ie\mathcal{A}lg$ such that $\pi j = i_S$, $\mu j = i_T \nu$, π is a trivial fibration, μ is a weak equivalence.

Write $\mathcal{L}'_T = \mathcal{L}_T(Q, \tau^T_Q, \epsilon^T)$ (see the end of 4.1.11). We can find a morphism of complexes $\epsilon^S : Q[-1] \rightarrow \mathcal{L}_S$ such that $\nu\epsilon^S$ is homotopic to ϵ^T (since ν is a quasi-isomorphism). By Remark at the end of 4.1.11, replacing our datum by an equivalent one, we can assume that $\nu\epsilon^S = \epsilon^F$. Let $\nu_S : S \rightarrow T$ be the morphism of algebras corresponding to ν. Consider the morphisms of complexes $\Theta_S \rightarrow \mathrm{Der}_{\nu_S}(S,T) \leftarrow \Theta_T$ which send derivations $\theta^S \in \Theta_S$, $\theta^T \in \Theta_T$ to $\nu_S\theta^S, \theta^T\nu_S \in \mathrm{Der}_{\nu_S}(S,T)$. Since S is cofibrant, Ω_S is a homotopically projective S-module; hence the morphism $\Theta_S \rightarrow \mathrm{Der}_{\nu_S}(S,T)$ is a quasi-isomorphism. Therefore one can find morphisms of graded modules $\tau^S_Q : Q \rightarrow \Theta_S$ and $\kappa_T : Q[1] \rightarrow \mathrm{Der}_{\nu_S}(S,T)$ such that $d\tau^S_Q = \tau_{\mathcal{L}_S}\epsilon^S$ and $d\kappa_T = \tau^T_Q \nu_S - \nu_S \tau^S_Q$.

We define i_S as the elementary cofibration $\mathcal{L}_S \rightarrow \mathcal{L}'_S := \mathcal{L}_S(Q, \tau^S_Q, \epsilon^S)$. Define $\mathcal{L}_U$ as the Lie algebroid whose morphisms to any $\mathcal{L}_F$ are the same as the triples (ϕ, χ, κ), where $\phi : \mathcal{L}_S \rightarrow \mathcal{L}_F$ is a morphism of Lie algebroids and $\chi : Q \rightarrow \mathcal{L}_F$, $\kappa : Q[1] \rightarrow \mathrm{Der}_{\phi_S}(S,F)$ are morphisms of graded modules, such that $d\chi = \phi\epsilon^S$ and $d\kappa = \tau_{\mathcal{L}_F}(\chi)\phi_S - \phi_S\tau^S_Q$. So one has an evident morphism $j : \mathcal{L}_S \rightarrow \mathcal{L}_U$ and morphisms $\mathcal{L}'_S \xleftarrow{\pi} \mathcal{L}_U \xrightarrow{\mu} \mathcal{L}'_T$ corresponding to the triples $(i_S, \chi_S, 0)$ and (ν, χ_T, κ_T) where χ_S, χ_T are the structure embeddings of Q for our elementary cofibrations. The promised relations between these morphisms are evident, so it remains only to check that π and μ are weak equivalences. We leave it as an exercise to the reader.

(iii) Let us return to (i). Our $\mathcal{L}_P$ is a semi-free Lie P-algebroid, so we have a filtration $0_P = \mathcal{L}_{P0} \subset \mathcal{L}_{P1} \subset \cdots$, $\bigcup \mathcal{L}_{Pn} = \mathcal{L}_P$, such that each $i_{Pn} : \mathcal{L}_{Pn} \hookrightarrow \mathcal{L}_{P\,n+1}$ is an elementary cofibration in $\mathcal{L}ie\mathcal{A}lg_P$. We will define by induction the morphisms $\mathcal{L}_{Rn} \xleftarrow{\pi_n} \mathcal{L}_{F_n} \xrightarrow{\mu_n} \mathcal{L}_{Tn}$, $i_{Rn} : \mathcal{L}_{Rn} \hookrightarrow \mathcal{L}_{R\,n+1}$, and $j_n : \mathcal{L}_{F_n} \rightarrow \mathcal{L}_{F_{n+1}}$ such that the obvious diagram is commutative (i.e., $\pi_{n+1}jn = i_{Rn}\pi_n$, $\mu_{n+1}j_n = i_{Pn}\mu_n$), i_{Rn} are elementary cofibrations, π_n are trivial fibrations, μ_n are weak equivalences. Passing to the inductive limit by i_{Rn}, j_n, i_{Pn}, we get the promised $\mathcal{L}_R \leftarrow \mathcal{L}_F \rightarrow \mathcal{L}_P$.

Step 0: Our R is cofibrant, so we can realize the homotopy class $[\varphi]$ by an actual morphism $\varphi : R \rightarrow P$. Set $\mathcal{L}_{R0} := 0_R$, $\mathcal{L}_{F_0} := 0_R$, $\pi_0 := id_R$, $\mu_0 = \varphi$.

Induction step: Suppose we have already defined $\mathcal{L}_{Rn} \xleftarrow{\pi_n} \mathcal{L}_{F_n} \xrightarrow{\mu_n} \mathcal{L}_{Pn}$. Let us apply the construction of (ii) to $\mathcal{L}_S = \mathcal{L}_{F_n}$, $\mathcal{L}_T = \mathcal{L}_{Pn}$, $\nu = \mu_n$, and $i_T = i_{Pn}$. The $\mathcal{L}_U$, j, and μ we get are our $\mathcal{L}_{F_{n+1}}$, j_n, and μ_{n+1}. Notice that the construction of (ii) depends on a choice of ϵ^S, τ^S_Q, and κ_T from loc. cit. subject to certain conditions, and we have to choose them properly in order to define π_{n+1}. Since $\pi_{F_n} : F_n \rightarrow R$ is a trivial fibration and R is cofibrant, the subcomplex $\Theta_{F_n,R} \subset \Theta_{F_n}$ of vector fields preserving π_n is quasi-isomorphic to the whole of Θ_{F_n}. This implies that for given ϵ^S we can choose τ^S_Q and κ_T so that $\tau^S_Q \in \Theta_{F_n,R}$, and we do it that way. Then τ_Q^S yields $\tau_Q^R : Q \to \Theta_R$. Now let i_{Rn} be the elementary cofibration $\mathcal{L}_{Rn} \hookrightarrow \mathcal{L}_{R\,n+1} := \mathcal{L}_{Rn}(Q, \tau_Q^R, \pi_n \epsilon^S)$ and let π_{n+1} be the composition of the trivial fibration $\pi : \mathcal{L}_U \to \mathcal{L}'_S = \mathcal{L}_{F_n}(Q, \tau_Q^S, \epsilon^S)$ from (ii) and the projection $\mathcal{L}_{F_n}(Q, \tau_Q^S, \epsilon^S) \to \mathcal{L}_{Rn}(Q, \tau_Q^R, \pi_n \epsilon^S)$ which equals π_n on $\mathcal{L}_{F_n}$ and id_Q on Q. We leave it to the reader to check that this is a trivial fibration, and we are done.

Surjectivity on morphisms: Suppose we have $\mathcal{L}_R, \mathcal{L}'_R \in \mathcal{L}ie\mathcal{A}lg_R$ and a morphism $\nu : \mathcal{L}_R \to \mathcal{L}'_R$ in $\mathcal{H}o\mathcal{L}ie\mathcal{A}lg$ which lifts $id_{[R]}$. We want to show that it comes from a morphism $\mathcal{L}_R \to \mathcal{L}'_R$ in $\mathcal{H}o\mathcal{L}ie\mathcal{A}lg_R$. One uses a lifting of homotopy argument.

Let $\pi : \mathcal{L}_T \to \mathcal{L}_R$ be a trivial fibration[13] in $\mathcal{L}ie\mathcal{A}lg$ such that $\mathcal{L}_T$ is cofibrant. One can find a morphism $\mu : \mathcal{L}_T \to \mathcal{L}'_R$ in $\mathcal{L}ie\mathcal{A}lg$ such that μ is homotopic to $\nu\pi$.

The morphism $\pi_T : T \to R$ is a trivial fibration, so it identifies R with T/I where $I \subset T$ is a contractible ideal. So $\mathcal{L}_T/I\mathcal{L}_T$ is a Lie R-algebroid. The morphism $\mu_T : T \to R$ is homotopic to π_T. We will show that μ is homotopic to a morphism $\kappa : \mathcal{L}_T \to \mathcal{L}'_R$ such that $\kappa_T : T \to R$ *equals* π_T. Then π, κ yield morphisms of Lie R-algebroids $\mathcal{L}_R \leftarrow \mathcal{L}_T/I\mathcal{L}_T \to \mathcal{L}'_R$. The left one is a trivial fibration, the composition is homotopic to ν, and we are done.

To define κ, we first choose a homotopy between π_T and μ_T, i.e., a cofibrant S, a trivial fibration $\psi : S \to T$ in $\mathcal{C}omu$, its sections $\gamma_\pi, \gamma_\mu : T \to S$, and a morphism $\rho : S \to R$ such that $\rho\gamma_\pi = \pi_T$, $\rho\gamma_\mu = \mu_T$.

Let $\mathcal{L}_Q$ be the colimit of the diagram $\mathcal{L}_T \leftarrow 0_T \xrightarrow{\gamma_\mu} 0_S$ in $\mathcal{L}ie\mathcal{A}lg$, and let $\mathcal{L}_T \xrightarrow{i_\mu} \mathcal{L}_Q \xleftarrow{\epsilon_S} 0_S$ be the structure morphisms. Notice that i_μ is a trivial cofibration since γ_μ is. There is a natural projection $p : \mathcal{L}_Q \to \mathcal{L}_T$ such that $p\epsilon_S$ equals $0_S \xrightarrow{\psi} 0_T \to \mathcal{L}_T$ and $pi_\pi = id_{\mathcal{L}_T}$. Our p is a trivial fibration, so the morphism $0_T \xrightarrow{\gamma_\pi} 0_S \to \mathcal{L}_Q$ extends to a morphism $i_\pi : \mathcal{L}_T \to \mathcal{L}_Q$ such that $pi_\pi = id_{\mathcal{L}_T}$. Finally, one has a morphism $\beta : \mathcal{L}_Q \to \mathcal{L}'_R$ such that $\beta i_\mu = \mu$ and $\beta\epsilon_S$ is the composition $0_S \xrightarrow{\rho} 0_R \to \mathcal{L}'_R$. Our κ is $\beta i_\pi : \mathcal{L}_T \to \mathcal{L}'_R$. □

4.1.14. Homotopy unital commutative algebras and modules. Let $\mathcal{C}om$ be the category of possibly non-unital commutative DG algebras.

We say that $E \in \mathcal{C}om$ is a *homotopy unit* algebra if the corresponding graded cohomology algebra $H^{\cdot}E$ is the unit algebra. Thus $H^{\neq 0}E = 0$, $H^0E = k$. A morphism of homotopy unit algebras is their morphism as commutative algebras which commutes with the identification $H^0 = k$ (or, equivalently, is a quasi-isomorphism). For every homotopy unit E one has canonical morphisms of homotopy unit algebras $E \leftarrow \tau_{\leq 0}E \to H^0E = k$. The tensor product of homotopy unit algebras is a homotopy unit algebra.

For $R \in \mathcal{C}om$ a *homotopy unit* in R is a morphism $E \to C$ such that E is a homotopy unit algebra and $k = H^0E \to H^{\cdot}C$ is a unit in the (graded) algebra $H^{\cdot}R$. A *homotopy unital* commutative algebra is a commutative algebra R equipped with a homotopy unit $i_R : E_R \to R$; we often denote it simply as R. A morphism of homotopy unital algebras $R \to R'$ is a pair (f, f_E) where $f : R \to R'$ is a morphism of commutative algebras, $f_E : E_R \to E_{R'}$ a morphism of the homotopy unit algebras, such that $fi_R = i_{R'}f_E$; we often abbreviate (f, f_E) to f. The category $\mathcal{C}om_{hu}$ of homotopy unital commutative algebras is a tensor category. One has an obvious fully faithful embedding of tensor categories $\mathcal{C}omu \hookrightarrow \mathcal{C}om_{hu}$.

[13]Which means that both $T \to R$ and $\mathcal{L}_T \to \mathcal{L}_R$ are surjective quasi-isomorphisms.

PROPOSITION. (i) $\mathcal{C}om_{hu}$ *is a closed model category with weak equivalences being quasi-isomorphisms and fibrations those morphisms* (f, f_E) *for which both* f, f_E *are surjective. Denote by* $\mathcal{H}o\mathcal{C}om_{hu}$ *the corresponding homotopy category.*

(ii) The functor $\mathcal{H}o\mathcal{C}omu \to \mathcal{H}o\mathcal{C}om_{hu}$ *is an equivalence of categories.*

Sketch of a proof. (i) Arguments of [**H**] work with obvious small modifications. One should only replace the definition of a standard cofibration from [**H**] 2.2.3(i) by the following: a morphism $f : R \to R'$ in $\mathcal{C}om_{hu}$ is a standard cofibration if $f_E : E_R \to E_{R'}$ and the morphism $R \underset{E_R}{*} E_{R'} \to R'$ are standard cofibrations in $\mathcal{C}om$. Here $R \underset{E_R}{*} E_{R'}$ is the coproduct of R and $E_{R'}$ over E_R in $\mathcal{C}om$.

(ii) We consider a sequence of fully faithful embeddings $\mathcal{C}om_u \subset \mathcal{C}om''_{hu} \subset \mathcal{C}om'_{hu} \subset \mathcal{C}om_{hu}$ each of which becomes an equivalence on the level of homotopy categories:

Let $\mathcal{C}om'_{hu} \subset \mathcal{C}om_{hu}$ be the subcategory of R with $E_R^{>0} = 0$. The left adjoint functor to this embedding is $(R, E_R) \mapsto (R, \tau_{\leq 0} E_R)$. It is clear that the corresponding homotopy categories are equivalent.

Let $\mathcal{C}om''_{hu} \subset \mathcal{C}om'_{hu}$ be the subcategory of R with $E_R = k$. The left adjoint functor to this embedding is $R \mapsto R \underset{E_R}{*} k$ (notice that for $R \in \mathcal{C}om'_{hu}$ there is a unique morphism $E_R \to k$). If R is cofibrant, then the morphism $R \to R \underset{E_R}{*} k$ is a weak equivalence, so the homotopy categories of $\mathcal{C}om''_{hu}$ and $\mathcal{C}om'_{hu}$ are equivalent.

Finally, consider the embedding $\mathcal{C}om_u \subset \mathcal{C}om''_{hu}$. For $R \in \mathcal{BV}''_{hu}$ the element $i_R(1) \in R$ is idempotent. It is clear that $R_u := i_R(1) \cdot R \in \mathcal{C}omu$, the functor $\mathcal{C}om''_{hu} \to \mathcal{C}omu$, $R \mapsto R_u$, is both left and right adjoint to the embedding, and the obvious arrows $R_u \rightleftarrows R$ are quasi-isomorphisms. It is clear that we get mutually inverse equivalences of the homotopy categories. $\square$

Suppose we have a homotopy unital commutative algebra R. An R-module M is said to be *homotopy unital* if $H^\cdot M$ is a unital $H^\cdot R$-module. This amounts to the fact that the multiplication by $1 \in H^0 R$ endomorphism of M, considered as an object of the derived category of R-modules, is equal to id_M.[14]

Denote the DG category of homotopy unital R-modules by $C(R)_{hu}$ and the corresponding derived category by $D(R)_{hu}$. A morphism $f : R \to R'$ of homotopy unital algebras yields an obvious exact DG functor $C(R')_{hu} \to C(R)_{hu}$. We leave it to the reader to show that if f is a quasi-isomorphism, then the corresponding functor $D(R')_{hu} \xrightarrow{\sim} D(R)_{hu}$ is an equivalence and also to check that for $R \in \mathcal{C}omu$ the category $D(R)_{hu}$ is canonically equivalent to the derived category of unital R-modules. Therefore, by the above proposition, from the homotopy point of view homotopy unital modules over homotopy unital algebras are the same as unital modules over unital algebras.

The above category $C(R)_{hu}$ is a tensor category in the obvious manner. For every $M \in C(R)_{hu}$ the morphism $R \otimes M \to M$, $r \otimes m \mapsto m$, is a quasi-isomorphism whose homotopy inverse is a morphism $M \to R \otimes M$, $m \mapsto \tilde{1} \otimes m$, where $\tilde{1} \in R^0$ is any lifting of $1 \in H^0 R$. Similarly, the morphism $M \to \mathrm{Hom}(R, M)$ which assigns to $m \in M$ the morphism $R \to M$, $r \mapsto rm$, is a quasi-isomorphism whose homotopy inverse is $f \mapsto f(\tilde{1})$. Thus one can compute morphisms and the tensor product in $D(R)_{hu}$ using, say, semi-free resolutions just as we do in the case of unital algebras.

[14]Indeed, if M is homotopy unital, then this is an idempotent automorphism in the derived category; hence it is the identity.

4.1.15. Homotopy unital BV algebras. Here is a BV version of the above definitions.

For a $C \in \mathcal{BV}$ a *homotopy unit* in C is a morphism of commutative algebras $E \to C_0$ such that E is a homotopy unit commutative algebra, the image of E is central in C,[15] and $k = H^0 E \to H^{\cdot}\mathrm{gr}\, C$ is a unit in $H^{\cdot}\mathrm{gr}\, C$. A *homotopy unital* BV algebra is $C \in \mathcal{BV}$ equipped with a homotopy unit $i_C : E_C \to C$. The category of homotopy unital BV algebras (cf. 4.1.14) is denoted by $\mathcal{BV}_{hu}$; it is a tensor category which contains $\mathcal{BV}_u$ as a full tensor subcategory.

PROPOSITION. (i) *$\mathcal{BV}_{hu}$ is a closed model category with weak equivalences being filtered quasi-isomorphisms and fibrations those morphisms (f, f_E) for which both $\mathrm{gr}\, f_C$, $\mathrm{gr}\, f_E$ are surjective.*

(ii) *One has an equivalence of the homotopy categories $\mathcal{H}o\mathcal{BV}_u \xrightarrow{\sim} \mathcal{H}o\mathcal{BV}_{hu}$.*

Proof. An immediate modification of the proof in 4.1.14. □

The construction from 4.1.9 generalizes to the homotopy unital setting as follows. For $R \in \mathcal{C}om_{hu}$ a *homotopy unital Lie R-algebroid* $\mathcal{L}$ is a Lie R-algebroid[16] such that $\mathcal{L}$ is a homotopy unital R-module and the image of $E_R \to R$ is annihilated by the action of $\mathcal{L}$. For such an $\mathcal{L}$ its *BV extension* is defined exactly as in 4.1.9 with an obvious modification (we demand that the ι-image of $E_R[-1]$ lies in the center of $\mathcal{L}^\flat$). One calls $(\mathcal{L}, \mathcal{L}^\flat)$ a *homotopy unital BV Lie R-algebroid*; we abbreviate it often to $\mathcal{L}^\flat$. The pairs $(R, \mathcal{L}^\flat)$ form a category $\mathcal{L}ie\mathcal{A}lg_{hu}^{BV}$. There is an obvious functor $\mathcal{BV}_{hu} \to \mathcal{L}ie\mathcal{A}lg_{hu}^{BV}$, $C \mapsto (C_0, C_1[-1])$ (we tacitly assume that $E_{C_0} := E_C$ and the Lie algebroid $\mathcal{L}$ is $\mathrm{gr}_1 C$). It admits a left adjoint $\mathcal{L}ie\mathcal{A}lg_{hu}^{BV} \to \mathcal{BV}_{hu}$, $(R, \mathcal{L}^\flat) \mapsto C_{BV}(R, \mathcal{L})^\flat$. We have an obvious morphism of 1-Poisson algebras $\mathrm{Sym}_R \mathcal{L} \to \mathrm{gr}\, C_{BV}(R, \mathcal{L})^\flat$ which is a quasi-isomorphism if $\mathcal{L}$ is homotopically R-flat. One defines $C_{BV}^L(R, \mathcal{L})^\flat \in \mathcal{H}o\mathcal{BV}_{hu} = \mathcal{H}o\mathcal{BV}_u$ as in 4.1.12.

4.1.16. Perfect complexes. Let D be an additive category which admits arbitrary direct sums. An object $P \in D$ is said to be *compact* if the functor $\mathrm{Hom}(P, \cdot)$ commutes with direct sums; i.e., for any family M_α of objects of D the obvious map $\oplus \mathrm{Hom}(P, M_\alpha) \to \mathrm{Hom}(P, \oplus M_\alpha)$ is an isomorphism.

EXERCISE. Suppose D is the category of S-modules where S is a plain associative algebra. Then compact objects of D are the same as finitely generated S-modules.

If D is a triangulated category, then its compact objects are also called *perfect* objects; they form a thick subcategory $D_{perf} \subset D$.

Let F be a commutative algebra (:= a commutative unital DG super algebra) having degrees ≤ 0. Consider the derived category $D(F)$ of DG F-modules. Notice that $D(F)$ is a t-category, so we have the subcategories $D(F)^{>a}, D(F)^{<b} \subset D(F)$ of F-modules acyclic in degrees $\leq a$, resp. $\geq b$. We say that a non-zero $P \in D(F)$ has *finite span* if $\mathrm{Hom}(P, D(F)^{>a}) = 0 = \mathrm{Hom}(P, D(F)^{<b})$ for some integers a, b; then its *span* is the interval $[b, a]$ formed by the smallest a and the largest b as above. Every perfect P has finite span.[17]

[15] I.e., C is an E-algebra.

[16] The axioms of a Lie R-algebroid for a non-unital R are the same as in the unital setting.

[17] This follows from the following observation: suppose we have $M_i \in D(F)^{>a_i}$ such that $a_i \to \infty$; then $\oplus M_i \xrightarrow{\sim} \Pi M_i$. The same is true for the b situation.

Notice that for every $f \in F^0$ whose image in H^0F is invertible, the functor $D(F_f) \to D(F)$ is an equivalence of categories (here F_f is the f-localization of F). Take any $\bar{f} \in H^0F$; we see that the category $D(F_f)$ for $f \in F$ lifting $\bar{f}$ does not depend on the choice of f; we denote it by $D(F_{\bar{f}})$. Thus $D(F)$ can be localized with respect to the Zariski topology of $\mathrm{Spec}(H^0F)$, so we can speak of properties of objects of $D(F)$ that hold Zariski locally on $\mathrm{Spec}(H^0F)$.

Notice that semi-free DG F-modules (see 4.1.5) with generators in degrees bounded from above are the same as DG F-modules which are, as mere graded F-modules, free with generators in degrees bounded from above.

Lemma. *For $P \in D(F)$ and an interval $[b,a]$ the next conditions are equivalent:*

(i) *P is perfect of span in $[b,a]$;*

(ii) *P is a retract of an object of $D(F)$ represented by a semi-free F-module $\tilde{P}$ with finitely many generators whose degrees are in $[b,a]$;*

(iii) *locally on* $\mathrm{Spec}(H^0F)$ *our P can be represented by a semi-free module $P^{\mathfrak{f}}$ with finitely many generators of degrees in $[b,a]$.*

Proof. (ii)⇒(i): Clear.

(i)⇒(ii): (a) First we construct a semi-free resolution $\phi : T \to P$ with finitely many generators in each degree $\leq a$. The construction goes as follows. Suppose we have already defined T; set $T_i := FT^{\geq a-i} \subset T$. This is a semi-free F-module with finitely many generators in degrees $[a-i,a]$, and $H^jT_i \to H^jP$ is an isomorphism for $j > a-i$ and is surjective for $j = a-i$. We will define T_i and $\phi_i := \phi|_{T_i}$ by induction by i.

The first step: We know that $H^{>a}P = 0$. Also H^aP is a finitely generated H^0F-module (by Exercise above), so one can find (ϕ_0, T_0). Induction step: Suppose we have (T_i, ϕ_i). Then $\mathcal{C}one(\phi_i)$ is perfect and its cohomology vanishes in degrees $\geq a-i$. Then T_{i+1} is obtained from T_i by adding finitely many free generators e_α in degree $a-i-1$, where $d(e_\alpha) \in T_i^{a-i}$ and $\phi_{i+1}(e_\alpha) \in P^{a-i-1}$ are chosen so that the cycles $(-d(e_\alpha), \phi_{i+1}(e_\alpha))$ generate $H^{a-i-1}\mathcal{C}one(\phi_i)$.

(b) So we have defined our T. Set $\tilde{P} = T_{a-b}$, $\varphi := \phi_{a-b}$; denote by ψ the composition $\tilde{P} \to P \to \tau_{\geq a}P$. Since $\mathcal{C}one(\psi) \in D(F)^{<a}$, one has $\mathrm{Hom}(P, \mathcal{C}one(\psi)) = 0$. Thus one can find $\rho : P \to \tilde{P}$ such that $\psi\rho$ is the projection $P \to \tau_{\geq a}P$. Since $\mathrm{Hom}(P, \tau_{<a}P) = 0$, one has $\varphi\rho = id_P$; q.e.d.

(i)⇒(iii): Choose a semi-free DG F-module $\tilde{P}$ such that we have a direct sum decomposition $\tilde{P} = P \oplus B$ in $D(F)$ and $H^i\tilde{P} = H^iP$ for $i > b$. Such a $\tilde{P}$ was constructed in step (i)⇒(ii) above. Then B is a perfect complex of span $[b,b]$.

Since $B \overset{L}{\underset{F}{\otimes}} H^0F$ is a perfect H^0F-complex of span $[b,b]$, it equals $\bar{Q}[-b]$ where $\bar{Q}$ is a finitely generated projective H^0F-module. Notice that $\bar{Q} = H^bB$. Passing to a Zariski localization, we can assume that $\bar{Q}$ is a free H^0F-module. Let Q be a free F-module with generators in degree b such that $H^bQ = \bar{Q} = H^bB$. This identification amounts to a morphism $Q \to B$ in $D(F)$ which is automatically a quasi-isomorphism (because a morphism of perfect F-modules which becomes an isomorphism after tensoring by H^0F is a quasi-isomorphism).

The morphisms $\tilde{P} \leftrightarrows B$ in $D(F)$ yield homotopy classes of morphisms of DG F-modules $\tilde{P} \leftrightarrows Q$. Choose some true morphisms from the homotopy classes; after possible further Zariski localization, we can assume that the composition $Q \to \tilde{P} \to Q$ is an isomorphism, so Q is a retract of $\tilde{P}$. The kernel $P^{\mathfrak{f}}$ of the retraction represents P. The F^0-module of degree b generators $(P^{\mathfrak{f}}/F^{<0}P^{\mathfrak{f}})^b$ is projective and hence locally free, and we are done.

(iii)⇒(i): Perfectness is a local property: Indeed, suppose P is perfect Zariski locally on $\mathrm{Spec}(H^0F)$. Then one can find a finite covering $\mathrm{Spec}\, F^0_{f_\alpha}$ of $\mathrm{Spec}\, F^0$ such that each $P_{f_\alpha} \in D(F_{f_\alpha})$ is perfect. For any DG F-module M consider its Čech complex $C(M)$ with respect to this covering; this is a resolution of M. If g is a finite product of f_α's, then the functor $M \mapsto \mathrm{Hom}_{D(F)}(P, M_g) = \mathrm{Hom}_{D(F_g)}(P_g, M_g)$ commutes with direct sums. Hence the functor $M \mapsto \mathrm{Hom}_{D(F)}(P, C(M)) = \mathrm{Hom}_{D(F)}(P, M)$ commutes with direct sums, and we are done.

The fact that the span of a perfect P is determined locally is clear from the following observations. The upper bound of the span is the largest a such that $H^aP \neq 0$. The lower bound is the largest b such that for every plain H^0F-module M the complex $\mathrm{Hom}_F(T, M)$ is acyclic in degrees $> -b$; here T is a resolution of P from the step (i)⇒(ii). □

Corollary. *For any perfect $P \in D(F)$ the complex $P^* := R\mathrm{Hom}(P, R) \in D(F)$ is perfect and the canonical morphism $P \to (P^*)^*$ is an isomorphism.* □

4.1.17. Perfect DG algebras. As above, let F be a commutative algebra having degrees ≤ 0.

We say that F is *perfect* if, as an object of the homotopy category of commutative DG algebras, it is a retract of a DG algebra $\tilde{F}$ which, as a mere graded commutative super algebra, is free with finitely many generators of degrees ≤ 0. Such F has *span* $\leq n$ if one can find $\tilde{F}$ with generators in degrees $\in [-n, 0]$.

The next proposition is a non-linear version of the lemma in 4.1.16. Let us make first two remarks:

(a) Notice that if $f \in F^0$ is such that its image in H^0F is invertible, then the morphism $F \to F_f$ is a quasi-isomorphism. Thus for any $\bar{f} \in H^0F$ the localizations F_f for $f \in F$ lifting $\bar{f}$ define the same object of the homotopy category of DG algebras; denote it by $F_{\bar{f}}$. So we can localize F, as an object of the homotopy category, with respect to the Zariski topology of $\mathrm{Spec}(H^0F)$, and one can speak of (homotopy) properties of F that hold Zariski locally on $\mathrm{Spec}(H^0F)$.

(b) F is semi-free if and only if it is free as a mere graded algebra.

Proposition. *For an algebra F having degrees ≤ 0 and an integer $n \geq 1$ the following conditions are equivalent:*

(i) *F is perfect of span $\leq n$;*

(ii) *Zariski locally on* $\mathrm{Spec}(H^0F)$*, our F is homotopy equivalent to a semi-free DG algebra with finitely many generators of degrees $-n, \ldots, 0$;*

(iii) *H^0F is a finitely generated algebra and the cotangent complex ${}^L\Omega_F \in D(F)$ (see 4.1.5) is perfect of span in $[-n, 0]$.*

Proof. It is clear that (i)⇒(iii)⇐(ii). Suppose F satisfies (iii); let us check (i) and (ii).

(a) Let us show that F admits a resolution $\phi : G \to F$ such that G is semi-free with finitely many generators in each (non-positive) degree.

We write $G = \bigcup G_i$ where G_i is the subalgebra of G generated by $G^{\geq -i}$, and we construct G_i by induction getting on the ith step a morphism of DG algebras $\phi_i = \phi|_{G_i} : G_i \to F$ such that $H^a(G_i) \xrightarrow{\sim} H^aF$ for $a > -i$, $H^{-i}(G_i) \to H^{-i}F$ is surjective, and G_i is semi-free with finitely many generators in degrees $-i, \ldots, 0$. Then G_{i+1} is obtained from G_i by adding finitely many free generators in degree

$-i-1$ and extending ϕ to them. To be able to do this, we need to know that $H^{-i-1}F$ is a finitely generated H^0F-module.[18]

Replacing F by an appropriate resolution, we can assume that F is obtained from G_i by adding (possibly infinitely many) semi-free generators in degrees $< -i$. The exact sequence of DG F-modules $0 \to \Omega_{G_i} \underset{G_i}{\otimes} F \to \Omega_F \to \Omega_{F/G_i} \to 0$ shows that Ω_{F/G_i} is perfect. It has generators in degrees $\leq -i-1$, so $H^{-i-1}\Omega_{F/G_i}$ is a finitely generated H^0F-module. If $i \geq 1$, then the universal derivation yields $F^{-i-1}/(d(F^{-i-2})+G_i^{-i-1}) \xrightarrow{\sim} H^{-i-1}\Omega_{F/G_i}$; hence $H^{-i-1}F$ is a finitely generated H^0F-module, and we are done. Consider the case $i=0$; set $\mathcal{I} := d(F^{-1}) \subset F^0 = G_0$ (which is a finitely generated polynomial algebra), so $H^0F = F^0/\mathcal{I}$. Then $F^{-1}/(d(F^{-2})+\mathcal{I}F^{-1}) \xrightarrow{\sim} H^{-1}\Omega_{F/G_0}$, so $F^{-1}/(d(F^{-2})+\mathcal{I}F^{-1})$ is a finitely generated H^0F-module. The exact sequence $0 \to H^{-1}F \to F^{-1}/d(F^{-2}) \xrightarrow{d} \mathcal{I} \to 0$ shows then that $H^{-1}F$ is a finitely generated H^0F-module; q.e.d.

From now on we assume, as is possible by (a), that F is a polynomial algebra with finitely many generators in each (non-positive) degree.

(b) Let B be any commutative DG algebra, $I \subset B$ a DG ideal. Recall that $n \geq 1$ is an integer such that the span of ${}^L\Omega_F$ lies in $[-n, 0]$.

Lemma. *If $H^iI = 0$ for every $i > -n$, then any morphism $\bar{\rho} : F \to B/I$ can be lifted to $\rho : F \to B$. If, in addition, $H^{-n}I = 0$, then such ρ is unique up to a homotopy.*

Proof. One has $B = \varprojlim B/I^m$, so one can construct a lifting, and a homotopy between two liftings, passing successively from B/I^m to B/I^{m+1}. Therefore we can assume that $I^2 = 0$. Replacing B by the pull-back of $B \to B/I$ by ρ, we can assume that $\bar{\rho} = id_F$. So B is an extension of F by a DG F-module I, and we want to construct a section.

Since F is semi-free, we can find a section $F \to B$ which is a morphism of graded algebras but may not commute with the differential d. Commuting it with d, we get a derivation $F \to I[1]$, i.e., a morphism of DG F-modules $c : \Omega_F \to I[1]$. One can modify our section to make it commute with d if and only if c is homotopic to zero, i.e., since Ω_F is semi-free, if c vanishes as an element of $\mathrm{Hom}_{D(F)}(\Omega_F, I[1])$. If it happens, then the homotopy classes of ρ form a $\mathrm{Hom}_{D(F)}(\Omega_F, I)$-torsor. We are done by the condition on n. □

(c) Let $F_n \subset F$ be the (DG) subalgebra generated by $F^{\geq -n}$ and let $\mathcal{I} \subset F$ be the DG ideal in F generated by $F^{<-n}$. By the lemma, the projection $\bar{\rho} : F \to F/\mathcal{I} = F_n/F_n \cap \mathcal{I}$ can be lifted to a morphism of DG algebras $\rho : F \to F_n$.

Again by the lemma (applied to $B = F$, $I = \mathcal{I}$) the composition $F \xrightarrow{\rho} F_n \to F$ is homotopic to id_F, so F is a retract of F_n in the homotopy category, and we have proved (i).

To prove (ii) consider step (i)⇒(iii) of the proof of the lemma in 4.1.16 for $P := \Omega_F$, $\tilde{P} := \Omega_{F_n} \underset{F_n}{\otimes} F$, $b := -n$. We can choose, after a Zariski localization of $\mathrm{Spec}(H^0F)$, a direct summand $P^{\mathfrak{f}}$ of $\tilde{P}$ as in loc. cit. Then the universal derivation identifies the F^0-module $(F_n/(F_n^{<0})^2)^{-n}$ of degree $-n$ generators of F_n with

[18] We assume that $i \geq 0$; one constructs G_0 in the evident way using the fact that H^0F is a finitely generated algebra.

$(\tilde{P}/F^{<0}\tilde{P})^{-n}$. Define $K \subset (F_n/(F_n^{<0})^2)^{-n}$ as the submodule that corresponds to $(P^{\mathfrak{f}}/F^{<0}P^{\mathfrak{f}})^{-n} \subset (\tilde{P}/F^{<0}\tilde{P})^{-n}$. Let $F^{\mathfrak{f}}$ be the subalgebra of F_n generated by $F_n^{>-n}$ and the preimage of K in F_n^{-n}. Since K is a free F^0-module, our $F^{\mathfrak{f}}$ is semi-free with finitely many generators in degrees $-n, \dots, 0$. Finally, the morphism $F^{\mathfrak{f}} \to F$ is a quasi-isomorphism. To see this, notice that $F^{\mathfrak{f}i} \xrightarrow{\sim} F^i$ for $i > -n$ and $H^{-n}\Omega_{F^{\mathfrak{f}}} \xrightarrow{\sim} H^{-n}\Omega_F$, and apply then the lemma from (b). $\square$

4.1.18. Perfect BV algebras. Let C be a 1-Poisson DG algebra C having degrees ≤ 0. We say that C is *perfect* if it is perfect as a commutative algebra and the pairing $\Omega_R \otimes \Omega_R \to R[1]$ defined by $\{\ \}$ yields a quasi-isomorphism ${}^L\Omega_R \xrightarrow{\sim} R\mathrm{Hom}_R({}^L\Omega_R, R)[1]$. Notice that such a C has span ≤ 1.

A filtered unital BV algebra C having degrees ≤ 0 is said to be *perfect* if the 1-Poisson algebra $\mathrm{gr}\, C$ is.

Suppose that $C = C^L_{BV}(R, \mathcal{L})^\flat$ for some BV Lie R-algebroid $\mathcal{L}^\flat$. Then C is perfect if and only if R is a perfect commutative algebra and the composition $\mathcal{L} \to \Theta_R = \mathrm{Hom}(\Omega_R, R) \to R\mathrm{Hom}({}^L\Omega_R, R)$ is a quasi-isomorphism. Here the first arrow comes from the $\mathcal{L}$-action on R.

Question. Is it true that the cohomology of any perfect BV algebra is finite-dimensional?

This happens for $C = C^L_{BV}(R, \mathcal{L})^\flat$ as above:

Proposition. *For such a C one has* $\dim H^{\cdot} C < \infty$.

Proof. By 4.1.13, replacing R by a quasi-isomorphic algebra, we can assume that R is a semi-free algebra sitting in degrees ≤ 0. Our objects have the Zariski local nature with respect to $\mathrm{Spec}\, R^0$, so it suffices to show that our cohomology is finite-dimensional locally on $\mathrm{Spec}\, R^0$. By 4.1.17 and 4.1.13 we are reduced to the situation when R is a semi-free algebra with finitely many generators in degrees 0 and -1. We can also assume that $\mathcal{L}$ is semi-free. Let us show that $C_{BV}(R, \mathcal{L})^\flat$ is quasi-isomorphic to the de Rham complex for a certain twisted right $\mathcal{D}$-module structure on R (cf. Example in 4.1.10).

We know that Ω_R is semi-free and $\Omega_R = {}^L\Omega_R$. So the morphism $\mathcal{L} \to \Theta_R$ is a quasi-isomorphism. Therefore the corresponding morphism of the de Rham-Chevalley DG algebras $DR(R) = \mathcal{C}_R(R, \Theta_R) \to \mathcal{C}_R(\mathcal{L})$ is a filtered quasi-isomorphism.

The conditions on R imply that R admits a right $\mathcal{D}$-module structure,[19] so Θ_R admits a BV extension $\Theta_R^\flat$ (see Remark (i) in 4.1.9). Its pull-back to $\mathcal{L}$ differs from $\mathcal{L}^\flat$ by some "classical" extension of $\mathcal{L}$ given by an appropriate cocycle of $\mathcal{C}_R(\mathcal{L})^{\geq 1}$. By Remark (ii) in 4.1.10 we can replace this cocycle by a homologous one without changing $C_{BV}(R, \mathcal{L})^\flat$. So we can assume that our cocycle actually comes from an odd 1-cocycle γ in $DR(R)^{\geq 1}$. Let P be $C_{BV}(R, \Theta_R)^\flat$ equipped with the twisted differential d_γ. The evident morphism of graded modules $C_{BV}(R, \mathcal{L})^\flat \to P$ commutes with the differential, and it is a filtered quasi-isomorphism.

It remains to show that $\dim H^{\cdot} P < \infty$. Our P is a twisted de Rham complex, hence a DG $DR(R)$-module. Forgetting about the differentials, one has $DR(R) = \varprojlim \mathrm{Sym}_R(\Omega_R[-1])/\mathrm{Sym}_R^{\geq n}(\Omega_R[-1])$, $P = \mathrm{Sym}_R(\Theta_R[1])$, and the $DR(R)$-action on

[19]Indeed, the dualizing module of R is isomorphic to a shift of a copy of R (as a DG R-module).

P is the usual convolution product. Notice that both P and $DR(R)$ are supported in finitely many degrees.

Consider P as a DG $DR(R^0)$-module. We have a finite decreasing filtration $F^i := \Omega^{\geq i}_{R^0} P$. It suffices to prove that the terms $E_2^{p,q}$ of the corresponding spectral sequence are finite-dimensional.

Set $\bar{P} := \mathrm{gr}^0 P$; this is a complex of free R^0-modules, and $\mathrm{gr}^i P = \Omega^i_{R^0} \underset{R^0}{\otimes} \bar{P}$. Thus $E_1^{p,q} := H^{p+q}\mathrm{gr}^p P = \Omega^p_{R^0} \otimes H^q \bar{P}$ and the differential d_1 is the de Rham differential for a $\mathcal{D}_{R^0}$-module structure on $H^q\bar{P}$. So it suffices to show that all $H^q\bar{P}$ are holonomic $\mathcal{D}_{R^0}$-modules. This amounts to the existence of a stratification Z_α of $\operatorname{Spec} R^0$ such that each complex $\bar{P}|_{Z_\alpha}$ has coherent cohomology. Now the proposition follows from the next two facts whose proof is left to the reader:[20]

- the complex of H^0R-modules $\bar{P} \otimes_{R^0} H^0 R$ is quasi-isomorphic to $(H^0R)[\dim R]$;
- the restriction of $\bar{P}$ to $\operatorname{Spec} R^0 \smallsetminus \operatorname{Spec} H^0 R$ is acyclic. □

4.2. The construction and first properties

Chiral homology is a "quantum" version of (the algebra of functions on) the space of global horizontal sections of an affine $\mathcal{D}_X$-scheme (i.e., the space of global solutions of a system of non-linear differential equations). To define it, recall that chiral algebras amount to factorization algebras which are $\mathcal{D}$-modules on Ran's space $\mathcal{R}(X)$ (see 3.4.1 and 3.4.9). Now the chiral homology of X with coefficients in a chiral algebra is the de Rham homology of $\mathcal{R}(X)$ with coefficients in the corresponding $\mathcal{D}$-module.

Below we make the above informal definition rigorous. The technical problem to deal with is the fact that $\mathcal{R}(X)$ is *not* an ind-scheme in the strict sense[21] (recall that the $\mathcal{R}(X)_i$ are not algebraic varieties for $i \geq 3$; see 3.4.2). So we are outside of the documented grounds of algebraic geometry, and one has to explain what the $\mathcal{D}$-modules on $\mathcal{R}(X)$ are and how to compute the de Rham homology. We know that $\mathcal{R}(X)$ is an ind-scheme in a broad sense: it is an inductive limit of the X^n's with respect to the *non-directed* family of all diagonal embeddings (see 3.4.1(ii)). Luckily this inductive system satisfies a number of specific properties (listed in 4.2.1) which permit us to handle sheaves on $\mathcal{R}(X)$ almost as easily as if it were an ind-scheme in the strict sense.

We begin with the definition of !-sheaves on $X^{\mathcal{S}}$ and consider some special types of complexes in 4.2.1. For a general formalism of sheaves on diagrams of spaces see [SGA 4] Exp. V^{bis}; we consider the dual setting of !-sheaves. The cohomology of !-sheaves is defined in 4.2.2. In 4.2.3 we describe a usual spectral sequence which computes the homology beginning with the homology of the configuration spaces strata of $\mathcal{R}(X)$. In 4.2.4 the !-sheaves on $X^{\mathcal{S}}$ are compared, in the case of the classical topology, with plain sheaves on $\mathcal{R}(X)$. The convolution tensor product $\otimes^*$ of !-sheaves is defined in 4.2.5. The $\mathcal{D}$-module setting and the de Rham homology are considered in 4.2.6. We explain that the de Rham homology can be computed using Dolbeault resolutions in 4.2.7, prove the compatibility with the $\otimes^*$ product of coefficients in 4.2.8, and prove a stabilization property in 4.2.9–4.2.10. The chiral homology of a chiral algebra is defined in 4.2.11 using the Chevalley-Coisin complex from 3.4.11. In 4.2.12 we show that the chiral homology can be computed by

[20] Hint: $\bar{P}$ carries a natural filtration with $\mathrm{gr}_a \bar{P}$ isomorphic to $\mathrm{Sym}^a(\Theta_{R/R^0}[1])[\dim R^0]$.

[21] I.e., $\mathcal{R}(X)$ is not a union of a directed system of closed subschemes.

certain functorial chiral chain complexes $\tilde{C}^{ch}$ and (more economic) $C^{ch}_{\mathcal{P}\mathcal{Q}}$. Another chiral chain complex C^{ch}_{log} is defined in 4.2.14 after a digression on forms with logarithmic singularities (this complex will not be used in subsequent sections, so the reader can skip 4.2.13 and 4.2.14). Section 4.2.15 contains a remark on non-quasi-coherent Dolbeault resolutions. In 4.2.16 we consider the zero homology H_0^{ch}. In the commutative case it coincides with the algebra of functions on the space of global solutions from 2.4.1–2.4.5. Compatibility with filtrations and chiral homology with coefficients are discussed, respectively, in 4.2.18 and 4.2.19.

4.2.1 Sheaves on $X^{\mathcal{S}}$. We begin with general definitions. Below, k is our base field of characteristic 0.

Let $\mathcal{I}$ be a category equivalent to a small one, and let $Y_{\mathcal{I}}$, $I \mapsto Y_I$, be an $\mathcal{I}$-diagram of topological spaces or of schemes. We assume that every Y_I has a finite cohomological dimension (to be able to consider painlessly infinite complexes of sheaves) and that for each $\varphi : I \to J$ the map $\varphi_Y : Y_I \to Y_J$ is a closed embedding. A *!-sheaf on* $Y_{\mathcal{I}}$ is a rule P that assigns to each $I \in \mathcal{I}$ a sheaf of k-vector spaces P_{Y_I} on Y_I and to each $\varphi : I \to J$ a morphism $\theta^{(\varphi)} : \varphi_Y . P_{Y_I} \to P_{Y_J}$. We demand that the θ's are compatible with the composition of the φ's and that $\theta^{(id)}$ is the identity map. In the scheme-theoretic setting we consider sheaves with respect to the étale topology. The !-sheaves form an abelian k-category $Sh^!(Y_{\mathcal{I}})$. We denote by $CSh^!(Y_{\mathcal{I}})$ the corresponding DG category of complexes and by $D(Y_{\mathcal{I}})$ the derived category.

We say that a complex P of !-sheaves on $Y_{\mathcal{I}}$ is *admissible* if for each morphism $\varphi : I \to J$ the morphism $\theta^{(\varphi)}$ yields a quasi-isomorphism $P_{Y_I} \xrightarrow{\sim} R\varphi^!_Y P_{Y_J}$. Admissible complexes form a full DG subcategory $CSh^!(Y_{\mathcal{I}})_{adm} \subset CSh^!(Y_{\mathcal{I}})$ closed under quasi-isomorphisms, and a full triangulated subcategory $D(Y_{\mathcal{I}})_{adm} \subset D(Y_{\mathcal{I}})$.

We apply the above considerations to $\mathcal{I} = \mathcal{S}^\circ$ and the diagram $X^{\mathcal{S}}$, $I \mapsto X^I$; the structure embeddings are diagonal ones $\Delta^{(I/J)} : X^J \hookrightarrow X^I$. Here X is either a separated topological space of finite cohomological dimension or a separated k-scheme of finite type. In the case of schemes we consider sheaves with respect to the étale topology.

The diagram $X^{\mathcal{S}}$ satisfies the following specific properties. Fix any $I \in \mathcal{S}$. Then

- The category $\mathcal{S}^\circ_{/I}$ of all arrows $I \twoheadrightarrow T$ is equivalent to a finite ordered set $Q(I)$.

- The diagonals X^T, $T \in Q(I)$, form a stratification of X^I.

- Let $j^{(I)} : U^{(I)} \hookrightarrow X^I$ be the complement to all strata X^T, $T \in Q(I)$, $T \neq I$. Then the action of Aut I on $U^{(I)}$ is free.

The arguments below work for any diagram that satisfies these properties.

Every !-sheaf P on $X^{\mathcal{S}}$ carries a canonical *Cousin filtration* $P_1 \subset P_2 \subset \cdots$ where $P_a \subset P$ is the smallest subsheaf such that $P_{aX^J} = P_{X^J}$ for $|J| \leq a$. We say that P is *nice* if for every I and n the evident morphism $\underset{T\in Q(I,a)}{\oplus} \Delta^{(I/T)}_{\cdot} \mathrm{gr}_a P_{X^T} \to \mathrm{gr}_a P_{X^I}$ is an isomorphism.

LEMMA. *Suppose that P is nice and admissible. Then for any I the identification $\mathrm{gr}_{|I|} P_{U^{(I)}} = P_{U^{(I)}}$ yields a quasi-isomorphism*

$$\mathrm{gr}_{|I|} P_{X^I} \xrightarrow{\sim} Rj^{(I)}_* P_{U^{(I)}}. \tag{4.2.1.1}$$

Proof. We use induction by $n := |I|$. Assume that our assertion is true for all I' of order $< n$. Consider a filtration $X_1^I \subset \cdots \subset X_n^I = X^I$ on X^I where X_a^I is the union of the diagonals of dimension $\le a$, so $X_a^I \smallsetminus X_{a-1}^I$ is the disjoint union of $U^{(T)}$, $T \in Q(I,a)$. It yields a filtration on P_{X^I} (as on an object of the derived category of sheaves on X^I) with successive quotients $\mathrm{gr}'_a P_{X^I} = \underset{T\in Q(I,a)}{\oplus} R(\Delta^{(I/T)} j^{(T)})_{\cdot} R(\Delta^{(I/T)} j^{(T)})^! P_{X^I} = \underset{T\in Q(I,a)}{\oplus} \Delta_{\cdot}^{(I/T)} R j_{\cdot}^{(T)} j^{(T)\cdot} R\Delta^{(I/T)!} P_{X^I}$. Since P_{aX^I} are supported on X_a^I, the identity morphism of P_{X^I} lifts canonically to a morphism ϕ in the filtered derived category from P_{X^I} equipped with the Cousin filtration to P_{X^I} equipped with the above filtration. Since P is nice, one has $\mathrm{gr}_a\phi = \underset{T\in Q(I,a)}{\oplus} \Delta_{\cdot}^{(I/T)}(\phi_T)$ where $\phi_T : \mathrm{gr}_a P_{X^T} \to R j_{\cdot}^{(T)} j^{(T)\cdot} R\Delta^{(I/T)!} P_{X^I}$ is the evident morphism. Since P is admisible, we can rewrite ϕ_T as the canonical morphism $P_{X^T} \to R j_{\cdot}^{(T)} j^{(T)\cdot} P_{X^T}$. By the induction assumption, ϕ_T is a quasi-isomorphism for $|T| < n$, so $\mathrm{gr}_a\phi$ is a quasi-isomorphism for $a < n$. Our filtration has length n and ϕ lifts the identity morphism, so $\mathrm{gr}_n\phi$, which is (4.2.1.1), is a quasi-isomorphism as well, q.e.d. □

Every $P \in CSh^!(X^{\mathcal{S}})$ admits a *canonical nice resolution* $\tilde{P} \to P$. Namely, $\tilde{P}_{X^I}$ is the homotopy direct limit $C(Q(I), \Delta_*^{(I/T)} P_{X^T})$ (see 4.1.1(iv)) where the structure morphisms are the embeddings $\theta^{(I/J)} : \Delta_*^{(I/J)} \tilde{P}_{X^J} \hookrightarrow \tilde{P}_{X^I}$ coming from the embedding $Q(J) \subset Q(I)$. It is clear that $\tilde{P}$ is nice, the DG functor $P \mapsto \tilde{P}$ is exact, and the obvious projection $\tilde{P} \to P$ is a quasi-isomorphism.

Notice that any quasi-isomorphism between nice complexes is automatically a *filtered* quasi-isomorphism with respect to the Cousin filtrations.

4.2.2. Cohomology with coefficients in a !-sheaf. A !-sheaf P on $X^{\mathcal{S}}$ yields an $\mathcal{S}^\circ$-diagram of vector spaces, $I \mapsto \Gamma(X^I, P) := \Gamma(X^I, P_{X^I})$. Denote by $\Gamma(X^{\mathcal{S}}, P)$ its inductive limit. The Cousin filtration on P defines a filtration on $\Gamma(X^{\mathcal{S}}, P)$.

We say that P is *handsome*[22] if it is nice and the following extra conditions hold:

(a) The morphisms $\Gamma(X^n, P_{X^n}) \to R\Gamma(X^n, P_{X^n})$ are quasi-isomorphisms.

(b) The projections $P_{X^n} \to \mathrm{gr}_n P_{X^n}$ admit splittings (that need not commute with the differential).

LEMMA. *If P is handsome, then* $\mathrm{gr}_n\Gamma(X^{\mathcal{S}}, P) \xrightarrow{\sim} \Gamma(X^n, \mathrm{gr}_n P)_{\Sigma_n}$ *and the morphisms* $\Gamma(X^n, \mathrm{gr}_n P) \to R\Gamma(X^n, \mathrm{gr}_n P)$ *are quasi-isomorphisms.*

Proof. By (b) the Cousin filtration on P splits (the splitting need not commute with the differential), so $\mathrm{gr}_n\Gamma(X^{\mathcal{S}}, P) = \mathrm{gr}_n\Gamma(X^{\mathcal{S}}, \mathrm{gr}_n P)$, and the first claim follows since $\mathrm{gr}_n P$ is nice. The second claim is checked by induction by n. □

REMARKS. (i) For any $P \in CSh^!(X)$ the nice complex $\tilde{P}$ automatically satisfies (b); if P satisfies (a), then $\tilde{P}$ is handsome.

(ii) Every $P \in CSh^!(X)$ admits a flabby resolution, i.e., a quasi-isomorphism $P \to P'$ such that P' is flabby (which means that each P'_{X^n} is a flabby complex of sheaves on X^n).[23]

[22] Cf. "Teddy Bear" from "When we were very young" by A. A. Milne.

[23] Recall that a complex of sheaves F on Y is flabby if for every closed $Z \subset Y$ the morphism $\Gamma_Z(Y, F) \to R\Gamma_Z(Y, F)$ is a quasi-isomorphism.

Let $CSh^!(X)^h \subset CSh^!(X^{\mathcal{S}})$ be the full DG subcategory of handsome complexes. The above remarks imply that its localization by quasi-isomorphisms coincides with $D(X^{\mathcal{S}})$. On the other hand, the lemma shows that the functor $\Gamma(X^{\mathcal{S}}, \cdot)$ sends quasi-isomorphisms in $CSh^!(X)^h$ to filtered quasi-isomorphisms. Therefore one has an exact functor $R\Gamma(X^{\mathcal{S}}, \cdot) : D(X^{\mathcal{S}}) \to DF(k)$.[24] We write $H^{\cdot}(X^{\mathcal{S}}, P) := H^{\cdot} R\Gamma(X^{\mathcal{S}}, P)$.

By Remark (i), for any $P \in CSh^!(X)$ one has $R\Gamma(X^{\mathcal{S}}, P) = \Gamma(X^{\mathcal{S}}, \tilde{P'})$ where $P \to P'$ is a resolution such that P' satisfies (a); e.g., P' is flabby.

4.2.3. Cohomology of configuration spaces. Set $\mathcal{R}(X)_n^{\circ} := U^{(n)}/\Sigma_n$;[25] this is the space of configurations of n-points on X. The projection $U^{(n)} \to \mathcal{R}(X)_n^{\circ}$ is an étale Σ_n-covering. For $P \in Sh^!(X)$ the sheaf $P_{U^{(n)}}$ is Σ_n-equivariant, so by descent it defines a sheaf $P_{\mathcal{R}(X)_n^{\circ}}$ on $\mathcal{R}(X)_n^{\circ}$. We write $\Gamma(\mathcal{R}(X)_n^{\circ}, P) := \Gamma(\mathcal{R}(X)_n^{\circ}, P_{\mathcal{R}(X)_n^{\circ}})$, etc.

Lemma. *For admissible P there is a canonical quasi-isomorphism*

$$\mathrm{gr}_n R\Gamma(X^{\mathcal{S}}, P) \xrightarrow{\sim} R\Gamma(\mathcal{R}(X)_n^{\circ}, P). \tag{4.2.3.1}$$

Proof. By Lemmas from 4.2.1, 4.2.2 one has $\mathrm{gr}_n R\Gamma(X^{\mathcal{S}}, P) \xrightarrow{\sim} R\Gamma(U^{(n)}, P)_{\Sigma_n}$. Now the trace map yields $R\Gamma(U^{(n)}, P)_{\Sigma_n} \xrightarrow{\sim} R\Gamma(\mathcal{R}(X)_n^{\circ}, P)$, and we are done. □

Consider the spectral sequence $E_r^{p,q}$ for the Cousin filtration converging to $H^{\cdot}(X^{\mathcal{S}}, P)$ (we call it the *Cousin* spectral sequence). If P is admissible then, by the lemma, one has

$$E_1^{p,q} = H^{p+q}(\mathcal{R}(X)_{-p}^{o}, P). \tag{4.2.3.2}$$

4.2.4. Comparison with sheaves on $\mathcal{R}(X)$. This section plays a purely motivational role.

Suppose that our X is a locally compact separated topological space of finite cohomological dimension. Consider the topological space $\mathcal{R}(X)$ defined in 3.4.1; the obvious projections $r_I : X^I \to \mathcal{R}(X)$ identify $\mathcal{R}(X)$ with the inductive limit of the diagram $X^{\mathcal{S}}$. We deal only with those sheaves of k-vector spaces on $\mathcal{R}(X)$ which are inductive limits of subsheaves supported on the $\mathcal{R}(X)_i$'s. Such sheaves form an abelian k-category $Sh(\mathcal{R}(X))$; one has the corresponding DG category of complexes $CSh(\mathcal{R}(X))$ and the derived category $D(\mathcal{R}(X))$. For $G \in Sh(\mathcal{R}(X))$ the space of those sections of G that are supported on some $\mathcal{R}(X)_i$ is denoted (by abuse of notation) by $\Gamma(\mathcal{R}(X), G)$. It is naturally filtered by the subspaces $\Gamma(\mathcal{R}(X), G)_n$ of sections supported on $\mathcal{R}(X)_n$ (the Cousin filtration). The functor Γ admits a right derived functor $R\Gamma(\mathcal{R}(X), \cdot) : D(\mathcal{R}(X)) \to DF(k)$ which can be computed by means of flabby resolutions.[26] We write $H^{\cdot}(\mathcal{R}(X), G) := H^{\cdot} R\Gamma(\mathcal{R}(X), G)$.

The maps $r_I : X^I \to \mathcal{R}(X)$ are finite, so the push-forward functor $r_{I*} : Sh(X^I) \to Sh(\mathcal{R}(X))$ is exact. It admits a right adjoint functor $r_I^! : Sh(\mathcal{R}(X)) \to Sh(X^I)$. We have a right exact functor $r_* : Sh^!(X^{\mathcal{S}}) \to Sh(\mathcal{R}(X))$ which sends P

[24] For us, the filtered derived category is formed by complexes F equipped with an increasing filtration $F_{\cdot}$ such that $F_a = 0$ for $a \ll 0$ and $\bigcup F_a = F$; the morphisms are morphisms of complexes preserving filtrations localized by filtered quasi-isomorphisms (:= the morphisms that induce quasi-isomorphisms between the successive quotients).

[25] Here $U^{(n)} := U^{(\{1,\dots,n\})} \subset X^n$, etc.

[26] $G \in CSh(\mathcal{R}(X))$ is said to be flabby if for every n the complex of sheaves $i_n^! G$ on $\mathcal{R}(X)_n$ is flabby; here $i_n : \mathcal{R}(X)_n \hookrightarrow \mathcal{R}(X)$.

to the inductive $\mathcal{S}^\circ$-limit of sheaves $r_{I*}P_{X^I}$. It admits a right adjoint which is a left exact functor $r^! : Sh(\mathcal{R}(X)) \to Sh^!(X^{\mathcal{S}})$; one has $r^! F_{X^I} := r_I^! F$.

The functor r_* admits a left derived functor $Lr_* : D(X^{\mathcal{S}}) \to D(\mathcal{R}(X))$; for nice P one has $Lr_* P \xrightarrow{\sim} r_* P$. Thus for any P one has $r_*\tilde{P} \xrightarrow{\sim} Lr_* P$. Similarly, $r^!$ admits a right derived functor $Rr^! : D(\mathcal{R}(X)) \to D(X^{\mathcal{S}})$. For a flabby complex of sheaves F on $\mathcal{R}(X)$ one has $r^! F \xrightarrow{\sim} Rr^! F$; moreover, the complex $r^! F$ is nice and admissible, and $r_* r^! F \xrightarrow{\sim} F$. If P is admissible, then $P \to Rr^! Lr_* P$ is a quasi-isomorphism.

So the functors Lr_* and $Rr^!$ are adjoint, and they yield mutually inverse equivalences

$$D(X^{\mathcal{S}})_{adm} \underset{Rr^!}{\overset{Lr_!}{\rightleftarrows}} D\mathcal{R}(X). \tag{4.2.4.1}$$

For any $P \in CSh^!(X^{\mathcal{S}})$ an obvious projection $\Gamma(X^{\mathcal{S}}, P) \to \Gamma(\mathcal{R}(X), r_* P)$ is compatible with the Cousin filtrations. If P is handsome, then this projection is an isomorphism of mere complexes;[27] moreover $\Gamma(\mathcal{R}(X), r_* P) \xrightarrow{\sim} R\Gamma(\mathcal{R}(X), r_* P)$. If, in addition, P is admissible, then this is a *filtered* quasi-isomorphism. We get

LEMMA. *For $P \in D(X^{\mathcal{S}})$ there is a natural morphism*

$$R\Gamma(X^{\mathcal{S}}, P) \to R\Gamma(\mathcal{R}(X), Lr_* P) \tag{4.2.4.2}$$

in $DF(k)$ which is an isomorphism in $D(k)$. If $P \in D(X^{\mathcal{S}})_{adm}$, then this is a filtered quasi-isomorphism. □

REMARK. The spectral sequence (4.2.3.2) becomes the usual spectral sequence for the Cousin filtration $\mathcal{R}(X)_\cdot$ converging to $H^\cdot(\mathcal{R}(X), Lr_* P)$.

NOTATION. Suppose we are in the scheme-theoretic setting. In view of (4.2.4.1) we will write $D(\mathcal{R}(X)) := D(X^{\mathcal{S}})_{adm}$; admissible complexes will be also called complexes of sheaves on $\mathcal{R}(X)$. Thus $D(\mathcal{R}(X))$ is a full subcategory of $D(X^{\mathcal{S}})$. The restriction of the functor $R\Gamma(X^{\mathcal{S}}, \cdot)$ to $D(\mathcal{R}(X))$ is denoted by $R\Gamma(\mathcal{R}(X), \cdot)$; the terms of its Cousin filtration by $R\Gamma(\mathcal{R}(X), \cdot)_n = R\Gamma(\mathcal{R}(X)_n, \cdot)$. One has $\mathrm{gr}_n R\Gamma(\mathcal{R}(X), P) \xrightarrow{\sim} R\Gamma(\mathcal{R}(X)_n^\circ, P)$ (see (4.2.3.1)), etc.

4.2.5. The convolution product. The category $Sh^!(X^{\mathcal{S}})$ carries a natural tensor structure. Namely, for a finite non-empty family P_α, $\alpha \in A$, of !-sheaves on $X^{\mathcal{S}}$ we define their *convolution tensor product* $\otimes^* P_\alpha$ as (cf. (3.4.10.1))

$$(\otimes^* P_\alpha)_{X^I} := \underset{I \twoheadrightarrow A}{\oplus} \underset{A}{\boxtimes} (P_{\alpha\, X^{I_\alpha}}), \tag{4.2.5.1}$$

where the structure arrows $\theta^{(\pi)}$ are the obvious ones. Our tensor category is denoted by $Sh^!(X^{\mathcal{S}})^*$.

The obvious natural morphisms $\nu_{\{P_\alpha\}} : \otimes \Gamma(X^{\mathcal{S}}, P_\alpha) \to \Gamma(X^{\mathcal{S}}, \otimes^* P_\alpha)$ are compatible with the Cousin filtrations; they make $\Gamma(X^{\mathcal{S}}, \cdot)$ a pseudo-tensor functor (see 1.1.6(ii))

$$\Gamma(X^{\mathcal{S}}, \cdot) : Sh^!(X^{\mathcal{S}})^* \to CF(k)^{\otimes}. \tag{4.2.5.2}$$

Notice that $\otimes^* P_\alpha$ is nice if the P_α are nice. The same is true for "handsome", so $R\Gamma(X^{\mathcal{S}}, \cdot)$ is also a pseudo-tensor functor.

[27] *Not* filtered ones!

LEMMA. *If X is quasi-compact, then $\Gamma(X^{\mathcal{S}},\cdot)$ and $R\Gamma(X^{\mathcal{S}},\cdot)$ are tensor functors.*

Proof. Our condition assures that $\Gamma(X^I,\otimes^* P_\alpha)=\underset{I\twoheadrightarrow A}{\oplus}\underset{A}{\otimes}\Gamma(X^{I_\alpha},P_\alpha)$, so $\nu_{\{P_\alpha\}}$ are filtered isomorphisms. The fact about $R\Gamma(X^{\mathcal{S}},\cdot)$ follows as above. $\square$

Notice that $P\mapsto\tilde{P}$ is naturally a pseudo-tensor functor, as follows from (4.1.2.3). Namely, the compatibility morphism $\otimes^*\tilde{P}_\alpha\to\widetilde{\otimes^* P_\alpha}$ (see 1.1.6(ii)) is formed by the maps $\boxtimes C(Q(I_\alpha),\Delta_*^{(I_\alpha/T_\alpha)}P_{\alpha X^{T_\alpha}})\to C(\Pi Q(I_\alpha),\boxtimes\Delta_*^{(I_\alpha/T_\alpha)}P_{\alpha X^{T_\alpha}})\subset(\widetilde{\otimes^* P_\alpha})_{X^I}$ where the arrow comes from (4.1.2.1). The projection $\tilde{P}\to P$ is a morphism of pseudo-tensor functors.

Let $\mathcal{B}$ be a DG k-operad. We see that if P is a $\mathcal{B}$ algebra in $CSh^!(X)^*$, then so is $\tilde{P}$. By (4.2.5.2) the complexes of sections $\Gamma(X^{\mathcal{S}},\tilde{P})$ and $\Gamma(X^{\mathcal{S}},P)$ are $\mathcal{B}$ algebras, and the morphism $\Gamma(X^{\mathcal{S}},\tilde{P})\to\Gamma(X^{\mathcal{S}},P)$ is a morphism of $\mathcal{B}$ algebras.

REMARK. The tensor product $\otimes^*$ comes very naturally in the $\mathcal{R}(X)$ setting of 4.2.4. Then the commutative semigroup structure on $\mathcal{R}(X)$ (see (iii) in 3.4.1) yields the convolution tensor structure on $Sh(\mathcal{R}(X))$: for $F,G\in Sh(\mathcal{R}(X))$ one has $F\otimes^*G:=\circ_*(F\boxtimes G)$ where $\circ:\mathcal{R}(X)\times\mathcal{R}(X)\to\mathcal{R}(X)$ is the "disjoint sum" operation. Notice that $\otimes^*$ is an exact functor (since $\circ$ is finite). If X is quasi-compact, then there is an obvious identification $\Gamma(\mathcal{R}(X),F\otimes^*G)=\Gamma(\mathcal{R}(X),F)\otimes\Gamma(\mathcal{R}(X),G)$, and the same for the $R\Gamma$'s. So $\Gamma(\mathcal{R}(X),\cdot)$ and $R\Gamma(\mathcal{R}(X),\cdot)$ are tensor functors. Now $r_*:Sh^!(X^{\mathcal{S}})^*\to Sh(\mathcal{R}(X))^*$, $Lr_*:D(X^{\mathcal{S}})^*\to D(\mathcal{R}(X))^*$ are tensor functors in the obvious way. This explains the name "convolution tensor product" for $\otimes^*$.

4.2.6. $\mathcal{D}$-complexes on $X^{\mathcal{S}}$. (i) Suppose we have any diagram $I\mapsto Y_I$ as in 4.2.1 where the Y_I are smooth algebraic varieties and morphisms are closed embeddings. In such a situation a *right $\mathcal{D}$-module* on Y is a rule that assigns to each I a right $\mathcal{D}$-module $M_I=M_{Y_I}$ on Y_I and to every $\varphi:I\to J$ a morphism $\theta^{(\varphi)}:\varphi_{Y*}M_I\to M_J$; we demand that $\theta^{(\varphi)}$ is compatible with the composition of the φ's and $\theta^{(id_I)}=id_{M_I}$. Right $\mathcal{D}$-modules on Y form an abelian k-category $\mathcal{M}(Y)$. A complex M of right $\mathcal{D}$-modules on Y (a.k.a. a *right $\mathcal{D}$-complex on Y*) is *admissible* if for each φ the morphism $M_I\to R\varphi_Y^!M_J$ defined by $\theta^{(\varphi)}$ is a quasi-isomorphism. Every complex quasi-isomorphic to an admissible complex is admissible.

Our prime diagram is $X^{\mathcal{S}}$ where X is our curve; here the above notion of right $\mathcal{D}$-module coincides with that from 3.4.10. Admissible right $\mathcal{D}$-complexes on $X^{\mathcal{S}}$ are also called *right $\mathcal{D}$-complexes on* $\mathcal{R}(X)$. The corresponding DG category is denoted by $C\mathcal{M}(\mathcal{R}(X))$ and the derived category is denoted by $D\mathcal{M}(\mathcal{R}(X))$ (cf. 4.2.4). We have fully faithful embeddings $C\mathcal{M}(\mathcal{R}(X))\hookrightarrow C\mathcal{M}(X^{\mathcal{S}})$, $D\mathcal{M}(\mathcal{R}(X))\hookrightarrow D\mathcal{M}(X^{\mathcal{S}})$. Below we skip the word "right" when it is clear that we are dealing with right $\mathcal{D}$-modules.

An increasing filtration on a $\mathcal{D}$-complex M is *admissible* if gr_iM are admissible complexes. The corresponding filtered derived category is denoted by $DF\mathcal{M}(\mathcal{R}(X))$.

(ii) Recall that the tensor category $\mathcal{M}^\ell(\mathcal{R}(X))$ acts on $\mathcal{M}(X^{\mathcal{S}})$ (see (3.4.10.6)), so the tensor DG category of complexes $C\mathcal{M}^\ell(\mathcal{R}(X))$ acts on $C\mathcal{M}(X^{\mathcal{S}})$. The action of the tensor DG subcategory of homotopically flat complexes (see 3.4.3) preserves admissible complexes and quasi-isomorphisms, so it yields an action on $D\mathcal{M}(\mathcal{R}(X))$.

(iii) When needed, resolutions of $\mathcal{D}$-complexes on $X^{\mathcal{S}}$ can be built by induction. Here is the induction step: Suppose we have $M\in C\mathcal{M}(X^{\mathcal{S}})$ and a quasi-isomorphic embedding of Σ_n-equivariant $\mathcal{D}$-complexes $\alpha_n : M_{X^n} \hookrightarrow F_{X^n}$ which is a quasi-isomorphism. Then there is a quasi-isomorphic embedding $M \hookrightarrow F$ of $\mathcal{D}$-complexes on $X^{\mathcal{S}}$ which is an isomorphism on $X^{<n}$ and coincides with α_n on X^n.[28]

In particular, every $M \in C\mathcal{M}(X^{\mathcal{S}})$ admits a flabby resolution F (which means that each F_{X^n} is flabby). This implies the next lemma and remark in the usual way.

Let $i_Y : Y_I \hookrightarrow X^I$, $I \in \mathcal{S}$, be closed subspaces such that for every diagonal embedding $X^I \subset X^J$ one has $Y_I = X^I \cap Y_J$. Let $j_V : V_I := X^I \smallsetminus Y_I$ be the complementary open embeddings. These spaces form $\mathcal{S}^\circ$-diagrams Y, V of closed embeddings. Let $\mathcal{M}(X^{\mathcal{S}})_Y \subset \mathcal{M}(X^{\mathcal{S}})$ be the full subcategory of $\mathcal{D}$-modules supported on Y. The functor j_V^* identifies $\mathcal{M}(V)$ with the quotient abelian category $\mathcal{M}(X^{\mathcal{S}})/\mathcal{M}(X^{\mathcal{S}})_Y$. It admits right adjoint $j_{V*} : \mathcal{M}(V) \to \mathcal{M}(X^{\mathcal{S}})$, $(j_{V*}N)_{X^I} = j_{V_I *}N_{V_I}$.

LEMMA. *The functor $D(\mathcal{M}(X^{\mathcal{S}})_Y) \to D\mathcal{M}(X^{\mathcal{S}})$ is fully faithful; its essential image is the thick subcategory $D\mathcal{M}(X^{\mathcal{S}})_Y \subset D\mathcal{M}(X^{\mathcal{S}})$ of complexes acyclic on V. The functor j_V^* induces an equivalence $D\mathcal{M}(X^{\mathcal{S}})/D\mathcal{M}(X^{\mathcal{S}})_Y \xrightarrow{\sim} D\mathcal{M}(V)$. It admits right adjoint Rj_{V*}; one has $(Rj_{V*}N)_{X^I} = Rj_{V_I *}N_{V_I}$. For any $M \in D\mathcal{M}(X^{\mathcal{S}})$ there is a canonical exact triangle $M_Y \to M \to Rj_{V*}j_V^* M$ where $M_Y \in D\mathcal{M}(X^{\mathcal{S}})_Y$. If M is admissible then so are all terms of the triangle.* $\square$

REMARK. Suppose each Y_I is smooth. By Kashiwara's lemma the functor $i_{Y*} : \mathcal{M}(Y) \to \mathcal{M}(X^{\mathcal{S}})$ identifies $\mathcal{M}(Y)$ with $\mathcal{M}(X^{\mathcal{S}})_Y$. It admits a left adjoint $i_Y^! : \mathcal{M}(X^{\mathcal{S}}) \to \mathcal{M}(Y)$. We have the right derived functor $Ri_Y^! : D\mathcal{M}(X^{\mathcal{S}}) \to D\mathcal{M}(Y)$, and M_Y from the above lemma equals $i_{Y*}Ri_Y^! M$.

(iv) A $\mathcal{D}$-complex M on $X^{\mathcal{S}}$ yields a complex $DR(M)$ of !-sheaves on $X^{\mathcal{S}}$, $DR(M)_{X^I} := DR(M_{X^I})$. The DG functor $DR : C\mathcal{M}(X^{\mathcal{S}}) \to CSh^!(X^{\mathcal{S}})$ preserves quasi-isomorphisms and sends admissible complexes to admissible complexes, so we have the exact functors $DR : D\mathcal{M}(X^{\mathcal{S}}) \to D(X^{\mathcal{S}})$, $D\mathcal{M}(\mathcal{R}(X)) \to D(\mathcal{R}(X))$. Set $\Gamma_{DR}(X^{\mathcal{S}}, M) := \Gamma(X^{\mathcal{S}}, DR(M))$ and $R\Gamma_{DR}(X^{\mathcal{S}}, M) := R\Gamma(X^{\mathcal{S}}, DR(M))$. These complexes carry the Cousin filtrations coming from the Cousin filtrations on Γ and $R\Gamma$. For an admissible M we write $H^i_{DR}(\mathcal{R}(X), M) := H^i R\Gamma_{DR}(\mathcal{R}(X), M)$, $H^i_{DR}(\mathcal{R}(X)_n, M) := H^i(R\Gamma_{DR}(\mathcal{R}(X), M)_n)$.

We will also consider the canonical nice resolution $\widetilde{DR}$ of DR (see 4.2.1) and write $\Gamma_{\widetilde{DR}}(X^{\mathcal{S}}, M) := \Gamma(X^{\mathcal{S}}, \widetilde{DR}(M))$.

REMARK. The complexes $DR(M)$ are usually *not* nice. For example, for any non-zero $N \in \mathcal{M}(X)$ the complex $DR(\Delta_*^{(\mathcal{S})}N)$ (see 3.4.10) is not nice.

If we are in the situation of the above lemma, then, applying $R\Gamma_{DR}$ to its exact triangle, we get an exact triangle (here $R\Gamma_{DR}(X^{\mathcal{S}}, M)_Y := R\Gamma_{DR}(X^{\mathcal{S}}, M_Y)$)

$$R\Gamma_{DR}(X^{\mathcal{S}}, M)_Y \to R\Gamma_{DR}(X^{\mathcal{S}}, M) \to R\Gamma_{DR}(X^{\mathcal{S}}, Rj_{V*}j_V^* M). \tag{4.2.6.1}$$

One also has $R\Gamma_{DR}(X^{\mathcal{S}}, Rj_{V*}j_V^* M) = R\Gamma_{DR}(V, j_V^* M)$.

(v) A $\mathcal{D}$-complex M on $X^{\mathcal{S}}$ also yields a complex $h(M)$ of !-sheaves on $X^{\mathcal{S}}$, hence the filtered complex $\Gamma(X^{\mathcal{S}}, h(M))$. The canonical projections $\widetilde{DR}(M) \to$

[28] Our conditions define F on $X^{\mathcal{S}_n}$. Suppose $|I| > n$. Define F_{X^I} as the quotient of $M_{X^I} \oplus (\bigoplus_{T \in Q(I,n)} \Delta_*^{(I/T)} F_T)$ modulo the sum of the images of the maps $(\theta^{(I/T)}, -\Delta_*^{(I/T)}(\alpha_n))$: $\Delta_*^{(I/T)} M_{X^T} \hookrightarrow M_{X^I} \oplus \Delta_*^{(I/T)} F_{X^T}$.

$DR(M) \to h(M)$ yield morphisms of filtered complexes

$$\Gamma_{\widetilde{DR}}(X^{\mathcal{S}}, M) \to \Gamma_{DR}(X^{\mathcal{S}}, M) \to \Gamma(X^{\mathcal{S}}, h(M)). \tag{4.2.6.2}$$

4.2.7. Dolbeault resolutions. One can represent $R\Gamma_{DR}(X^{\mathcal{S}}, \cdot)$ by appropriate functorial complexes.

Namely, let $\mathcal{Q} = \mathcal{Q}_X$ be a Dolbeault $\mathcal{D}_X$-algebra (see 4.1.3). According to 3.4.9 and 3.4.20, it gives rise to a commutative factorization DG algebra which we denote also by $\mathcal{Q}$ by abuse of notation. Thus for every $n \geq 1$ we have a commutative DG $\mathcal{D}_{X^n}$-algebras $\mathcal{Q}_{X^n}$. We will call such a factorization algebra a *Dolbeault* $\mathcal{D}_{\mathcal{R}(X)}$-*algebra*.

LEMMA. *Every $\mathcal{Q}_{X^n}$ is a Dolbeault $\mathcal{D}_{X^n}$-algebra.*

Proof. Condition (a) of 4.1.3 holds since it obviously holds on the level of the Cousin complexes. Condition (b) follows from Remark (i) in 3.4.4. By construction, $\mathcal{Q}_{X^n}$ is a $\mathcal{Q}_X^{\boxtimes n}$-algebra, so one has an affine morphism $\operatorname{Spec} \mathcal{Q}_{X^n} \to (\operatorname{Spec} \mathcal{Q}_X)^n$. The latter is an affine scheme which implies condition (c). $\square$

Let M be a right $\mathcal{D}$-complex M on $X^{\mathcal{S}}$. The above lemma together with (4.1.4.1) show that $DR(M \otimes \mathcal{Q})$ satisfies condition (a) of 4.2.2. Thus, by Remark (i) in loc. cit., the complex $\widetilde{DR}(M \otimes \mathcal{Q})$ is handsome, so we have a canonical filtered quasi-isomorphism

$$\Gamma_{\widetilde{DR}}(X^{\mathcal{S}}, M \otimes \mathcal{Q}) \xrightarrow{\sim} R\Gamma_{DR}(X^{\mathcal{S}}, M). \tag{4.2.7.1}$$

4.2.8. $\otimes^*$ compatibilities. According to 3.4.10 the category $\mathcal{M}(X^{\mathcal{S}})$ carries two canonical tensor structures $\otimes^*$ and $\otimes^{ch}$.

The de Rham functor is a pseudo-tensor functor in an evident way:

$$DR : C\mathcal{M}(X^{\mathcal{S}})^* \to CSh^!(X^{\mathcal{S}})^*. \tag{4.2.8.1}$$

LEMMA. *The functor*

$$\Gamma_{DR}(X^{\mathcal{S}}, \cdot) : C\mathcal{M}(X^{\mathcal{S}})^* \to CF(k)^{\otimes} \tag{4.2.8.2}$$

is actually a tensor functor.

Proof. $\Gamma_{DR}(X^{\mathcal{S}}, \cdot)$ is a pseudo-tensor functor by (4.2.8.1) and (4.2.5.2). We want to show that the compatibility arrows (see 1.1.6(ii)) are isomorphisms; this follows from the fact that $\Gamma(X^{\sqcup I_\alpha}, DR(\boxtimes M_{\alpha\, X^{I_\alpha}})) = \otimes\Gamma(X^{I_\alpha}, DR(M_{\alpha\, X^{I_\alpha}}))$. $\square$

Composing DR and $\Gamma_{DR}(X^{\mathcal{S}}, \cdot)$ with $P \mapsto \tilde{P}$ (see 4.2.1 and 4.2.5), we get pseudo-tensor functors

$$\widetilde{DR} : C\mathcal{M}(X^{\mathcal{S}})^* \to CSh^!(X^{\mathcal{S}})^*, \tag{4.2.8.3}$$

$$\Gamma_{\widetilde{DR}}(X^{\mathcal{S}}, \cdot) : C\mathcal{M}(X^{\mathcal{S}})^* \to CF(k)^{\otimes}. \tag{4.2.8.4}$$

Let $\mathcal{Q} \in C\mathcal{M}^\ell(\mathcal{R}(X))$ be any commutative factorization algebra. Then $M \mapsto M \otimes \mathcal{Q}$ (see 4.2.6(ii)) is a pseudo-tensor endofunctor of $\mathcal{M}(X^{\mathcal{S}})^*$. Namely, the compatibility morphisms $\otimes^*(M_\alpha \otimes \mathcal{Q}) \to (\otimes^* M_\alpha) \otimes \mathcal{Q}$ (see 1.1.6(ii)) are formed by the maps $\boxtimes(M_{X^{I_\alpha}} \otimes \mathcal{Q}_{X^{I_\alpha}}) = (\boxtimes M_{X^{I_\alpha}}) \otimes (\boxtimes \mathcal{Q}_{X^{I_\alpha}}) \to (\boxtimes M_{X^{I_\alpha}}) \otimes \mathcal{Q}_{X^{\sqcup I_\alpha}}$ where the arrow is the tensor product of the identity map for $\boxtimes M_{X^{I_\alpha}}$ and the factorization product for $\mathcal{Q}$.

Suppose that $\mathcal{Q}$ is a Dolbeault algebra. Composing (4.2.8.4) with $\cdot \otimes \mathcal{Q}$, we get a pseudo-tensor functor

$$\Gamma_{\widetilde{DR}}(X^{\mathcal{S}}, \cdot \otimes \mathcal{Q}) : C\mathcal{M}(X^{\mathcal{S}})^* \to CF(k)^{\otimes}. \tag{4.2.8.5}$$

Since the compatibility morphisms for $\cdot \otimes \mathcal{Q}$ are quasi-isomorphisms, it descends, by (4.2.7.1), to a *tensor* functor

$$R\Gamma_{DR}(X^{\mathcal{S}}, \cdot) : D\mathcal{M}(X^{\mathcal{S}})^* \to CF(k)^{\otimes}. \tag{4.2.8.6}$$

Notice that $h : \mathcal{M}(X^{\mathcal{S}})^* \to Sh^!(X^{\mathcal{S}})^*$ is evidently a pseudo-tensor functor, so composing it with $\Gamma(X^{\mathcal{S}}, \cdot)$, we get a pseudo-tensor functor

$$\Gamma(X^{\mathcal{S}}, h(\cdot)) : C\mathcal{M}(X^{\mathcal{S}})^* \to CF(k)^{\otimes}. \tag{4.2.8.7}$$

Morphisms (4.2.6.2) are actually morphisms of pseudo-tensor functors.

4.2.9. Cousin $\mathcal{D}$-complexes. Repeating the construction from 4.2.3, we see that a $\mathcal{D}$-complex M on $X^{\mathcal{S}}$ defines a complex $M_{\mathcal{R}(X)^o_n}$ of $\mathcal{D}$-modules on every $\mathcal{R}(X)^o_n$. One has $DR(M_{\mathcal{R}(X)^o_n}) = DR(M)_{\mathcal{R}(X)^o_n}$, so, by (4.2.3.1), for $M \in D\mathcal{M}(\mathcal{R}(X))$ we get a canonical quasi-isomorphism

$$\mathrm{gr}_n R\Gamma_{DR}(\mathcal{R}(X), M) \xrightarrow{\sim} R\Gamma_{DR}(\mathcal{R}(X)^o_n, M) \tag{4.2.9.1}$$

and the Cousin spectral sequence $E_r^{p,q}$ converging to $H^{\cdot}_{DR}(\mathcal{R}(X), M)$ with the first term

$$E_1^{p,q} = H^{p+q}_{DR}(\mathcal{R}(X)^o_{-p}, M). \tag{4.2.9.2}$$

There is an exact fully faithful functor

$$j^{(n)}_{\mathcal{R}*} : \mathcal{M}(\mathcal{R}(X)^o_n) \hookrightarrow \mathcal{M}(X^{\mathcal{S}}) \tag{4.2.9.3}$$

defined by $(j^{(n)}_{\mathcal{R}*}N)_{X^I} := \underset{T \in Q(I,n)}{\oplus} \Delta_*^{(I/T)} j_*^{(I)} N_{U^{(T)}}$, $N \in \mathcal{M}(\mathcal{R}(X)^o_n)$. Here $N_{U^{(T)}}$ is the pull-back of N by the étale projection $U^{(T)} \to \mathcal{R}(X)^o_n$. One has $N = (j^{(n)}_{\mathcal{R}*}N)_{\mathcal{R}(X)^o_n}$.

Notice that for N as above and a left $\mathcal{D}$-module L on $\mathcal{R}(X)$ one has $L \otimes j^{(n)}_{\mathcal{R}*}N = j^{(n)}_{\mathcal{R}*}(L_{\mathcal{R}(X)^o_n} \otimes N)$, so the image of $j^{(n)}_{\mathcal{R}*}$ is preserved by the action of the tensor category $\mathcal{M}^{\ell}(\mathcal{R}(X))$.

A *Cousin $\mathcal{D}$-complex* on $\mathcal{R}(X)$ is a complex M of right $\mathcal{D}$-modules on $X^{\mathcal{S}}$ such that $M^{-n} \in j^{(n)}_{\mathcal{R}*}(\mathcal{M}(\mathcal{R}(X)^o_n))$ (in particular, $M^a = 0$ for $a \geq 0$). Such M is automatically admissible. Cousin $\mathcal{D}$-complexes form an abelian category $\mathcal{C}ous(\mathcal{R}(X))$.

Remark. Clearly, the functor $\mathcal{C}ous(\mathcal{R}(X)) \to D\mathcal{M}(\mathcal{R}(X))$ is a fully faithful embedding. In fact, $\mathcal{C}ous(\mathcal{R}(X))$ is the core of a certain canonical t-structure on $D\mathcal{M}(\mathcal{R}(X))$.

Let M be a Cousin $\mathcal{D}$-complex. According to (4.2.9.1) we have a canonical quasi-isomorphism

$$\mathrm{gr}_n R\Gamma_{DR}(\mathcal{R}(X), M) \xrightarrow{\sim} R\Gamma_{DR}(\mathcal{R}(X)^o_n, M^{-n}). \tag{4.2.9.4}$$

Since $\dim \mathcal{R}(X)^o_n = n$, the cohomology groups $H^i_{DR}(\mathcal{R}(X), M)$ vanish for $i > 0$.

4.2.10. Lemma. *The morphisms $H^a_{DR}(\mathcal{R}(X)_n, M) \to H^a_{DR}(\mathcal{R}(X), M)$ are surjective for $a > -n$ and are isomorphisms for $a > -n+1$. If X is affine, then this is true, respectively, for $a > -n-1$ and $a > -n$.*

Proof. We use (4.2.9.4). If X is affine, then so are the varieties $\mathcal{R}(X)^o_m$. So the groups $H^a \mathrm{gr}_m R\Gamma_{DR}(\mathcal{R}(X), M)$ vanish for $a > -m$, which implies our statement. If X is compact, then we consider a coordinate projection $U^{(m)} \to X$. It is affine and X is a curve, so $H^a_{DR}(U^{(m)}, M^{-m}) = 0$ for $a > 1$. Since $H^a_{DR}(\mathcal{R}(X)^o_m, M^{-m}) \subset H^a_{DR}(U^{(m)}, M^{-m})$, we see that $H^a \mathrm{gr}_m R\Gamma_{DR}(\mathcal{R}(X), M) = 0$ for $a > -m+1$, and we are done. □

4.2.11. Chiral homology: definition. *For the rest of this chapter we assume (if not explicitly stated otherwise) that X is proper and connected. We work in the DG setting skipping the letters "DG" whenever possible, so "chiral algebra" means "DG chiral super algebra"; a chiral algebra which sits in degree 0 is called a "plain chiral algebra".*

Let A be a not necessary unital chiral algebra on X. Consider the corresponding Chevalley-Cousin complex $C(A) \in C\mathcal{M}(X^{\mathcal{S}})$ as defined in 3.4.11. This complex is obviously admissible, so $C(A)$ is a $\mathcal{D}$-complex on $\mathcal{R}(X)$. If A is a plain chiral algebra, then $C(A)$ is a Cousin complex. We define the *chiral homology* of X with coefficients in A or, simply, the chiral homology of A as the de Rham cohomology of $C(A)$:

$$C^{ch}(X, A) := R\Gamma_{DR}(\mathcal{R}(X), C(A)), \quad H^{ch}_a(X, A) := H^{-a} C^{ch}(X, A). \tag{4.2.11.1}$$

Since C^{ch} preserves quasi-isomorphisms, it can be considered as a functor on the homotopy category $\mathcal{H}o\mathcal{CA}(X)$ (see 3.3.13).

One has $C(A)_{\mathcal{R}(X)^o_n} = (\mathrm{Sym}^n(A[1]))_{\mathcal{R}(X)^o_n}$ (here Sym^n is the exterior symmetric power), so the Cousin spectral sequence (4.2.9.2) converging to $H^{ch}_n(X, A)$ looks as

$$E^1_{p,q} = H^{-p-q}_{DR}(\mathcal{R}(X)^o_p, \mathrm{Sym}^p(A[1])). \tag{4.2.11.2}$$

Remark. The Cousin filtration on $C^{ch}(X, A)$ seems not to be a part of any fundamental structure and plays mere technical role. In most cases the spectral sequence is highly non-degenerate (of course, it degenerates when $\mu_A = 0$) and of no help for computations.

Following the notation from 2.1.12, for a plain chiral algebra A we can rewrite (4.2.11.2) as

$$E^1_{p,q} = H^{DR}_{p+q}(\mathcal{R}(X)^o_p, \Lambda^p_{ext} A). \tag{4.2.11.3}$$

Here the $\mathcal{D}$-module $\Lambda^p_{ext} A$ on $\mathcal{R}(X)^o_p$ is $\mathrm{Sym}^p(A[1])_{\mathcal{R}(X)^o_p}[-p]$. Notice that the vector spaces $H^{ch}_a(X, A)$ vanish for $a < 0$.

4.2.12. Chiral chain complexes. For a chiral algebra A we have defined $C^{ch}(X, A)$ as an object of the derived category. Often it is important to represent it by means of some actual functorial complexes; we refer to any such construction as a *chiral chain complex*. One defines a chiral chain complex replacing $DR(C(A))$ by a quasi-isomorphic handsome complex (see 4.2.2). Two nuisances had to be dealt with: the global one (each complex $DR(C(A)_{X^n})$ needs to be resolved in order to compute $R\Gamma_{DR}(X, C(A)_{X^n})$) and the local one (the complexes $DR(C(A))$ are not nice). The global problem is treated by means of Dolbeault resolutions (or their non-quasi-coherent version; see 4.2.15). To make $DR(C(A))$ nice, one can either replace DR by its canonical nice resolution $\widetilde{DR}$ or use a modified de Rham functor from 2.2.10. One gets chiral chain complexes denoted, respectively, by $\tilde{C}^{ch}(X,A)_{\mathcal{Q}}$ and $C^{ch}(X,A)_{\mathcal{P}\mathcal{Q}}$. Another possibility is to use forms with logarithmic singularities along the diagonals; it leads to the complex $C^{ch}_{log}(X,A)_{\mathcal{Q}}$ to be discussed in 4.2.14.

Let us define the first chiral chain complex $\tilde{C}^{ch}$. Choose a Dolbeault $\mathcal{D}_X$-algebra $\mathcal{Q}$ and set $A_{\mathcal{Q}} := A \otimes \mathcal{Q}$. Our complex is $\tilde{C}^{ch}(X,A)_{\mathcal{Q}} := \Gamma_{\widetilde{DR}}(X^{\mathcal{S}}, C(A_{\mathcal{Q}}))$. Since $C(A_{\mathcal{Q}}) = C(A) \otimes \mathcal{Q}$, we have a canonical filtered quasi-isomorphism (see (4.2.7.1))

$$\tilde{C}^{ch}(X,A)_{\mathcal{Q}} \xrightarrow{\sim} C^{ch}(X,A). \tag{4.2.12.1}$$

Unfortunately, the complex $\tilde{C}^{ch}(X,A)_{\mathcal{Q}}$ is unpleasantly huge. Indeed, its subquotient $\mathrm{gr}_n \tilde{C}^{ch}(X,A)_{\mathcal{Q}}$ comprises, apart from relevant sections of $A_{\mathcal{Q}}^{\boxtimes n}$ over $U^{(n)}$, a pile of contractible debris from lower dimensional strata. The chiral chain complexes $C^{ch}(X,A)_{\mathcal{P}\mathcal{Q}}$ we are going to consider next have the advantage of being reasonably small.

To define it, we need to choose, apart from $\mathcal{Q}$, a (non-unital) commutative $\mathcal{D}_X$-algebra resolution $\epsilon_{\mathcal{P}} : \mathcal{P} \to \mathcal{O}_X$ such that $\mathcal{P}^{>0} = 0$ and each $\mathcal{P}^a$ is $\mathcal{D}_X$-flat (see 2.2.10). For example, one can take $\mathcal{P}$ from Example in 2.2.10.

Consider the non-unital chiral algebra $A_{\mathcal{P}\mathcal{Q}} := A \otimes \mathcal{P} \otimes \mathcal{Q}$. The promised chiral chain complex is $C^{ch}(X,A)_{\mathcal{P}\mathcal{Q}} := \Gamma(X^{\mathcal{S}}, h(C(A_{\mathcal{P}\mathcal{Q}})))$.

Proposition. *There is a canonical filtered quasi-isomorphism*

$$C^{ch}(X,A)_{\mathcal{P}\mathcal{Q}} \xrightarrow{\sim} C^{ch}(X,A). \tag{4.2.12.2}$$

Proof. The quasi-isomorphisms $A \to A_{\mathcal{Q}} \leftarrow A_{\mathcal{P}\mathcal{Q}}$ yield one $C^{ch}(X,A) \xrightarrow{\sim} C^{ch}(X, A_{\mathcal{P}\mathcal{Q}})$. Consider a canonical morphism $p : DR(C(A_{\mathcal{P}\mathcal{Q}})) \to h(C(A_{\mathcal{P}\mathcal{Q}}))$ in $CSh^!(X^{\mathcal{S}})$. We will show that (i) p is a quasi-isomorphism (thus $h(C(A_{\mathcal{P}\mathcal{Q}}))$ is admissible), and (ii) $h(C(A_{\mathcal{P}\mathcal{Q}}))$ is handsome. Now (i) implies that $C^{ch}(X, A_{\mathcal{P}\mathcal{Q}}) \xrightarrow{\sim} R\Gamma(X^{\mathcal{S}}, h(C(A_{\mathcal{P}\mathcal{Q}})))$, and (ii) implies that $C^{ch}(X,A)_{\mathcal{P}\mathcal{Q}} \xrightarrow{\sim} R\Gamma(X^{\mathcal{S}}, h(C(A_{\mathcal{P}\mathcal{Q}})))$ (see 4.2.2). Our (4.2.12.2) is the composition.

(i) Consider the Cousin filtration on $C(A_{\mathcal{P}\mathcal{Q}})_{X^I}$. It splits (in a way that does not respect the differential), so it suffices to show that $DR(\mathrm{gr}\, C(A_{\mathcal{P}\mathcal{Q}})_{X^I}) \to h(\mathrm{gr}\, C(A_{\mathcal{P}\mathcal{Q}})_{X^I})$ is a quasi-isomorphism. This follows since $\mathcal{P}^{\boxtimes T}$ is $\mathcal{D}_{X^T}$-flat, for $\mathrm{gr}\, C(A_{\mathcal{P}\mathcal{Q}})_{X^I}$ is a direct sum of complexes $\Delta_*^{(I/T)}((j_*^{(T)} j^{(T)*}(A_{\mathcal{Q}}[1]^{\boxtimes T})) \otimes \mathcal{P}^{\boxtimes T})$.

(ii) $h(C(A_{\mathcal{P}\mathcal{Q}}))$ is evidently nice and satisfies condition (b) of 4.2.2. To check condition (a) of loc. cit., it suffices, by the above argument, to verify that each term of the complex $h(j_*^{(I)} j^{(I)*}(A_{\mathcal{Q}}[1]^{\boxtimes I}) \otimes \mathcal{P}^{\boxtimes I})$ has no higher cohomology. In fact, for every $\mathcal{Q}^{\boxtimes I}[\mathcal{D}_{X^I}]$-module N and a $\mathcal{D}_{X^I}$-flat $\mathcal{D}_{X^I}$-module R the sheaf $h(N \otimes R)$ has no higher cohomology. Indeed, the complex $DR(N \otimes R)$ is a *left* resolution of $h(N \otimes R)$, and each term of this resolution has no higher cohomology by condition (c) of 4.1.3. $\square$

Lemma. *One has*

$$\mathrm{gr}_n C^{ch}(X,A)_{\mathcal{P}\mathcal{Q}} \xrightarrow{\sim} \Gamma(U^{(n)}, h((A_{\mathcal{P}\mathcal{Q}}[1])^{\boxtimes n}))_{\Sigma_n}. \tag{4.2.12.3}$$

Proof. It suffices to show that

$$h(j_*^{(n)}j^{(n)*}(A_{\mathcal{PQ}})^{\boxtimes n}) \xrightarrow{\sim} j_*^{(n)}j^{(n)*}h((A_{\mathcal{PQ}})^{\boxtimes n}). \tag{4.2.12.4}$$

We will check this replacing the complex $(A_{\mathcal{PQ}})^{\boxtimes n}$ by each of its terms. As in the end of the proof of the proposition, these are direct sums of modules of type $N\otimes R$ where R is $\mathcal{D}_{X^I}$-flat; hence $DR(N\otimes R)$ is a left resolution of $h(N\otimes R)$ and $DR(j_*^{(n)}j^{(n)*}N\otimes R)$ is a left resolution of $h(j_*^{(n)}j^{(n)*}N\otimes R)$. Since the latter complex equals $j_*^{(n)}j^{(n)*}DR(N\otimes R)=Rj_*^{(n)}j^{(n)*}DR(N\otimes R)$, we see that $h(j_*^{(n)}j^{(n)*}N\otimes R)\xrightarrow{\sim} Rj_*^{(n)}j^{(n)*}h(N\otimes R)$. Thus, being a mere sheaf, $h(j_*^{(n)}j^{(n)*}N\otimes R)$ equals $j_*^{(n)}j^{(n)*}h(N\otimes R)$, and we are done. □

Therefore we have an identification of mere graded modules

$$C^{ch}(X,A)_{\mathcal{PQ}} = \bigoplus_{n\geq 1}\Gamma(U^{(n)},h((A_{\mathcal{PQ}}[1])^{\boxtimes n}))_{\Sigma_n}, \tag{4.2.12.5}$$

the Cousin filtration is the filtration by n.

The canonical morphism $A_{\mathcal{PQ}}\to A_{\mathcal{Q}}$ provides filtered quasi-isomorphisms

$$C^{ch}(X,A)_{\mathcal{PQ}} \leftarrow \tilde{C}^{ch}(X,A_{\mathcal{P}})_{\mathcal{Q}} \to \tilde{C}^{ch}(X,A)_{\mathcal{Q}} \tag{4.2.12.6}$$

which compare the two types of chiral chain complexes.

Remark. One can define the chiral homology functor directly by formulas (4.2.12.1) or (4.2.12.2). One has to show then that our complexes as objects of the filtered derived category do not depend on the auxiliary choice of $\mathcal{Q}$ or $\mathcal{P}$, $\mathcal{Q}$. This follows from the lemma in 2.2.10 (or rather the remarks after it) and the second lemma in 4.1.3.

4.2.13. A digression on forms with logarithmic singularities. *The material of 4.2.13 and 4.2.14 will not be used in the subsequent sections and can be skipped.*

We will construct another chiral chain complex $C^{ch}_{log}(X,A)_{\mathcal{Q}}$ using forms with logarithmic singularities along the diagonal divisor. This section collects some basic facts about the logarithmic de Rham complex.

For $I\in\mathcal{S}$ consider the de Rham DG algebra DR_{X^I} which is contained in the larger DG algebra $j_*^{(I)}j^{(I)*}DR_{X^I}$ of forms with possible singularities along the diagonal divisor. Let $DR^{log}_{X^I}\subset j_*^{(I)}j^{(I)*}DR_{X^I}$ be the DG subalgebra generated by $\mathcal{O}_{X^I}$ and 1-forms df/f where $f=0$ is an equation of a component of the diagonal divisor. It carries an increasing filtration $W_0\subset W_1\subset\cdots$ where $W_0=DR_{X^I}$ and W_a is the DR_{X^I}-submodule generated by the products of $\leq a$ forms df/f as above.

Proposition. (i) *The embedding $\iota : DR^{log}_{X^I}\hookrightarrow j_*^{(I)}j^{(I)*}DR_{X^I}$ is a quasi-isomorphism.*

(ii) *There is a canonical identification (cf. (3.1.10.1))*

$$\mathrm{gr}^W_a DR^{log}_{X^I} \xrightarrow{\sim} \bigoplus_{T\in Q(I,|I|-a)} \Delta^{(I/T)}_{\cdot} DR_{X^T}\otimes \mathcal{L}ie^*_{I/T}\otimes(\lambda_I/\lambda_T)[-a]. \tag{4.2.13.1}$$

Proof. Recall (see 3.1.7 and (3.1.10.1)) that $j_*^{(I)}j^{(I)*}\omega_{X^I}=j_*^{(I)}j^{(I)*}\omega_X^{\boxtimes I}\otimes\lambda_I\in\mathcal{M}(X^I)$ has a canonical filtration $W_\cdot$ with $\mathrm{gr}^W_{-\ell}j_*^{(I)}j^{(I)*}\omega_{X^I}=\bigoplus_{T\in Q(I,\ell)}\Delta_*^{(I/T)}\omega_{X^T}\otimes\mathcal{L}ie^*_{I/T}\otimes(\lambda_I/\lambda_T)$. For the corresponding filtration on the de Rham complex one has $\mathrm{gr}_\ell^W DR(j_*^{(I)} j^{(I)*}\omega_{X^I}) = \underset{T\in Q(I,-\ell)}{\oplus} DR(\Delta_*^{(I/T)}\omega_{X^T})\otimes \mathcal{L}ie^*_{I/T}\otimes(\lambda_I/\lambda_T)$. Set $\Phi_T := \Delta_\cdot^{(I/T)} DR(\omega_{X^T})\otimes \mathcal{L}ie^*_{I/T}\otimes(\lambda_I/\lambda_T)$. The usual quasi-isomorphic embeddings $\Delta_\cdot^{(I/T)}DR(\omega_{X^T}) \hookrightarrow DR(\Delta_*^{(I/T)}\omega_{X^T})$ define a quasi-isomorphic embedding $\Phi_{\ell X^I} := \underset{T\in Q(I,-\ell)}{\oplus}\Phi_T \hookrightarrow \mathrm{gr}_\ell^W DR(j_*^{(I)}j^{(I)*}\omega_{X^I})$.

One has $DR(j_*^{(I)}j^{(I)*}\omega_{X^I}) = j_*^{(I)}j^{(I)*}DR_{X^I}[I]$. Set $DR^{log}(\omega_{X^I}) := DR^{log}_{X^I}[I]$, so we have an embedding $\iota : DR^{log}(\omega_{X^I}) \hookrightarrow DR(j_*^{(I)}j^{(I)*}\omega_{X^I})$. Define the W filtration on $DR^{log}(\omega_{X^I})$ by $W_\ell DR^{log}(\omega_{X^I}) := (W_{\ell+|I|}DR^{log}_{X^I})[I]$. We will show that ι is compatible with the W filtrations and

$$\mathrm{gr}_\ell^W\iota : \mathrm{gr}_\ell^W DR^{log}(\omega_{X^I}) \xrightarrow{\sim} \Phi_{\ell X^I}\subset \mathrm{gr}_\ell^W DR(j_*^{(I)}j^{(I)*}\omega_{X^I}). \tag{4.2.13.2}$$

This implies the proposition.

Let us check that ι sends W_ℓ to W_ℓ and that the image of $\mathrm{gr}_\ell^W\iota$ lies in $\Phi_{\ell X^I}$. By induction we can assume that the statement is known for every $\ell'<\ell$. A section of $W_\ell DR^{log}(\omega_{X^I})$ is a linear combination of forms of type $\mu = \nu\wedge df_1/f_1\wedge\cdots\wedge df_a/f_a$ where $a=|I|+\ell$, the f_i are local equations of some irreducible components of the diagonal divisor, and ν is a regular form. We can assume that the divisors $f_i=0$ have normal crossings (otherwise our form lies in $W_{\ell-1}$); let X^T be their intersection. If μ has top degree, then it evidently lies in $W_\ell j_*^{(I)}j^{(I)*}\omega_{X^I}$. Its image $\bar\mu$ in $\mathrm{gr}_\ell^W j_*^{(I)}j^{(I)*}\omega_{X^I}$ is killed by multiplication by each f_i; hence $\bar\mu\in\Phi_T$. If μ is not of top degree, then it can be written as a convolution of a similar form of top degree with a polyvector field along the fibers of the projection $(f_i): X^I\to\mathbb{A}^a$; hence $\mu\in W_\ell DR(j_*^{(I)}j^{(I)*}\omega_{X^I})$ and its image in $\mathrm{gr}_\ell^W j_*^{(I)}j^{(I)*}\omega_{X^I}$ lies in Φ_T.

The surjectivity of $\mathrm{gr}_\ell^W\iota : \mathrm{gr}_\ell^W DR^{log}(\omega_{X^I})\to\Phi_{\ell X^I}$ follows from the above argument together with the fact that the $\mathrm{Aut}(I/T)$-module $\mathcal{L}ie^*_{I/T}\otimes(\lambda_I/\lambda_T)$ is irreducible.

It remains to show that $\mathrm{gr}_\ell^W\iota$ is injective. By surjectivity it suffices to check that for $\mu\in DR^{log}(\omega_{X^I})\cap W_\ell DR(j_*^{(I)}j^{(I)*}\omega_{X^I})$ its image $\bar\mu\in\mathrm{gr}_\ell^W DR(j_*^{(I)}j^{(I)*}\omega_{X^I})$ lies in $\Phi_{\ell X^I}$. We will prove this by induction by $|I|$. The case $\ell\le -|I|$ is evident, so we can assume that $\ell>-|I|$.

Consider the residue map $r: j_*^{(I)}j^{(I)*}\omega_{X^I}\to \underset{T\in Q(I,|I|-1)}{\oplus}\Delta_*^{(I/T)}j_*^{(T)}j^{(T)*}\omega_{X^T}$. It sends the subcomplex $DR^{log}(\omega_{X^I})\subset DR(j_*^{(I)}j^{(I)*}\omega_{X^I})$ to the sum of subcomplexes $\Delta_\cdot^{(I/T)}DR^{log}(\omega_{X^T})\subset DR(\Delta_*^{(I/T)}j_*^{(T)}j^{(T)*}\omega_{X^T})$.

The kernel of r equals $\omega_{X^I}=W_{-|I|}j_*^{(I)}j^{(I)*}\omega_{X^I}$ and r is strictly compatible with W filtrations (see 3.1.6 and 3.1.7). Therefore $\mathrm{gr}_\ell^W r$ is injective. Thus $\Phi_{\ell X^I} = r^{-1}(\oplus\Delta_\cdot^{(I/T)}\Phi_{\ell X^T})$. So we need to check that $r(\bar\mu)\in\oplus\Delta_\cdot^{(I/T)}\Phi_{\ell X^T}$. Since $r(\mu)\in\oplus\Delta_\cdot^{(I/T)}DR^{log}(\omega_{X^T})$, this follows from the induction assumption. □

Remark. Another way to prove the above proposition is to notice that we can assume that $X=\mathbb{A}^1$ and use then the Orlik-Solomon theorem [**OS**].

4.2.14. Now we can define the promised chiral chain complex $C^{ch}_{log}(X,A)_{\mathcal{Q}}$. *We assume that A is unital.* Let $A^\ell_{X^{\mathcal{S}}}$ be the corresponding factorization algebra (see 3.4.9). This is a left $\mathcal{D}$-module on $X^{\mathcal{S}}$, and one has a canonical isomorphism $C(\omega)\otimes A^\ell_{X^{\mathcal{S}}}\xrightarrow{\sim} C(A)$ of right $\mathcal{D}$-modules on $X^{\mathcal{S}}$ (see (3.4.13.1)).

Consider the Cousin complex $C(\omega)_{X^I}$, $I \in \mathcal{S}$. As a mere graded module, it is the sum of components $\Delta_*^{(I/T)} j_*^{(I/T)} j^{(I/T)*}(\omega_X[1])^{\boxtimes T} = \Delta_*^{(I/T)} j_*^{(I/T)} j^{(I/T)*}\omega_{X^T}[|T|]$, $T \in Q(I)$. Then $\Delta_\cdot^{(I/T)} DR^{log}(\omega_{X^T})[|T|] \subset DR(\Delta_*^{(I/T)} j_*^{(I/T)} j^{(I/T)*}\omega_{X^T})[|T|]$ form a DG DR_{X^I}-submodule of $DR(C(\omega)_{X^I})$ which we denote by $DRC^{log}_{X^I}$. When I varies, we get a !-subcomplex $DRC^{log} \subset DR(C(\omega))$.

Now consider DG DR_{X^I}-modules $DR^{log}(A_{X^I}) := DR^{log}(\omega_{X^I}) \otimes A^\ell_{X^I}$ and $DRC^{log}(A)_{X^I} := DRC^{log}_{X^I} \otimes A^\ell_{X^I}$. Since $A^\ell_{X^I}$ is flat along the diagonals, one has $\Delta_\cdot^{(I/T)} DR^{log}(\omega_{X^T})[T] \otimes A^\ell_{X^I} \xrightarrow{\sim} \Delta_\cdot^{(I/T)} DR^{log}(A_{X^T})[T]$, so there is a canonical identification of mere graded modules

$$DRC^{log}(A)_{X^I} \xrightarrow{\sim} \underset{T \in Q(I)}{\oplus} \Delta_\cdot^{(I/T)} DR^{log}(A_{X^T})[T]. \tag{4.2.14.1}$$

The obvious embedding $DR^{\log}(\omega_{X^I}) \hookrightarrow DR(j_*^{(I)} j^{(I)*}\omega_{X^I})$ yields a morphism $DR^{log}(A_{X^I}) \to DR(j_*^{(I)} j^{(I)*} A_{X^I}) = DR(j_*^{(I)} j^{(I)*}(A_X[1])^{\boxtimes I})[-|I|]$ which is an injective quasi-isomorphism by (4.2.13.2) since $A^\ell_{X^I}$ is flat along the diagonals. Similarly, we have a quasi-isomorphic embedding $DRC^{log}(A)_{X^I} \hookrightarrow DR(C(A))_{X^I}$. When I varies, we get a quasi-isomorphic embedding of !-complexes on $X^{\mathcal{S}}$

$$DRC^{log}(A) \hookrightarrow DR(C(A)). \tag{4.2.14.2}$$

The !-complex $DRC^{log}(A)$ is admissible by (4.2.14.1) and (i) in the proposition in 4.2.13, and nice according to (4.2.14.1).[29] Choose a Dolbeault $\mathcal{D}_X$-algebra $\mathcal{Q}$ and set $A_{\mathcal{Q}} := A \otimes \mathcal{Q}$. Then $DRC^{log}(A_{\mathcal{Q}})$ is handsome.[30] Set $C^{ch}_{log}(X, A)_{\mathcal{Q}} := \Gamma(X^{\mathcal{S}}, DRC^{log}(A_{\mathcal{Q}}))$. According to 4.2.2, the arrow (4.2.14.2) defines a filtered quasi-isomorphism

$$C^{ch}_{log}(X, A)_{\mathcal{Q}} \xrightarrow{\sim} C^{ch}(X, A). \tag{4.2.14.3}$$

The chiral chain complexes $C^{ch}_{log}(X, A)_{\mathcal{Q}}$ and $\tilde{C}^{ch}(X, A)_{\mathcal{Q}}$ (see 4.2.12) are connected by natural quasi-isomorphisms. Consider the morphisms $DRC^{log}(A_{\mathcal{Q}}) \leftarrow \widetilde{DRC}^{\,log}(A_{\mathcal{Q}}) \to \widetilde{DR}(C(A_{\mathcal{Q}}))$ where the left arrow is the canonical nice resolution (see 4.2.1) and the right one comes from (4.2.14.2). Applying $\Gamma(X^{\mathcal{S}}, \cdot)$, we get the promised quasi-isomorphisms

$$C^{ch}_{log}(X, A)_{\mathcal{Q}} \leftarrow \Gamma(X^{\mathcal{S}}, \widetilde{DRC}^{\,log}(A_{\mathcal{Q}})) \to \tilde{C}^{ch}(X, A)_{\mathcal{Q}}. \tag{4.2.14.4}$$

4.2.15. Variant. Sometimes it is convenient to use instead of $\mathcal{Q}$ some non-quasi-coherent Dolbeault-style algebras; see 4.1.4. Namely (cf. 3.4.2), suppose that for each $I \in \mathcal{S}$ we are given a Dolbeault-style $\mathcal{D}_{X^I}$-algebra $\mathcal{Q}_{X^I}$ and for each $\pi : J \twoheadrightarrow I$ a horizontal morphism of unital DG $\mathcal{O}_{X^I}$-algebras $\nu^{(\pi)} = \nu^{(J/I)} : \Delta^{(J/I)*}\mathcal{Q}_{X^J} \to \mathcal{Q}_{X^I}$; one assumes that the $\nu^{(\pi)}$ are compatible with the composition of the π's, $\nu^{(id_I)} = id_{\mathcal{Q}_{X^I}}$. Let us call such datum a *Dolbeault-style* $\mathcal{D}_{\mathcal{R}(X)}$-*algebra*. For example, for $k = \mathbb{C}$ and X compact the classical Dolbeault algebras on X^I (see 4.1.4) form a Dolbeault-style $\mathcal{D}_{\mathcal{R}(X)}$-algebra. Of course, any Dolbeault $\mathcal{D}_{\mathcal{R}(X)}$-algebra (see 4.2.7) is automatically a Dolbeault-style $\mathcal{D}_{\mathcal{R}(X)}$-algebra.

Now such $\mathcal{Q}$ defines a resolution $C(A)_{\mathcal{Q}}$ of $C(A)$. As a mere graded (non-quasi-coherent) $\mathcal{D}_{X^I}$-module, $C(A)_{\mathcal{Q}X^I}$ is equal to the direct sum of components

[29] Notice that $DR(C(A))$ is *not* nice.

[30] Properties (a) and (b) from 4.2.2 are evident.

$\Delta_*^{(I/T)}(j_*^{(T)}j^{(T)*}(A[1])^{\boxtimes T}\otimes Q_{X^T})$; the definition of the differential is left to the reader. If $\mathcal{Q}$ is a Dolbeault $\mathcal{D}_{\mathcal{R}(X)}$-algebra, then $C(A)_{\mathcal{Q}}=C(A)\otimes\mathcal{Q}=C(A_{\mathcal{Q}})$.

One can use $C(A)_{\mathcal{Q}}$ in the same way that we have used the Dolbeault resolutions in the previous sections. Therefore we have the corresponding chiral chain complexes $\widetilde{C}^{ch}(X,A)_{\mathcal{Q}}:=\Gamma(X^{\mathcal{S}},\widetilde{DR}(C(A)_{\mathcal{Q}}))$, $C^{ch}(X,A)_{\mathcal{P}\mathcal{Q}}:=\Gamma(X^{\mathcal{S}},h(C(A_{\mathcal{P}})_{\mathcal{Q}}))$, $C^{ch}(X,A)_{\mathcal{Q}}:=\Gamma(X^{\mathcal{S}},DRC^{log}\otimes\mathcal{Q}\otimes A^{\ell}_{X^{\mathcal{S}}})$, etc. The details are left to the reader.

4.2.16. The 0th chiral homology. For a *plain* chiral algebra A set $\langle A\rangle=\langle A\rangle(X):=H_0^{ch}(X,A)$. By construction and 3.4.12 one has

$$\langle A\rangle=\varinjlim H_0^{DR}(X^I,A_{X^I}) \tag{4.2.16.1}$$

(the inductive limit of the $\mathcal{S}^{\circ}$-system of vector spaces). It follows from 4.2.10 that

$$\langle A\rangle=\mathrm{Coker}(H^1_{DR}(U,j^*A^{\boxtimes 2})\to H^1_{DR}(X,A)) \tag{4.2.16.2}$$

where the arrow comes from the chiral product $\mu:j_*j^*A\boxtimes A\to\Delta_*A$.

Suppose A is commutative. Then, by (4.2.16.2) and 2.4.5, $\langle A\rangle$ coincides with the same noted vector space from 2.4.1. Therefore $\langle A\rangle$ is a commutative unital algebra. Its product $\cdot$ is[31] the quotient map of the composition $H_0^{DR}(X,A)^{\otimes 2}\xrightarrow{\boxtimes}H_0^{DR}(X\times X,A^{\boxtimes 2})\to H_0^{DR}(X\times X,A_{X^2})\to\langle A\rangle$; here the last arrow is the canonical morphism and the middle one comes since $A^{\boxtimes 2}\subset A_{X^2}$.

Example. For the unit chiral algebra ω the identification $H_0^{ch}(X,\omega)=\langle\omega\rangle\xrightarrow{\sim}k$ comes from the trace isomorphisms $H_0^{DR}(X^I,\omega_{X^I})\xrightarrow{\sim}k$.

For any unital chiral algebra A we denote by $1^{ch}=1^{ch}_A\in\langle A\rangle$ the image of $1\in\langle\omega\rangle=k$ by 1_A.

4.2.17. The construction of 4.2.16 can be rendered to the DG setting as follows. For a DG super chiral algebra A let $A^{\heartsuit}$ be a copy of A considered as a plain super chiral algebra equipped with an extra $\mathbb{Z}$-grading and an odd derivation δ of degree 1 and square 0. Set $\langle A\rangle:=\langle A^{\heartsuit}\rangle$; the $\mathbb{Z}$-grading and δ make it a super complex. Similarly, $C^{ch}(X,A^{\heartsuit})$ is naturally a complex in the abelian category $C\mathcal{V}ect^s$ of super complexes. Let $H^aC^{ch}(X,A^{\heartsuit})\in C\mathcal{V}ect^s$ be its cohomology and $\tau^C_{\le 0}C^{ch}(X,A^{\heartsuit})$ the corresponding truncation. Since $H^0C^{ch}(X,A^{\heartsuit})=\langle A\rangle$, one has a projection $\tau^C_{\le 0}C^{ch}(X,A^{\heartsuit})\to\langle A\rangle$.

A complex in $C\mathcal{V}ect^s$ is the same as a super bicomplex, and we can pass to the total super complex. Then $C^{ch}(X,A^{\heartsuit})$ becomes $C^{ch}(X,A)$; denote by $\tau^C_{\le 0}C^{ch}(X,A)$ the total complex of $\tau^C_{\le 0}C^{ch}(X,A^{\heartsuit})$. Since $H^{>0}C^{ch}(X,A^{\heartsuit})=0$, the map $\tau^C_{\le 0}C^{ch}(X,A)\to C^{ch}(X,A)$ is a quasi-isomorphism. So the above projection yields a canonical morphism in the derived category

$$\phi_A:C^{ch}(X,A)\to\langle A\rangle. \tag{4.2.17.1}$$

Question. Is it true that C^{ch} is equal to the left derived functor of the functor $\langle\ \rangle$? In other words, can one find for every A a morphism of chiral algebras $A'\to A$ which is a quasi-isomorphism and such that (4.2.17.1) for A' is a quasi-isomorphism?

If A is commutative, then $\langle A\rangle$ is a commutative DG algebra in a natural way, and the canonical morphism of $\mathcal{D}_X$-modules $A^{\ell}\to\langle A\rangle\otimes\mathcal{O}_X$ is a morphism of DG

[31] See, e.g., the proof of 2.4.5.

commutative $\mathcal{D}_X$-algebras which identifies the right-hand side with the maximal constant $\mathcal{D}_X$-algebra quotient of A_X^ℓ (see 2.4.1–2.4.5).

We will see in 4.6.1 that the above question has a positive answer if we restrict ourselves to commutative chiral algebras.

4.2.18. Compatibility with filtrations. A filtration $A_0 \subset A_1 \subset \cdots$ on a (not necessary unital) chiral algebra A (see 3.3.12) yields an admissible filtration $C(A)_0 \subset C(A)_1 \subset \cdots$ on $C(A)$. Namely, for $I \in \mathcal{S}$ set $(A[1])_n^{\boxtimes I} := \sum \underset{i\in I}{\otimes} (A_{l_i}[1]) \subset (A[1])^{\boxtimes I}$, the summation is over the set of all collections $(l_i) \in \mathbb{Z}_{\geq 0}^I$ such that $\sum\limits_{i\in I} l_i \leq n$. Now one has $C(A)_{nX^I} := \underset{T\in Q(I)}{\oplus} \Delta_*^{(I/T)} j_*^{(T)} j^{(T)*}(A[1])_n^{\boxtimes T}$ (see (3.4.11.1)).

Our filtration on $C(A)$ yields a filtration on $C^{ch}(X,A)$. Since $\operatorname{gr} C(A) = C(\operatorname{gr} A)$, one has

$$\operatorname{gr} C^{ch}(X,A) = C^{ch}(X, \operatorname{gr} A). \tag{4.2.18.1}$$

So one has a spectral sequence converging to $H^{ch}_\cdot(X,A)$ with

$$E^1_{p,q} = H^{ch}_{p+q}(X, \operatorname{gr} A)^p. \tag{4.2.18.2}$$

Here the upper index p is the grading on $H^{ch}(X, \operatorname{gr} A)$ that comes from the grading $\operatorname{gr}.A$.

The compatibility with filtrations can be seen on the level of concrete chiral chain complexes $\tilde{C}^{ch}(X,A)_{\mathcal{Q}}$ and $C^{ch}(X,A)_{\mathcal{PQ}}$ from 4.2.12. Here it is convenient to choose the Dolbeault algebra $\mathcal{Q}$ so that each component $\mathcal{Q}^i$ is $\mathcal{O}_X$-flat. Then $A_n \otimes \mathcal{Q}$, $A_n \otimes \mathcal{P} \otimes \mathcal{Q}$ form filtrations on $A_{\mathcal{Q}}$ and $A_{\mathcal{PQ}}$. The corresponding filtrations on the Chevalley-Cousin complexes satisfy $\operatorname{gr} C(A_{\mathcal{Q}}) = C(\operatorname{gr} A_{\mathcal{Q}})$ and $\operatorname{gr} C(A_{\mathcal{PQ}}) = C(\operatorname{gr} A_{\mathcal{PQ}})$. They yield filtrations on $\tilde{C}^{ch}(X,A)_{\mathcal{Q}}$, $C^{ch}(X,A)_{\mathcal{PQ}}$, and

$$\operatorname{gr}.\tilde{C}^{ch}(X,A)_{\mathcal{Q}} = \tilde{C}^{ch}(X, \operatorname{gr}.A)_{\mathcal{Q}}, \quad \operatorname{gr}.C^{ch}(X,A)_{\mathcal{PQ}} = C^{ch}(X, \operatorname{gr}.A)_{\mathcal{PQ}} \tag{4.2.18.3}$$

(the second equality needs an argument similar to the one used in the proof of the proposition in 4.2.12; the details are left to the reader).

Example. Every A carries a (non-unital) filtration $A_0 = 0$, $A_1 = A$.[32] The corresponding filtration on $C^{ch}(X,A)_{\mathcal{PQ}}$ is the Cousin filtration.

4.2.19. Chiral homology with coefficients. (i) Let A be a (not necessarily unital) chiral algebra and $\{M_s\}$, $s \in S$, a finite family of (possibly non-unital) chiral A-modules.

Consider the chiral algebra $A^{\{M_s\}} := A \oplus (\oplus M_s[-1])$ (see 3.3.5(i)). The homotheties of M_s define a $\mathbb{G}_m^S$-action on $A^{\{M_s\}}$. Therefore $C(A^{\{M_s\}})$ is a $\mathbb{Z}^S$-graded complex. Denote by $C(A, \{M_s\})$ its component of degree 1^S. Set

$$C^{ch}(X, A, \{M_s\}) := R\Gamma_{DR}(\mathcal{R}(X), C(A, \{M_s\})) \tag{4.2.19.1}$$

and $H^{ch}_a(X, A, \{M_s\}) = H^{ch}_a(A, \{M_s\}) := H^{-a} C^{ch}(X, A, \{M_s\})$; this is a *chiral homology of A with coefficients in $\{M_s\}$*. In other words, $C^{ch}(X, A, \{M_s\})$ is the component of degree 1^S of $C^{ch}(X, A^{\{M_s\}})$. If $S = \emptyset$, we get the chiral homology of A. Our complex is equipped with the *Cousin filtration* which is the translation by $|S|$ of the filtration induced by the Cousin filtration of $C^{ch}(X, A^{\{M_s\}})$.

[32] Notice that this filtration is commutative (see 3.3.12).

If all the M_s are equal to M, we write $C^{ch}(X,A,M_S) := C^{ch}(X,A,\{M_s\})$; if $|S|=1$, we write simply $C^{ch}(X,A,M)$.

Let us describe $C^{ch}(X,A,\{M_s\})$ a bit more explicitly. Denote by $\mathcal{S}_S$ the category whose objects are non-empty finite sets equipped with an embedding $S \hookrightarrow I$; morphisms are surjections identical on S. We have an $\mathcal{S}_S^{\circ}$-diagram of closed embeddings $X^{\mathcal{S}_S}$, $I \mapsto X^I$ which carries a $\mathcal{D}$-complex $C'(A,\{M_s\})$ defined as follows. The graded $\mathcal{D}$-module $C'(A,\{M_s\})^{\cdot}_{X^I}$ has an additional grading by the subset $Q(I,S) \subset Q(I)$ that consists of all $I \twoheadrightarrow T$ in $\mathcal{S}_S$. For such T the corresponding component is $\Delta_*^{(I/T)} j_*^{(T)} j^{(T)*}((\boxtimes M_s)\boxtimes (A[1])^{\boxtimes T\smallsetminus S})[|T\smallsetminus S|]$. The differential is the sum of two components: the first one comes from μ_A and μ_{M_s} and the second one comes from the differentials on A and M_s. We have an obvious canonical identification $R\Gamma_{DR}(X^{\mathcal{S}_S}, C'(A,\{M_s\})) = C^{ch}(X,A,\{M_s\})$.

We can also use the chiral chain complexes from 4.2.12. Namely, let us define $\tilde{C}^{ch}(X,A,\{M_s\})_{\mathcal{Q}}$, $C^{ch}(X,A,\{M_s\})_{\mathcal{P}\mathcal{Q}}$ as the components of degree 1^S of the complexes $\tilde{C}^{ch}(X,A^{\{M_s\}})_{\mathcal{Q}}$, $C^{ch}(X,A^{\{M_s\}})_{\mathcal{P}\mathcal{Q}}$. Then (4.2.12.1) and (4.2.12.2) yield canonical quasi-isomorphisms

$$C^{ch}(X,A,\{M_s\})_{\mathcal{P}\mathcal{Q}} \xrightarrow{\sim} C^{ch}(X,A,\{M_s\}) \xleftarrow{\sim} \tilde{C}^{ch}(X,A,\{M_s\})_{\mathcal{Q}}. \tag{4.2.19.2}$$

As a mere graded module, our $C^{ch}(X,A,\{M_s\})_{\mathcal{P}\mathcal{Q}}$ is a direct sum of components $C^{ch}_n(X,A,\{M_s\})_{\mathcal{P}\mathcal{Q}} := \Gamma(X^S\times X^n, h(j_*^{(S+n)} j^{(S+n)*}(\boxtimes(M_{s\mathcal{P}\mathcal{Q}})\boxtimes(A_{\mathcal{P}\mathcal{Q}}[1])^{\boxtimes n})))_{\Sigma_n}$, $n\geq 0$, where $M_{s\mathcal{P}\mathcal{Q}} := M_s\otimes\mathcal{P}\otimes\mathcal{Q}$.

The chiral chain complexes are functorial with respect to chiral operations in the following sense. For $S\twoheadrightarrow T$ and a T-family of A-modules N_t each operation $\varphi\in P^{ch}_{A\,S/T}(\{M_s\},\{N_t\}) := \otimes P^{ch}_{A\,S_t}(\{M_s\},N_t)$ (see 3.3.4) yields a morphism $C^{ch}(\varphi) : C^{ch}(X,A,\{M_s\})_{\mathcal{P}\mathcal{Q}} \to C^{ch}(X,A,\{N_t\})_{\mathcal{P}\mathcal{Q}}$; one has $C^{ch}(\varphi\psi) = C^{ch}(\varphi)C^{ch}(\psi)$. The same is true for $\tilde{C}^{ch}$ complexes.

(ii) Consider the case when $S\neq\emptyset$ and each M_s is supported at a single closed point $x_s\in X$. We can assume that these points are pairwise different (otherwise $C^{ch}(X,A,\{M_s\})=0$). Let $j_S : U_S := X\smallsetminus\{x_s\}\hookrightarrow X$ be the complement. Notice that in the definition of the chiral chain complex there occur now only *affine* varieties, so there is no need for using the Dolbeault $\mathcal{D}_X$-algebra $\mathcal{Q}$. We also do not need to use $\mathcal{P}$ to compute the de Rham cohomology of M_s. Therefore we see that $C^{ch}(X,A,\{M_s\})$ can be represented by a smaller complex $C^{ch}(X,A,\{M_s\})_{\mathcal{P}}$ with components

$$C^{ch}_n(X,A,\{M_s\})_{\mathcal{P}} := (\otimes h(M_s))\otimes\Gamma(U_S^{(n)}, h((A_{\mathcal{P}}[1])^{\boxtimes n}))_{\Sigma_n} \tag{4.2.19.3}$$

where $U_S^{(n)}$ is the complement to the diagonal divisor on $(U_S)^n$, $n\geq 0$. The differential comes from the chiral product on $A_{\mathcal{P}}$ and the $A_{\mathcal{P}}$-module structure on M_s[33] in the usual manner, using the fact that sections of $h((A_{\mathcal{P}}[1])^{\boxtimes n})$ over $U_S^{(n)}$ are the same as sections of $h(j_*^{(n)}j^{(n)*}(j_{S*}j_S^*A_{\mathcal{P}}[1])^{\boxtimes n})$ over the whole of X^n (see (4.2.12.4)). We can consider M_s as $j_{S*}j_S^*A$-modules (see 3.6.3) and one has

$$C^{ch}(X,A,\{M_s\})_{\mathcal{P}} = C^{ch}(X, j_{S*}j_S^*A,\{M_s\})_{\mathcal{P}}. \tag{4.2.19.4}$$

Sometimes it is convenient to identify M_s with the corresponding A_s^{as}-module $h(M_s)$ (see 3.6.7) and to write $C^{ch}(X,A,\{h(M_s)\}) := C^{ch}(X,A,\{M_s\})$, etc.

[33] Defined by the projection $id_A\otimes\epsilon_{\mathcal{P}} : A_{\mathcal{P}}\to A$.

The complexes $C^{ch}(X,A,\{M_s\})_{\mathcal{PQ}}$ and $C^{ch}(X,A,\{M_s\})_{\mathcal{P}}$ are connected by evident natural quasi-isomorphisms

$$C^{ch}(X,A,\{M_s\})_{\mathcal{PQ}} \to C^{ch}(X,A_{\mathcal{Q}},\{M_{s\mathcal{Q}}\})_{\mathcal{P}} \leftarrow C^{ch}(X,A,\{M_s\})_{\mathcal{P}}. \tag{4.2.19.5}$$

Suppose in addition that A is a plain chiral algebra and the M_s are plain A-modules. The spectral sequence converging to $H^{ch}_n(X,A,\{M_s\})$ for the Cousin filtration is

$$E^1_{p,q} = (\otimes h(M_s)) \otimes H^{DR}_{p+q}(U^{(p)}_S, A)^{sgn}_{\Sigma_p} \tag{4.2.19.6}$$

where the right indices mean skew-coinvariants of the action of the symmetric group. Since $U^{(p)}_S$ is affine, $E^1_{p,q}$ vanishes unless $p \geq q \geq 0$. In particular, one has $H^{ch}_{<0}(X,A,\{M_s\}) = 0$ and $H^{ch}_0(X,A,\{M_s\})$ is the space of the coinvariants of the Lie algebra $\Gamma(U_S, h(A))$ acting on $\otimes h(M_s)$.

(iii) The above constructions make sense for families of A-modules. Namely, suppose M_s is a Y_s-family of A-modules where $Y_s = \operatorname{Spec} R_s$; i.e., M_s is an $R_s \otimes A$-module. Then $C^{ch}(X,A,\{M_s\})_{\mathcal{PQ}}$ is naturally an $\otimes R_s$-module, i.e., an $\mathcal{O}$-module on $\prod Y_s$. This construction is compatible with the flat base change, so one can take for Y_s any scheme (or an algebraic stack). If the M_s are $\mathcal{D}_{Y_s}$-modules (in a way compatible with the A-action), then $C^{ch}(X,A,\{M_s\})_{\mathcal{PQ}}$ is a $\mathcal{D}$-module on $\prod Y_s$.

For example, for any A-module M the $\mathcal{D}_{X\times X}$-module $\Delta_* M$ can be considered as an X-family of A-modules where A acts along the second variable. Therefore the A-modules $\{M_s\}$, $s \in S$, yield a complex of $\mathcal{D}_{X^S}$-modules

$$\mathcal{C}^{ch}(X,A,\{M_s\})_{\mathcal{PQ}} := C^{ch}(X,A,\{\Delta_* M_s\})_{\mathcal{PQ}}. \tag{4.2.19.7}$$

One has $\mathcal{C}^{ch}(X,A,\{M_s\})_{\mathcal{PQ}} = j^{(I)}_* j^{(I)*} \mathcal{C}^{ch}(X,A,\{M_s\})_{\mathcal{PQ}}$ and $C^{ch}(X,A,\{M_s\})_{\mathcal{PQ}} = \Gamma(X^S, h\mathcal{C}^{ch}(X,A,\{M_s\})_{\mathcal{PQ}}) = R\Gamma_{DR}(X^S, \mathcal{C}^{ch}(X,A,\{M_s\})_{\mathcal{PQ}})$. If all M_s are equal to M, we write $\mathcal{C}^{ch}(X,A,M_S)_{\mathcal{PQ}} := \mathcal{C}^{ch}(X,A,\{M_s\})_{\mathcal{PQ}}$.

As in the end of (i), our complexes are functorial with respect to chiral operations: every φ as in loc. cit. yields a morphism of complexes of $\mathcal{D}_{X^S}$-modules

$$\mathcal{C}^{ch}(\varphi) : \mathcal{C}^{ch}(X,A,\{M_s\})_{\mathcal{PQ}\,X^S} \to \Delta^{(S/T)}_* \mathcal{C}^{ch}(X,A,\{N_t\})_{\mathcal{PQ}\,X^T} \tag{4.2.19.8}$$

and $\mathcal{C}^{ch}(\varphi\psi) = \mathcal{C}^{ch}(\varphi)\mathcal{C}^{ch}(\psi)$ in the obvious sense.

Remark. As in (ii), in the definition of the $\mathcal{D}_{X^S}$-complexes $\mathcal{C}^{ch}(X,A,\{M_s\})$ there is no need to use Q (for $S \neq \emptyset$).

4.3. The BV structure and products

The principal result of this section is theorem 4.3.6 which says that the chiral homology functor commutes with the tensor product. In the case of commutative algebras and 0th chiral homology this becomes an obvious statement (see 4.2.16 and (ii) in the lemma in 2.4.1): the functor which assigns to a $\mathcal{D}_X$-scheme the space of its horizontal sections commutes with the direct products. The key tool is a canonical Batalin-Vilkovisky algebra structure on the chiral complex C^{ch} defined in 4.3.1. Its "classical" counterpart is an 1-Poisson algebra structure on $C^{ch}(R)$ for a coisson algebra R. If R is any commutative chiral algebra and A a chiral R-algebra, then $C^{ch}(X,R)$ is a homotopy commutative algebra and $C^{ch}(X,A)$ is a homotopy $C^{ch}(X,R)$-module (see 4.3.2 and 4.3.4). In 4.3.3 we show that the higher chiral homology of the unit chiral algebra ω is trivial by an adaptation of the topological argument of the proposition in 3.4.1. The compatibility with tensor products is proven in 4.3.6 by a closely related argument. After the homotopy algebra preliminaries of 4.3.7 and 4.3.8 we show in 4.3.9 that the chiral homology of chiral R-algebras is compatible with the base change of R. This implies, in particular, that chiral homology is compatible with relative tensor products and direct products (see 4.3.10 and 4.3.11). It would be very interesting to understand if the chiral homology of a chiral R-algebra has local origin with respect to Spec $\langle R\rangle$ (see 4.3.13 for a more general question); a weaker result is established in 4.3.12.

In 4.3.3–4.3.13 all chiral algebras are assumed to be unital.

4.3.1. The BV structure. For a review of basic facts on homotopy Batalin-Vilkovisky algebras see 4.1.6–4.1.15.

PROPOSITION. (i) *For a (not necessary unital) chiral algebra A the chiral chain complex $C^{ch}(X,A)_{\mathcal{PQ}}$ from 4.2.12 is naturally a BV algebra.*

(ii) *Suppose that A is commutative. Then the above BV structure is commutative; i.e., $C^{ch}(X,A)_{\mathcal{PQ}}$ is a DG commutative algebra. Therefore $C^{ch}(X,A)$ has a canonical structure of the homotopy commutative algebra.*

(iii) *A coisson bracket on A yields a 1-Poisson bracket (see 1.4.18) on the chiral chain complex. If A_t is a quantization of the coisson structure (see 3.3.11), then the BV algebras $C^{ch}(X,A_t)_{\mathcal{PQ}}$ form a BV quantization of the 1-Poisson algebra $C^{ch}(X,A)_{\mathcal{PQ}}$.*

(iv) *More generally, an n-coisson bracket on A (see 1.4.18) yields an $n+1$-Poisson bracket on the chiral chain complex.*

Proof. (i) Recall that the Chevalley-Cousin complex $C(A)$ is the Chevalley complex of the Lie algebra $\Delta_*^{(\mathcal{S})}A$ in the tensor DG category $C\mathcal{M}(X^{\mathcal{S}})^{ch}$ (see 3.4.11). Thus (see 4.1.6) $C(A)$ carries a canonical structure of the BV algebra with respect to $\otimes^{ch}$.[34] The same is true for $C(A_{\mathcal{PQ}})$. By (3.4.10.4) these are automatically BV algebras with respect to $\otimes^*$. We are done by (4.2.8.7).

(ii) is clear. The 1-Poisson structure on $C^{ch}(X,A)_{\mathcal{PQ}}$ for coisson A comes from a natural 1-Poisson structure on the commutative algebra $C(A)$ in the tensor category $C\mathcal{M}(X^{\mathcal{S}})^*$; the definition is left to the reader. The n-coisson brackets are treated similarly. □

REMARKS. (i) Consider the embedding $\Gamma(X,h(A^{Lie}_{\mathcal{PQ}}))=C_1^{ch}(X,A)_{\mathcal{PQ}}[-1]\hookrightarrow C^{ch}(X,A)_{\mathcal{PQ}}[-1]$. This is a morphism of Lie algebras, so it extends naturally to a morphism $\bar{C}(\Gamma(X,h(A^{Lie}_{\mathcal{PQ}})))\to C^{ch}(X,A)_{\mathcal{PQ}}$ of BV algebras (see 4.1.8(a)). We get a canonical morphism of filtered complexes[35]

$$\bar{C}(R\Gamma_{DR}(X,A^{Lie}))\to C^{ch}(X,A). \tag{4.3.1.1}$$

Explicitly, on the nth component of the Chevalley complex it is the composition $\mathrm{Sym}^n(\Gamma(X,h(A_{\mathcal{PQ}}))[1])\xrightarrow{\sim}\Gamma(X^n,h((A_{\mathcal{PQ}}[1])^{\boxtimes n}))_{\Sigma_n}\to\Gamma(U^{(n)},h((A_{\mathcal{PQ}}[1])^{\boxtimes n}))_{\Sigma_n}$.

(ii) The above proposition (and its proof with (4.2.8.7) replaced by (4.2.8.4)) remains true for the chiral chain complex $\tilde{C}^{ch}(X,A)_{\mathcal{Q}}$, and quasi-isomorphisms (4.2.12.6) are compatible with the BV structure.

(iii) The chiral chain complex $C^{ch}_{log}(X,A)_{\mathcal{Q}}$ from 4.2.14 is *not* a BV algebra for general A. However it is a commutative BV algebra if A is commutative. Indeed,

[34] See Remark in 3.4.11 for an explicit description of the BV operations.

[35] See 4.5.1 for the discussion of the homotopy Lie algebra structure on $R\Gamma_{DR}(X,A^{Lie})$.

by (4.2.8.1) the complex $DR(C(A_{\mathcal{Q}}))$ is always a BV algebra in $CSh^!(X^{\mathcal{S}})^*$. If A is commutative then $DRC^{log}(A_{\mathcal{Q}})$ is a subalgebra of $DR(C(A_{\mathcal{Q}}))$ (which is a commutative BV algebra). Now apply (4.2.5.2).

Quasi-isomorphisms (4.2.14.4) are morphisms of commutative BV algebras.

4.3.2. Further remarks. (i) For a plain commutative unital chiral algebra R the isomorphism $H_0^{ch}(X,R) \xrightarrow{\sim} \langle R\rangle$ (see 4.2.16) identifies the product on $H_0^{ch}(R)$ with the product on $\langle R\rangle$ from 2.4.1. More generally, for any commutative R the morphism $\phi_R : C^{ch}(X,R) \to \langle R\rangle$ from (4.2.17.1) is naturally a morphism in the homotopy category of commutative algebras.

(ii) Let $R \to A$ be a central morphism of chiral algebras, so A is a chiral R-algebra. Then, in the same way as above, $C^{ch}(X,A)_{\mathcal{PQ}}$ is a $C^{ch}(X,R)_{\mathcal{PQ}}$-module. In fact, it is a BV $C^{ch}(X,R)_{\mathcal{PQ}}$-algebra. Thus $C^{ch}(X,A)$ carries a canonical structure of the homotopy $C^{ch}(X,R)$-module; i.e., it lifts canonically to an object of the homotopy category of $C^{ch}(X,R)$-modules.

(iii) If A is a chiral algebra and $\{M_s\}$ a finite family of A-modules, then the complex $C^{ch}(X,A,\{M_s\})_{\mathcal{PQ}}$ is a BV $C^{ch}(X,A)_{\mathcal{PQ}}$-module.[36] Here are a couple of ways to use this structure:

(a) Suppose $N_s \subset M_s$ are $\mathcal{D}_X$-submodules such that $\mathcal{C}ent(N_s) = A$ (see 3.3.7). The image of $\Gamma(U^{(S)}, h(\boxtimes N_{s\mathcal{PQ}})) \to \Gamma(U^{(S)}, h(\boxtimes M_{s\mathcal{PQ}})) \subset C^{ch}(X,A,\{M_s\})_{\mathcal{PQ}}$ consists of $C^{ch}(X,A)_{\mathcal{PQ}}$-central elements, so we have a morphism of complexes $\cdot : C^{ch}(X,A)_{\mathcal{PQ}} \otimes \Gamma(U^{(S)}, h(\boxtimes N_{s\mathcal{PQ}})) \to C^{ch}(X,A,\{M_s\})_{\mathcal{PQ}}$, hence a canonical morphism $C^{ch}(X,A) \otimes R\Gamma_{DR}(U^{(S)}, \boxtimes N_s) \to C^{ch}(X,A,\{M_s\})$.

(b) Suppose we are in situation (ii) and each M_s is a central R-module (see 3.3.7). Then $C^{ch}(X,A,\{M_s\})_{\mathcal{PQ}}$ is a $C^{ch}(X,R)_{\mathcal{PQ}}$-module; hence $C^{ch}(X,A,\{M_s\})$ is naturally a homotopy $C^{ch}(X,R)$-module.

(iv) Suppose A is any (not necessary unital) chiral algebra equipped with a *commutative* filtration $A_0 \subset A_1 \subset \cdots$ (see 3.3.12). Then the corresponding filtration on $C^{ch}(X,A)_{\mathcal{PQ}}$ (see 4.2.18) is compatible with the BV structure, so $C^{ch}(X,A)$ is canonically an object of the homotopy category of filtered BV algebras $\mathcal{H}o\mathcal{BV}$ (see 4.1.6) which we denote by $C^{ch}(X,A.)$ by abuse of notation.

(v) Let R be a commutative chiral algebra, $\mathcal{L}$ a Lie* R-algebroid. Then the $C^{ch}(X,R)_{\mathcal{PQ}}$-module $C^{ch}(X,R,\mathcal{L})_{\mathcal{PQ}}$ is naturally a Lie $C^{ch}(X,R)_{\mathcal{PQ}}$-algebroid.[37] Indeed, $\mathrm{Sym}_R\mathcal{L}$ is a coisson algebra (see 1.4.18, Example (ii)); its bracket shifts the $\mathbb{Z}$-grading by -1. So, by (iii) in the proposition in 4.3.1, $C^{ch}(X,\mathrm{Sym}_R\mathcal{L})_{\mathcal{PQ}}$ is a 1-Poisson algebra. Now $C^{ch}(X,R,\mathcal{L})_{\mathcal{PQ}}$ is the component of $C^{ch}(X,\mathrm{Sym}_R\mathcal{L})_{\mathcal{PQ}}[-1]$ of degree 1.[38] The Lie algebroid structure is the restriction of the 1-Poisson structure.

4.3.3. PROPOSITION. (i) *For the unit chiral algebra ω one has $H^{ch}_{\neq 0}(X,\omega) = 0$, so there is a canonical identification $C^{ch}(X,\omega) \xrightarrow{\sim} \langle\omega\rangle = k$.*

(ii) *For any unital chiral algebra A the multiplication by the generator $1^{ch} \in H_0^{ch}(X,\omega)$ is the identity endomorphism of $C^{ch}(X,A)$.*

REMARK. Statement (i) follows from the contractibility of $\mathcal{R}(X)$ in the classical topology (see the proposition in 3.4.1) combined with 4.2.4 and the usual

[36] This follows from (i) in the proposition in 4.3.1 and the definition of $C^{ch}(X,A,\{M_s\})_{\mathcal{PQ}}$ in 4.2.19.

[37] In the non-unital setting.

[38] See 4.6.4 for a description of the whole $C^{ch}(X,\mathrm{Sym}_R\mathcal{L})_{\mathcal{PQ}}$.

comparison between the de Rham and topological homology. The argument below is an algebraic version of the topological proof of the proposition in 3.4.1.

For the chiral homology of ω with arbitrary coefficients see (4.4.7.2).

Proof. Statement (ii) requires an extra construction to be introduced in the beginning of the proof of theorem 4.3.6; it will be proven in part (iii) of loc. cit. We deal now with statement (i).

Let tr be the projection $C^{ch}(X,\omega) \to H_0^{ch}(X,\omega) = k$ and $tr_1, tr_2 : C^{ch}(X,\omega)^{\otimes 2} \to C^{ch}(X,\omega)$ the maps $tr_1(a\otimes b) := tr(b)a$, $tr_2(a\otimes b) := tr(a)b$.

We will define a morphism $\delta : C^{ch}(X,\omega) \to C^{ch}(X,\omega)^{\otimes 2}$ such that $\cdot\delta$, $tr_1\delta$, and $tr_2\delta$ are the identity morphisms of $C^{ch}(X,\omega)$.

This yields the vanishing. Indeed, suppose $H_n^{ch}(X,\omega) \neq 0$ for some $n > 0$. Take the smallest such n; then one has $H^{-n}C^{ch}(X,\omega)^{\otimes 2} = H_0^{ch}(X,\omega)\otimes H_n^{ch}(X,\omega) \oplus H_n^{ch}(X,\omega)\otimes H_0^{ch}(X,\omega)$. For $h \neq 0 \in H_n^{ch}(X,\omega)$ write $\delta(h) = h_1\otimes 1^{ch} + 1^{ch}\otimes h_2$. Then $h_i = tr_i\delta(h) = h$; hence $h = \cdot\delta(h) = 2(1^{ch}\cdot h)$. Since $(1^{ch})^2 = 1^{ch}$, we come to a contradiction.

To define δ, it is convenient to represent $C^{ch}(X,\omega)$ not by complexes from 4.2.12, but by means of the Cousin resolution, i.e., as the homotopy direct limit of the $\mathcal{S}^\circ$-diagram of complexes $I \mapsto C_I := \Gamma(X^I, DR(\mathcal{C}_{X^I}))$ where $\mathcal{C}_{X^I}$ is the (whole) Cousin resolution of $\omega_{X^I}[|I|] = (\omega[1])^{\boxtimes I}$. So C_I has degrees in the interval $[-2|I|, 0]$ and we have a canonical trace map $tr_C : C_I \to k$. Now $C^{ch}(X,\omega)^{\otimes 2}$ can be represented as the homotopy direct limit of the $\mathcal{S}^\circ\times\mathcal{S}^\circ$-diagram $I,J \mapsto C_I\otimes C_J$. Consider another such diagram $I,J \mapsto C_{I,J} := \Gamma(X^I\times X^J, DR(\mathcal{C}_{X^I\times X^J}))$. There is an obvious quasi-isomorphic embedding of the diagrams $C_I\otimes C_J \hookrightarrow C_{I,J}$. The map $tr_{C1} := id_{C_I}\otimes tr_C : C_I\otimes C_J \to C_I$ extends in the usual way to the morphism of diagrams $tr_{C1} : C_{I,J}\to C_I$ and the same for tr_{C2}. Similarly, the exterior tensor product map $\circ_C : C_I\otimes C_J \to C_{I\sqcup J}$ extends to $\circ_C : C_{I,J} \xrightarrow{\sim} C_{I\sqcup J}$.

Let $\delta_C : C_I \to C_{I,I}$ be the morphism of Cousin complexes defined by the diagonal embedding $X^I \hookrightarrow X^I\times X^I$. Let us represent $C^{ch}(X,\omega)^{\otimes 2}$ as the homotopy direct limit of the diagram $C_{I,J}$. Our δ is the morphism defined by δ_C.

It remains to check that the compositions $tr_i\delta$ and $\cdot\delta$ are identity morphisms. Notice that $\cdot$ and tr_i come from the morphisms of diagrams $\circ_C$, tr_{Ci}. Since $tr_{Ci}\delta_C$ is the identity map, the morphism $tr_i\delta$ is the identity. The morphism of diagrams $\circ_C\delta_C$ is not the identity, but it becomes canonically homotopic to the identity after passing to the homotopy limit (by the definition of the homotopy direct limit). □

4.3.4. The unital setting. *From now until the end of* 4.3 *all chiral algebras are assumed to be unital.*

Suppose that R is a commutative chiral algebra. Then, by 4.3.3, the morphism $1_R : C^{ch}(X,\omega)_{\mathcal{PQ}} \to C^{ch}(X,R)_{\mathcal{PQ}}$ is a homotopy unit in $C^{ch}(X,R)_{\mathcal{PQ}}$ (see 4.1.14). Therefore, by the proposition in 4.1.14, $C^{ch}(X,R)$ lifts canonicaly to an object of the homotopy category of unital commutative algebras, which we denote again by $C^{ch}(X,R)$ by abuse of notation.

If A is a chiral R-algebra, then $C^{ch}(X,A)_{\mathcal{PQ}}$ is a homotopy unital $C^{ch}(X,R)$-module (see 4.1.14). Thus $C^{ch}(X,A)$ lives naturally in the derived category of the unital $C^{ch}(X,R)$-modules. If $\{M_s\}$ is a finite family of unital A-modules such that each M_s is a central R-module, then $C^{ch}(X,A,\{M_s\})$ is a unital $C^{ch}(X,R)$-module.

In particular, if $1_R^{ch} \in H_0^{ch}(X,R)$ vanishes, then $C^{ch}(X,A) = 0$. An example:

LEMMA. *Let $j_U : U \hookrightarrow X$ be an open subset of X, $U \neq X$. Then for any chiral algebra A one has $C^{ch}(X, j_{U*}j_U^* A) = 0$.*

Proof. Notice that $j_{U*}j_U^* A$ is a chiral $j_{U*}\mathcal{O}_U$-algebra, and $H_0^{ch}(X, j_{U*}\mathcal{O}_U) = \langle j_{U*}\mathcal{O}_U \rangle = 0$. □

REMARKS. (i) For non-commutative unital A the vanishing of $1^{ch} \in H_0^{ch}(X, A)$ need *not* imply that $C^{ch}(X, A) = 0$.

(ii) In the above lemma the condition that A is unital is essential: for example, if $\mu_A = 0$, then $H^{ch}_{\cdot}(X, j_{U*}j_U^* A) \supset H_{DR}^{\cdot+1}(U, A)$.

If a chiral algebra A is equipped with a unital commutative filtration, then $C^{ch}(X, A)_{\mathcal{PQ}}$ is a filtered BV algebra (see 4.3.2), and the map $1_A : C^{ch}(X, \omega)_{\mathcal{PQ}} \to C^{ch}(X, A)_{\mathcal{PQ}}$ is a homotopy unit. Thus $C^{ch}(X, A)$ lifts canonically to an object of $\mathcal{H}o\mathcal{BV}_u$ (see 4.1.6 and the proposition in 4.1.15) which we denote by $C^{ch}(X, A.)$ by abuse of notation.

4.3.5. Suppose we have a finite family of chiral algebras $\{A_i\}_{i \in I}$; set $A := \otimes A_i$. For every $i \in I$ the canonical morphism $\nu_i : A_i \to A$ (see 3.4.15) yields a morphism of complexes $C^{ch}(X, A_i) \to C^{ch}(X, A)$. One has a natural morphism of complexes $\circ_I : \otimes C^{ch}(X, A_i) \to C^{ch}(X, A)$, $\otimes a_i \mapsto \cdot_I(\otimes \nu_i(a_i))$, where $\cdot_I \in BV_I$ is the I-fold product.[39] It is a morphism of BV algebras. It is clear that $\circ_I$ define an extension of C^{ch} to a DG pseudo-tensor functor $C^{ch}(X, \cdot) : \mathcal{CA}(X)^{\otimes} \to \mathcal{BV}^{\otimes}$.

To get interesting objects, we should infuse $\mathcal{P} \otimes \mathcal{Q}$ as was done in 4.2.12. Replacing ν_i by $\nu_{i\mathcal{PQ}} := \otimes id_{\mathcal{P}\otimes\mathcal{Q}} : A_{i\mathcal{PQ}} \to A_{\mathcal{PQ}}$, we get

$$\circ_I : \otimes C^{ch}(X, A_i)_{\mathcal{PQ}} \to C^{ch}(X, A)_{\mathcal{PQ}}. \tag{4.3.5.1}$$

These morphisms define a pseudo-tensor extension of our functor

$$C^{ch}(X, \cdot)_{\mathcal{PQ}} : \mathcal{CA}(X)^{\otimes} \to \mathcal{BV}^{\otimes}. \tag{4.3.5.2}$$

Passing to homotopy categories, we get canonical morphisms

$$\circ_I : \otimes C^{ch}(X, A_i) \to C^{ch}(X, \otimes A_i) \tag{4.3.5.3}$$

which define a pseudo-tensor extension

$$C^{ch} : \mathcal{H}o\mathcal{CA}(X)^{\otimes} \to D(k)^{\otimes}. \tag{4.3.5.4}$$

If we play with the tensor homotopy category of chiral algebras equipped with commutative (unital) filtrations, then (4.3.5.2) defines a pseudo-tensor extension of the functor $(A, A.) \mapsto C^{ch}(X, A.)$ with values in $\mathcal{H}o\mathcal{BV}_u^{\otimes}$ (see 4.3.4).

4.3.6. THEOREM. *If the A_i are pairwise homotopically $\mathcal{O}_X$-Tor-independent,*[40] *then the canonical morphism (4.3.5.3) is a quasi-isomorphism.*

Together with 4.3.3, this shows that C^{ch} *is a unital tensor functor on the tensor category of homotopically $\mathcal{O}_X$-flat chiral algebras.*

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[39] $\circ_I$ commutes with the differential since $\{\nu_i(a_i), \nu_j(a_j)\} = 0$ for every $i \neq j$.

[40] I.e., $A_i \overset{L}{\otimes} A_j \xrightarrow{\sim} A_i \otimes A_j$ for every $i \neq j \in I$.

REMARK. Suppose that the A_i carry filtrations as in 4.2.18, and let us equip $\otimes A_i$ with the tensor product of this filtrations. Then (4.3.5.1) is a morphism of filtered complexes which is a filtered quasi-isomorphism if the $\operatorname{gr} A_i$ are pairwise homotopically $\mathcal{O}_X$-Tor-independent.[41]

Proof. It suffices to consider the case of two algebras A, B.

Set $C^{ch}(A) := C^{ch}(X, A)_{\mathcal{PQ}}$; define $C^{ch}(B)$, $C^{ch}(A \otimes B)$ similarly. We want to show that the morphism $\circ = \circ_{A,B} : C^{ch}(A) \otimes C^{ch}(B) \to C^{ch}(A \otimes B)$ of (4.3.5.1) is a quasi-isomorphism. To do this, we will construct a certain diagram

$$C^{ch}(A) \otimes C^{ch}(B) \xrightarrow{i} C^{ch}(A, B) \underset{\delta}{\overset{\tilde{\circ}}{\rightleftarrows}} C^{ch}(A \otimes B) \tag{4.3.6.1}$$

such that i is a quasi-isomorphism, $\circ = \tilde{\circ} i$, $\tilde{\circ}\delta$ is the identity map for $C^{ch}(A \otimes B)$, and $\delta\circ$ is a quasi-isomorphism. This will clearly do the job.

(i) For $I, J \in \mathcal{S}$ let $C(A, B)_{X^I \times X^J}$ be the Cousin complex of $C(A)_{X^I} \boxtimes C(B)_{X^J}$ with respect to the diagonal stratification of $X^{I \sqcup J}$. It looks as follows (cf. 3.4.11). As a mere graded $\mathcal{D}$-module, $C(A, B)_{X^I \times X^J}$ is a direct sum of components labeled by $T \in Q(I \sqcup J)$. The T-component is equal to $\Delta_*^{(I \sqcup J/T)} j_*^{(T)} j^{(T)*} \underset{t \in T}{\boxtimes} F_t$ where $F_t \in C\mathcal{M}(X)$ is $A[1]$ if $t \notin \pi_T(J)$, $B[1]$ if $t \notin \pi_T(I)$, or $(A \otimes B)[1]$ otherwise. The differential comes from the chiral products of A, B, $A \otimes B$, the chiral pairing $\in P_2^{ch}(\{A, B\}, A \otimes B)$, and differentials of A, B, $A \otimes B$ in the usual way (see 3.4.11). We have the obvious morphisms of $\mathcal{D}$-complexes $C(A)_{X^I} \boxtimes C(B)_{X^J} \to C(A, B)_{X^I \times X^J} \to C(A \otimes B)_{X^{I \sqcup J}}$. The left arrow is a quasi-isomorphism since A, B are homotopically $\mathcal{O}_X$-Tor-independent. Our $C(A, B)_{X^I \times X^J}$ form a $\mathcal{D}$-complex on the $\mathcal{S}^\circ \times \mathcal{S}^\circ$-diagram $X^{\mathcal{S}} \times X^{\mathcal{S}}$ in the obvious way (see 4.2.1 and 4.2.6), and the above arrows are morphisms of such complexes.

To compute the cohomology, we modify $C(A, B)_{X^I \times X^J}$ replacing it by a quasi-isomorphic complex $C(A, B, \mathcal{PQ})_{X^I \times X^J}$ which is again a direct sum of $Q(I \sqcup J)$ components where the T-component is $\Delta_*^{(I \sqcup J/T)} j_*^{(T)} j^{(T)*} \underset{t \in T}{\boxtimes} (F_t \otimes \mathcal{P} \otimes \mathcal{Q})$, and the differential is defined in the obvious way. We have the similar morphisms $C(A_{\mathcal{PQ}})_{X^I} \boxtimes C(B_{\mathcal{PQ}})_{X^J} \to C(A, B, \mathcal{PQ})_{X^I \times X^J} \to C((A \otimes B)_{\mathcal{PQ}})_{X^{I \sqcup J}}$ where the left arrow is a quasi-isomorphism.

(ii) We define $C^{ch}(A, B)$ as the naive direct limit of the $\mathcal{S}^\circ \times \mathcal{S}^\circ$-diagram of complexes $\Gamma(X^I \times X^J, h(C(A, B, \mathcal{PQ})_{X^I \times X^J}))$. The above arrows yield morphisms of complexes $i = i_{A,B} : C^{ch}(A) \otimes C^{ch}(B) \to C^{ch}(A, B)$ and $\tilde{\circ} = \tilde{\circ}_{A,B} : C^{ch}(A, B) \to C^{ch}(A \otimes B)$ of (4.3.6.1). It is clear that $\tilde{\circ} i = \circ$.

As a mere graded vector space our $C^{ch}(A, B)$ decomposes into a direct sum of subspaces $C^{ch}_{m,n}(A, B) := \Gamma(X^m \times X^n, h(C(A, B, \mathcal{PQ})_{X^m \times X^n}))_{\Sigma_m \times \Sigma_n}$. The differential is compatible with the corresponding bifiltration. As in 4.2.12, one shows that $H_{DR}^{\cdot}(X^I \times X^J, C(A, B, \mathcal{PQ})_{X^I \times X^J}) \xrightarrow{\sim} H^{\cdot}\Gamma(X^I \times X^J, h(C(A, B, \mathcal{PQ})_{X^I \times X^J}))$. Therefore i is a (bifiltered) quasi-isomorphism.

The obvious embeddings $\Delta_* C((A \otimes B)_{\mathcal{PQ}})_{X^I} \hookrightarrow C(A, B, \mathcal{PQ})_{X^I \times X^I}$, where $\Delta : X^I \to X^I \times X^I$ is the diagonal, define a morphism $\delta = \delta_{A,B} : C^{ch}(A \otimes B) \to C^{ch}(A, B)$ of (4.3.6.1). It is clear that $\tilde{\circ}$ is left inverse to δ. To finish the proof, it remains to check that $\delta\circ$ is a quasi-isomorphism.

[41] This follows from 4.3.6 and (4.2.18.1) since $\operatorname{gr}_{\cdot}(\otimes A_i) = \otimes \operatorname{gr}_{\cdot} A_i$.

(iii) Let us prove first the statement (ii) in the proposition in 4.3.3. Consider the above picture for $B = \omega$. By (i) in the proposition in 4.3.3, we have a canonical identification $C^{ch}(X, A) \xrightarrow{\sim} C^{ch}(X, A) \otimes C^{ch}(X, \omega)$. Its composition with $\circ$ is the multiplication by 1^{ch} map. Since δ is a right inverse to $\circ$, the multiplication by 1^{ch} admits a right inverse. Since $(1^{ch})^2 = 1^{ch}$, the multiplication by 1^{ch} is an idempotent. Thus it is the identity map; q.e.d.

Therefore we know that $\circ_{A,\omega}$ and $\delta_{A,\omega}$ are mutually inverse quasi-isomorphisms, as well as $\circ_{\omega,B}$ and $\delta_{\omega,B}$.

(iv) Let us return to the general situation. Consider a natural morphism of complexes $\kappa : C^{ch}(A, \omega) \otimes C^{ch}(\omega, B) \to C^{ch}(A, B)$ defined by the maps

$$C(A, \omega, \mathcal{PQ})_{X^I \times X^J} \boxtimes C(\omega, B, \mathcal{PQ})_{X^{I'} \times X^{J'}} \to C(A, B, \mathcal{PQ})_{X^{I \sqcup I'} \times X^{J \sqcup J'}}$$

which are the composition of the exterior product maps with the morphisms $1_A : \omega \to A$ along the I'-variables and $1_B : \omega \to B$ along the J-variables.

Our κ is a quasi-isomorphism. To see this, it suffices to check that the composition $\kappa(i_{A,\omega} \otimes i_{\omega,B}) : C^{ch}(A) \otimes C^{ch}(\omega) \otimes C^{ch}(\omega) \otimes C^{ch}(B) \to C^{ch}(A, B)$ is a quasi-isomorphism. The latter map is equal to the composition $i_{A,B}(\circ_{A,\omega} \otimes \circ_{\omega,B})\sigma$, where σ is the transposition of the middle multiples $C^{ch}(\omega)$, and we are done.

One checks immediately that the composition $\delta_{A,B} \circ_{A,B} : C^{ch}(A) \otimes C^{ch}(B) \to C^{ch}(A, B)$ is equal to $\kappa(\delta_{A,\omega} \otimes \delta_{\omega,B})$. Thus it is a quasi-isomorphism; q.e.d. $\square$

4.3.7. Resolutions of commutative $\mathcal{D}_X$-algebras. In this section we deal with commutative unital DG $\mathcal{D}_X$-algebras and call them simply $\mathcal{D}_X$-algebras. If X is affine, then $\mathcal{D}_X$-algebras form naturally a closed model category (with quasi-isomorphisms as weak equivalences and surjective morphisms as fibrations; arguments of [**H**] work in this situation). When X is proper, this is no longer true literally. We will not use seriously the formalism of closed model categories, but just some constructions that we are going to recall now.

Let $\varphi : R \to F$ be a morphism of $\mathcal{D}_X$-algebras. We say that φ (or F) is *homotopically R-flat* if F is homotopically R-flat as a DG R-module.[42] It is *elementary* if one can find a $\mathbb{Z}$-graded $\mathcal{D}_X$-submodule $V \subset F$ such that V is a locally projective $\mathcal{D}_X$-module, $R \otimes \mathrm{Sym} V \xrightarrow{\sim} F$, and $d_F(V) \subset R$. Finally, F is *R-semi-free* if one can find a sequence of $R[\mathcal{D}_X]$-subalgebras $F_0 \subset F_1 \subset \cdots$, $\bigcup F_i = F$, such that $R \to F_0$ and all $F_i \to F_{i+1}$ are elementary morphisms. If F is R-semi-free, then it is homotopically R-flat. For any φ its *resolution* is a morphism of $R[\mathcal{D}_X]$-algebras $G \to F$ which is a quasi-isomorphism. A resolution is homotopically R-flat or R-semi-free if G is. Resolutions of φ form a category in the obvious way.

LEMMA. (i) *Any φ admits an R-semi-free resolution.*

(ii) *The groupoid obtained from the category of homotopically R-flat resolutions by localization is contractible.*

Proof. (i) One constructs an R-semi-free resolution $\psi : G \to F$ as follows. Notation: let $G_0 \subset G_1 \subset \cdots$ be a sequence of subalgebras of G as above, $V_i \subset G_i$ the corresponding $\mathbb{Z}$-graded $\mathcal{D}_X$-submodules, $\psi_i := \psi|_{V_i}$, and $d_i := d_G|_{V_i} : V_i^{\cdot} \to G_{i-1}^{\cdot+1}$. We will define (V_i, d_i, ψ_i) by induction by i. Then G_i equals $G_{i-1} \otimes \mathrm{Sym}(V_i)$ as a $\mathbb{Z}$-graded $R[\mathcal{D}_X]$-algebra, its differential d_{G_i} is determined by d_i and $d_{G_{i-1}}$, and ψ_i together with $\psi|_{G_{i-1}}$ determines $\psi|_{G_i}$ (for $i = 0$ it is ψ_0 and φ).

[42] I.e., for every acyclic DG R-module N the complex $N \underset{R}{\otimes} F$ is acyclic; see [**Sp**].

First we choose a locally projective $\mathbb{Z}$-graded $\mathcal{D}_X$-module $V_0 = V_0^{\cdot}$ and a morphism of $\mathbb{Z}$-graded $\mathcal{D}_X$-modules $\psi_0 : V_0 \to F$ such that $d_F\psi_0 = 0$ and the corresponding morphism $\bar{\psi}_0 : V_0^{\cdot} \to H^{\cdot}(F)$ is surjective. Set $G_0 := R \otimes \mathrm{Sym}(V_0)$ (we consider V_0 as a complex with zero differential).

Now suppose we have defined $(G_i, \psi|_{G_i})$, $i \geq 0$. Then $\psi|_{G_i}$ is surjective on the cohomology (since ψ_0 is). Choose a locally projective $\mathbb{Z}$-graded $\mathcal{D}_X$-module V_{i+1} and morphisms of $\mathcal{D}_X$-modules $d_{i+1}^{\cdot} : V_{i+1}^{\cdot} \to G_i^{\cdot+1}$, $\psi_{i+1}^{\cdot} : V_{i+1}^{\cdot} \to F^{\cdot}$ such that $d_{G_i} d_{i+1}^{\cdot} = 0$, $d_F\psi_{i+1} = \psi|_{G_i} d_{i+1}$, and the maps $\bar{d}_{i+1}^a : V_{i+1}^{\cdot} \to \mathrm{Ker}(H^{\cdot+1}G_i \to H^{\cdot+1}F)$ are surjective.

Remarks. (a) A variant of the induction procedure in the proof of the first part of the lemma establishes the following fact: Suppose we have R-algebras M, N such that for every a the maps $H^aR \to H^aM, H^aN$ are surjective and have equal kernels. Then M, N admit a simultaneous R-semi-free resolution; i.e., there exists an R-semi-free algebra L together with morphisms of R-algebras $L \to M, N$ which are quasi-isomorphisms.

(b) As is clear from the proof, an R-semi-free resolution G of φ can be chosen so that the V_i are isomorphic to a direct sum of $\mathcal{D}_X$-modules of type $\mathcal{L}_{\mathcal{D}} = \mathcal{L} \otimes \mathcal{D}_X$ where $\mathcal{L}$ is a line bundle on X of degree bounded from above by any constant.[43] In particular, we can choose it so that $\Gamma(X, h(V_i)) = 0$. We can also assume that $d(V_0) = 0$.

(ii) Notice that for every finite family of homotopically R-flat resolutions $\{G_\alpha\}$ of F one can find another homotopically R-flat resolution K together with morphisms $G_\alpha \to K$. Namely, consider F as a $\underset{R}{\otimes} G_\alpha$-algebra, and take for K a homotopically $\underset{R}{\otimes} G_\alpha$-flat resolution of F.

To finish the proof, it suffices to prove that for every morphisms $\zeta, \zeta' : M \to N$ of homotopically R-flat resolutions the corresponding arrows in the groupoid coincide.[44] To do this, it suffices to find another homotopically R-flat resolution L and morphisms $\eta : L \to M$, $\xi, \xi' : M \to L$, $\chi : L \to N$ such that $\eta\xi = \eta\xi' = id_M$, $\chi\xi = \zeta$, $\chi\xi' = \zeta'$. Consider M, N as $M \underset{R}{\otimes} M$-algebras via the morphisms $m \otimes m' \mapsto mm'$ and $m \otimes m' \mapsto \zeta(m)\zeta'(m')$. Our L is the corresponding simultaneous $M \underset{R}{\otimes} M$-semi-free resolution (see Remark (a) above), η, χ are the corresponding quasi-isomorphisms, and ξ, ξ' come from the maps $M \to L \underset{R}{\otimes} M$, $m \mapsto m \otimes 1, 1 \otimes m$. $\square$

Exercise. Show that statement (ii) of the lemma remains valid if we replace "homotopically R-flat" by "R-semi-free."

4.3.8. Base change. Let $\varphi : R \to F$ be a morphism of commutative chiral algebras. It yields the base change functor $\varphi^* : \mathcal{CA}(X, R) \to \mathcal{CA}(X, F)$, $A \mapsto \varphi^*A = A \underset{R}{\otimes} F$. If F is homotopically R-flat, then our functor preserves quasi-isomorphisms and hence defines a functor between the homotopy categories

$$\varphi^* : \mathcal{H}o\mathcal{CA}(X, R) \to \mathcal{H}o\mathcal{CA}(X, F). \tag{4.3.8.1}$$

[43] It suffices to take $\mathcal{L}$ equal to tensor powers of a given negative line bundle.

[44] To show that this fact implies the lemma, one repeats the argument of part (iii) of the proof of the second lemma in 4.1.3.

To treat a non-flat φ, we have to change our homotopy categories. Anyway, the homotopy category $\mathcal{H}o\mathcal{CA}(X,R)$ is *not* a right object for it may change if we replace R by a quasi-isomorphic algebra. To dispatch this nuisance, one considers the category $\mathcal{CA}(X,R)\tilde{}$ of pairs $(A,R')=(A,R',\theta)$ where $\theta : R'\to R$ is a morphism of commutative $\mathcal{D}_X$-algebras which is a quasi-isomorphism, A a chiral R'-algebra. Its localization with respect to quasi-isomorphisms is denoted by $\mathcal{H}o\mathcal{CA}(X,R)\tilde{}$. There is an obvious functor $\mathcal{H}o\mathcal{CA}(X,R)\to\mathcal{H}o\mathcal{CA}(X,R)\tilde{}$.

Now any morphism of commutative $\mathcal{D}_X$-algebras $\varphi : R\to F$ yields a functor

$$L\varphi^* : \mathcal{H}o\mathcal{CA}(X,R)\tilde{}\to\mathcal{H}o\mathcal{CA}(X,F)\tilde{}. \tag{4.3.8.2}$$

Namely, $L\varphi^*$ sends (A,R') to $(A\underset{R'}{\otimes}G,G)$ where $R'\to G\to F$ is any homotopically R'-flat resolution of F. According to the lemma in 4.3.7, this is a well-defined object of $\mathcal{H}o\mathcal{CA}(X,F)\tilde{}$. The functors $L\varphi^*$ are compatible with the composition of the φ's.

For any (A,R') as above, $C^{ch}(X,A)$ is a unital homotopy $C^{ch}(X,R')$-module (see 4.3.2 and 4.3.4). Since $\theta : C^{ch}(X,R')\to C^{ch}(X,R)$ is a quasi-isomorphism, it identifies the corresponding derived categories of $C^{ch}(X,R)$- and $C^{ch}(X,R')$-modules, and we can consider $C^{ch}(X,A)$ as a homotopy $C^{ch}(X,R)$-module. Its $C^{ch}\varphi$ base change is a $C^{ch}(X,F)$-module

$$L(C^{ch}\varphi)^*C^{ch}(X,A) := C^{ch}(X,A)\overset{L}{\underset{C^{ch}(X,R)}{\otimes}}C^{ch}(X,F) \tag{4.3.8.3}$$

(see e.g. [**H**] for details). There is a canonical *base change* morphism

$$\beta_\varphi : L(C^{ch}\varphi)^*C^{ch}(X,A)\to C^{ch}(X,L\varphi^*A) \tag{4.3.8.4}$$

in the derived category of homotopy $C^{ch}(X,F)$-modules. To define it, we can assume that F is homotopically R-flat. The morphism of chiral R-algebras $A\to A\underset{R}{\otimes}F$ yields a morphism of homotopy $C^{ch}(X,R)$-modules $C^{ch}(X,A)\to C^{ch}(X,A\underset{R}{\otimes}F)$, hence, by adjunction, a morphism $L(C^{ch}\varphi)^*C^{ch}(X,A)\to C^{ch}(X,A\underset{R}{\otimes}F)$ of homotopy $C^{ch}(X,F)$-modules, which is our β_φ.

4.3.9. THEOREM. *The base change map (4.3.8.4) is a quasi-isomorphism.*

Proof. We can assume that $R'=R$ and, by (i) in the lemma in 4.3.7, that F is R-semi-free. Since chiral homology commutes with inductive limits, it suffices to consider the case of an elementary morphism φ.[45]

Let $V\subset F$ be as in 4.3.7. We have a filtration $R\otimes\mathrm{Sym}^{\le a}V$ on F, so $\mathrm{gr}\,F$ equals $R\otimes\mathrm{Sym}\,V$ as a DG $\mathcal{D}_X$-algebra (we consider V as a complex with zero differential). It defines filtrations on $C^{ch}(X,F)$, hence on $L\varphi^*C^{ch}(X,A) := C^{ch}(X,A)\overset{L}{\underset{C^{ch}(X,R)}{\otimes}}C^{ch}(X,F)$ and on $C^{ch}(X,A\underset{R}{\otimes}F)$ (see 4.2.18). The base change morphism is compatible with filtrations,[46] so it suffices to check that $\mathrm{gr}\,\beta_\varphi$ is a quasi-isomorphism. But, by (4.2.18.1), $\mathrm{gr}\,\beta_\varphi$ is the base change morphism for $R\to\mathrm{gr}\,F=R\otimes\mathrm{Sym}\,V$, so we have reduced our problem to the situation when $V\subset F$

[45]For if we have two composable morphisms φ and our statement holds for each of them, then it holds for the composition.

[46]Recall that this means that we have a canonical morphism in the filtered derived category.

is killed by the differential. Then $A \underset{R}{\otimes} F = A \otimes \mathrm{Sym}\, V$, so 4.3.6 provides canonical isomorphisms $C^{ch}(X,F) = C^{ch}(X,R) \otimes C^{ch}(X, \mathrm{Sym}\, V)$ and $C^{ch}(X, A \underset{R}{\otimes} F) = C^{ch}(X,A) \otimes C^{ch}(X, \mathrm{Sym}\, V)$. They identify β_φ with the identity map for $C^{ch}(X,A)$ $\otimes C^{ch}(X, \mathrm{Sym}\, V)$, and we are done. □

Here are some corollaries. Let R be a commutative chiral algebra, $\{A_i\}_{i\in I}$ a finite collection of chiral R-algebras. Suppose that R is homotopically $\mathcal{O}_X$-flat and that the A_i are pairwise Tor-$\mathcal{O}_X$-independent. Then $\otimes A_i$ is a chiral $R^{\otimes I}$-algebra; set $\overset{L}{\underset{R}{\otimes}} A_i := L\delta^*(\otimes A_i)$ where $\delta : R^{\otimes I} \to R$ is the product map. One has the following relative form of 4.3.6:

4.3.10. Corollary. *There is a canonical isomorphism of the homotopy $C^{ch}(X,R)$-modules*

$$\overset{L}{\underset{C^{ch}(X,R)}{\otimes}} C^{ch}(X, A_i) \xrightarrow{\sim} C^{ch}(X, \overset{L}{\underset{R}{\otimes}} A_i). \tag{4.3.10.1}$$

Proof. Use 4.3.9 for δ and $\otimes A_i$ together with 4.3.6. □

4.3.11. Corollary. *The chiral homology commutes with direct products: for any finite collection of chiral algebras A_i the projection morphisms yield an isomorphism*

$$C^{ch}(X, \Pi A_i) \xrightarrow{\sim} \Pi C^{ch}(X, A_i). \tag{4.3.11.1}$$

The inverse map comes from the obvious (non-unital) morphisms $A_j \to \Pi A_i$.

Remark. The assumption that our chiral algebras are unital is essential here (consider the case of $\mu_{A_i} = 0$).

Proof. The latter map is right inverse to (4.3.11.1), so it is enough to check that (4.3.11.1) is an isomorphism.

It suffices to consider the case when all A_i equal $\mathcal{O}_X$: Indeed, for arbitrary A_i we can consider ΠA_i as a chiral $\mathcal{O}_X^I$-algebra. Then (4.3.11.1) follows from 4.3.9 for $A = \Pi A_i$, $R = \mathcal{O}_X^I$, $F = \mathcal{O}_X$.

Case $A_i = \mathcal{O}_X$: We know that $H_0^{ch}(X, \mathcal{O}_X^I) = \langle \mathcal{O}_X^I \rangle = k^I$ (see the end of 4.2.16). Applying 4.3.9 to $R = \mathcal{O}_X^I$, $A = F = \mathcal{O}_X$, and $R \to A, F$ a projection map, we see that the higher chiral homology of $\mathcal{O}_X^I$ vanishes. Indeed, by 4.3.9 and 4.3.3, the first non-trivial $H_a^{ch}(X,R)$, $a > 0$, yields non-trivial $H_{a+1}^{ch}(X, A \overset{L}{\underset{R}{\otimes}} F)$, which contradicts the vanishing of the higher chiral homology of $A \overset{L}{\underset{R}{\otimes}} F = \mathcal{O}_X$. We are done. □

4.3.12. Here is a more general statement.

Let $\varphi : R \to F$ be an étale morphism of plain commutative $\mathcal{D}_X$-algebras, A any chiral R-algebra. Set $A_F := A \underset{R}{\otimes} F$. The chiral homologies $H_a^{ch}(X,A)$ are $\langle R\rangle$-modules, and the $H_a^{ch}(X, A_F)$ are $\langle F\rangle$-modules (see 4.2.16), so we have a canonical morphism

$$H_\cdot^{ch}(X,A) \underset{\langle R\rangle}{\otimes} \langle F\rangle \to H_\cdot^{ch}(X, A_F). \tag{4.3.12.1}$$

In particular, for $A = R$ we get a morphism

$$H^{ch}_{\cdot}(X,R) \underset{\langle R\rangle}{\otimes} \langle F\rangle \to H^{ch}_{\cdot}(X,F). \tag{4.3.12.2}$$

PROPOSITION. *The morphism $\langle\varphi\rangle : \langle R\rangle \to \langle F\rangle$ is étale, and (4.3.12.1) and (4.3.12.2) are isomorphisms.*

Proof. (i) $\langle\varphi\rangle$ is étale: Set $S := \mathrm{Spec}\langle R\rangle$, $T := \mathrm{Spec}\langle F\rangle$, $S_X := S \times X$, etc. We have closed embeddings $S_X \hookrightarrow \mathrm{Spec}\, R$, $T_X \hookrightarrow \mathrm{Spec}\, F$. Consider $\varphi|_{S_X}$: $\mathrm{Spec}\, F|_{S_X} \to S_X$; then T_X is the maximal constant closed $\mathcal{D}_X$-subscheme of $\mathrm{Spec}\, F|_{S_X}$. Since $\varphi|_{S_X}$ is étale, it is also an open subscheme, and we are done.

(ii) By 4.3.9 it suffices to consider the case $A = R$, i.e., (4.3.12.2).

(iii) Set $C := C^{ch}(X,R)$, $D := C^{ch}(X,F)$. These are commutative unital homotopy algebras having degrees ≤ 0. Set $\bar{C} := H_0C$, $\bar{D} := H_0D$. We have a morphism $C \to D$. We want to show that the corresponding morphism $H\tilde{\cdot}C := (H_{\cdot}C) \underset{\bar{C}}{\otimes} \bar{D} \to H_{\cdot}D$ is an isomorphism.

Since F is R-flat, 4.3.10 implies that $D \overset{L}{\underset{C}{\otimes}} D = C^{ch}(X, F \underset{R}{\otimes} F)$. Since F/R is étale, one has a $\mathcal{D}_X$-algebra decomposition $F \underset{R}{\otimes} F = F \times Q$ where the projection $F \underset{R}{\otimes} F \to F$ is the product map. Set $E := C^{ch}(X,Q)$, $\bar{E} := H_0E$. By 4.3.11 one has $D \overset{L}{\underset{C}{\otimes}} D = D \times E$ where the projection $D \overset{L}{\underset{C}{\otimes}} D \to D$ is the product map.

One has a spectral sequence converging to $H_{\cdot}(D \overset{L}{\underset{C}{\otimes}} D)$ with the second term equal to $Tor_p^{H_{\cdot}C}(H_{\cdot}D, H_{\cdot}D)_q$. The above decomposition then gives a spectral sequence converging to $H_{\cdot}D$ with $E^2_{p,q} = Tor_p^{H\tilde{\cdot}C}(H_{\cdot}D, H_{\cdot}D)_q$. Notice that the map $E^2_{0,\cdot} \to H_{\cdot}D$ is just the product map $H \cdot D \underset{H\tilde{\cdot}C}{\otimes} H_{\cdot}D \to H_{\cdot}D$ and hence it is surjective, i.e., $E^{\infty}_{>0,q} = 0$.

Suppose that $H\tilde{_a}C \neq H_aD$ for some $a \geq 1$; consider the first such a. We have $E^2_{p,q} = 0$ for $p \geq 1$ and $q \leq 2a-1$. This implies $E^2_{0,a} = H_aD$. On the other hand, $E^2_{0,a} = (H_{\cdot}D \underset{H\tilde{\cdot}C}{\otimes} H_{\cdot}D)_a$ which is the cokernel of the diagonal map $H\tilde{_a}C \to H_aD \oplus H_aD$. Thus $H\tilde{_a}C \to H_aD$ is surjective. We have $E^2_{1,2a} = \mathrm{Ker}(H\tilde{_a}C \to H_aD)$. Since $E^2_{1,2a} = E^{\infty}_{1,2a} = 0$, we arrive at a contradiction. $\square$

4.3.13. QUESTIONS. Let R be a plain commutative $\mathcal{D}_X$-algebra. Is it true that for any chiral R-algebra A its chiral homology has a local origin with respect to the Zariski or étale topology of $\mathrm{Spec}\langle R\rangle$? More generally, is this true if A is a chiral $R_{\mathcal{D}if}$-algebra (see 3.9.4)? Can one define the chiral homology for chiral algebras on any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ or on $\mathcal{Y}_{\mathcal{D}if}$ (see loc. cit.)?

4.4. Correlators and coinvariants

We begin with the definition of correlator functions for a plain chiral algebra A; these functions (and the differential equations they satisfy) are of primary interest for mathematical physicists (see, e.g., [**BPZ**]). In general, the "correlator-style" approach to chiral homology stems from the following observation: for any finite subset $\{x_s\} \subset X$ the complement to the subspace $\mathcal{R}(X)_{(x_s)} \subset \mathcal{R}(X)$ whose points are finite subsets containing $\{x_s\}$, is acyclic (see 4.4.2). It permits us to identify the "absolute" chiral homology of A with the chiral homology of A with coefficients in fibers $A^\ell_{x_s}$ (see 4.4.3). In particular, $\langle A\rangle$ identifies with coinvariants of the action of the Lie algebra $h(U_S,A)$ on $\otimes A^\ell_{x_s}$ where $U_S := X\smallsetminus\{x_s\}$ (see 4.4.4), so $\langle A\rangle$ is dual to the "space of conformal blocks" (see [**FBZ**] 8.2). To compute the coinvariants, it suffices to consider instead of the whole of $h(U_S,A)$ the space $h(U_S,P)$ where $P\subset A$ is any sub-$\mathcal{D}$-module generating A (see 4.4.5).[47] A relative version of this statement (when our points vary) is discussed in 4.4.6; we briefly mention the example of the Knizhnik-Zamolodchikov equation (see [**KZ**], [**EFK**], and Chapter 12 of [**FBZ**]). In 4.4.7 the chiral homology of the unit chiral algebra ω with arbitrary coefficients is computed. In 4.4.8 we show that the chiral homology functor commutes with adding of unit. In 4.4.9 a spectral sequence (similar to the Hochschild-Serre spectral sequence) for computing the chiral homology of a chiral algebra with a given subalgebra is constructed.

Since the early days of conformal field theory, the geometry of $\mathcal{R}(X)$ was used in order to write down explicit integral formulas for some correlators (the *Feigin-Fuchs integrals*), see [**DoF**]. It is used in [**BFS**] to construct geometrically the category of representations of a quantum group. We do not touch these subjects.

4.4.1 Correlators. Let us return to 4.2.16, so A is a plain chiral algebra. We see that for every $S\in\mathcal{S}$ one has a canonical morphism of $\mathcal{D}_{X^S}$-modules

$$\langle\ \rangle_I : A^\ell_{X^S}\to\langle A\rangle\otimes\mathcal{O}_{X^S}. \tag{4.4.1.1}$$

These morphisms are compatible with pull-backs to the diagonals, so they form a morphism $\langle\ \rangle : A^\ell_{\mathcal{R}(X)}\to\langle A\rangle\otimes\mathcal{O}_{\mathcal{R}(X)}$ of the left $\mathcal{D}$-modules on $\mathcal{R}(X)$ (see 3.4.2 for the terminology). So for a finite subset $\{x_s\}\subset X$, $s\in S$, and $a\in A^\ell_{(x_s)}=\underset{s\in S}{\otimes}A^\ell_{x_s}$ we have $\langle a\rangle\in\langle A\rangle$.

The compatibility with respect to the restriction to the diagonals implies that for every $\{x_t\}\subset X\smallsetminus\{x_s\}$ one has

$$\langle a\otimes(\otimes 1_{x_t})\rangle=\langle a\rangle \tag{4.4.1.2}$$

where 1 is the unit section of A^ℓ. In particular, $1^{ch}=\langle\otimes 1_t\rangle$ (see 4.2.16).

Restricting $\langle\ \rangle$ to the complement $\mathcal{R}(X)^o_n$ of the diagonal divisor on $\mathrm{Sym}^n X$, we get the *n-point correlator morphisms*

$$\langle\ \rangle_n : (\mathrm{Sym}^n A^\ell)_{\mathcal{R}(X)^o_n}\to\langle A\rangle\otimes\mathcal{O}_{\mathcal{R}(X)^o_n}. \tag{4.4.1.3}$$

Exercise. Show that the following diagram commutes:

$$\begin{array}{ccc} j_*j^*A\boxtimes A & \longrightarrow & \langle A\rangle\otimes j_*j^*\omega\boxtimes\omega \\ \downarrow & & \downarrow \\ \Delta_*A & \longrightarrow & \langle A\rangle\otimes\Delta_*\omega \end{array} \tag{4.4.1.4}$$

Here the horizontal arrows are the correlator morphisms and the vertical ones are the chiral products for A and ω.

[47] A particular case of this statement when P is a Lie* subalgebra of A was considered in [**FBZ**] 8.3.

Remark. Suppose that the 2-point correlator pairing is non-degenerate; i.e., for every non-zero (local) section $a(x) \in A^{\ell}$ there exists another section $b(y)$ such that $\langle a(x)b(y)\rangle \neq 0$. Then the chiral algebra structure on the $\mathcal{O}_X$-module A is uniquely determined by the vector space $\langle A\rangle$ and the 2- and 3-point correlator maps. Indeed, we know that the chiral algebra structure is determined by the sub-$\mathcal{D}_{X\times X}$-module $A^{\ell}_{X\times X} \subset j_*j^*A_X \boxtimes A_X$ together with the identification $\Delta^* A_{X\times X} \xrightarrow{\sim} A^{\ell}_X$. Now $A^{\ell}_{X\times X}$ consists of all sections $a(x,y) \in j_*j^*A_X \boxtimes A_X$ such that for every section $b(z) \in A^{\ell}$ the correlator $\langle a(x,y)b(z)\rangle$ is regular at the divisor $x = y$. If a is such a section, then $a(x,x) \in A^{\ell}$ is determined by the condition $\langle a(x,x)b(z)\rangle = \langle a(x,y)b(z)\rangle_{x=y}$ for any $b(z) \in A^{\ell}_X$.

4.4.2. Let A be a unital chiral algebra and $\{x_s\} \subset X$, $s \in S$, a finite non-empty subset. Consider the subspace $\mathcal{R}(X)_{(x_s)} \hookrightarrow \mathcal{R}(X)$ of those points tfor which the corresponding finite subsets of X contain $\{x_s\}$. We will consider the complex $C^{ch}(X,A)_{(x_s)}$ defined as the de Rham cohomology of $\mathcal{R}(X)$ with support in $\mathcal{R}(X)_{(x_s)}$ and coefficients $C(A)$.

The construction of the cohomology with support was explained in (iii) and (iv) in 4.2.6. Precisely, let $X^{\mathcal{S}}_{(x_s)} \hookrightarrow X^{\mathcal{S}}$ be the r-preimage of $\mathcal{R}(X)_{(x_s)}$; this is a diagram of closed subvarieties of $X^{\mathcal{S}}$. By (iii) in 4.2.6 we have an admissible $\mathcal{D}$-complex $C(A)_{(x_s)} := C(A)_{X^{\mathcal{S}}_{(x_s)}} \in D\mathcal{M}(X^{\mathcal{S}})_{X^{\mathcal{S}}_{(x_s)}} \subset D\mathcal{M}(X^{\mathcal{S}})$ equipped with a canonical morphism $C(A)_{(x_s)} \to C(A)$. Now one defines $C^{ch}(X,A)_{(x_s)}$ as $R\Gamma_{DR}(X^{\mathcal{S}}, C(A)_{(x_s)}) = R\Gamma_{DR}(X^{\mathcal{S}}, C(A))_{X^{\mathcal{S}}_{(x_s)}}$.

Proposition. *The canonical morphism $C(A)_{(x_s)} \to C(A)$ yields a quasi-isomorphism*

$$C^{ch}(X,A)_{(x_s)} \xrightarrow{\sim} C^{ch}(X,A). \tag{4.4.2.1}$$

Proof. Let $j_{V_{(x_s)}} : V_{(x_s)} \hookrightarrow X^{\mathcal{S}}$ be the complement of $X^{\mathcal{S}}_{(x_s)}$, so we have the exact triangle $C(A)_{(x_s)} \to C(A) \to Rj_{V_{(x_s)}*}j^*_{V_{(x_s)}}C(A)$. Let us show that the de Rham cohomology of $\mathcal{R}(X) \smallsetminus \mathcal{R}(X)_{(x_s)}$ with coefficients in $C(A)$ vanishes; i.e., the complex $R\Gamma_{DR}(X^{\mathcal{S}}, Rj_{V_{(x_s)}*}j^*_{V_{(x_s)}}C(A))$ is acyclic.

Notice that $\mathcal{R}(X) \smallsetminus \mathcal{R}(X)_{(x_s)} = \bigcup_{s\in S} \mathcal{R}(U_s)$ where $U_s \subset X$ is the complement to x_s. Any intersection of these subsets is the complement U_T to a *non-empty* subset $\{x_t\}_{t\in T}$ of X. We can write $Rj_{V_{(x_s)}*}j^*_{V_{(x_s)}}C(A)$ as the Čech resolution $\check{C}$ which is the total complex of a bicomplex whose nth column is isomorphic to $\bigoplus_{|T|=n+1} C(j_{T*}j^*_T A)$ where $j_T : U_T \hookrightarrow X$. By the lemma in 4.3.4 one has $R\Gamma_{DR}(\mathcal{R}(X), C(j_{T*}j^*_T A)) = C^{ch}(X, j_{T*}j^*_T A) = 0$; q.e.d. □

4.4.3. Let us describe the complex $C(A)_{(x_s)}$. Consider the embeddings $i_s : \{x_s\} \hookrightarrow X$ and $j_s : U_s := X \smallsetminus \{x_s\} \hookrightarrow X$. For any $s \in S$ we have an A-module $\tilde{A}_s := \mathcal{C}one(A \to j_{s*}j^*_s A)$ acyclic off x_s. The morphism of A-modules $\tilde{A}_s \to \operatorname{Coker}(A \to j_{s*}j^*_s A) = i_{s*}A^{\ell}_{x_s}$ is a quasi-isomorphism when A is homotopically $\mathcal{O}_X$-flat at x_s. Consider the complex $C(A, \{\tilde{A}_s\}) \in C\mathcal{M}(X^{\mathcal{S}})$ (see 4.2.19).

Proposition. *There is a canonical quasi-isomorphism in $D\mathcal{M}(X^{\mathcal{S}})$*

$$C(A, \{\tilde{A}_s\}) \xrightarrow{\sim} C(A)_{(x_s)}. \tag{4.4.3.1}$$

Therefore, passing to de Rham cohomology and applying (4.4.2.1), we get a canonical quasi-isomorphism

$$C^{ch}(X, A, \{\tilde{A}_s\}) \xrightarrow{\sim} C^{ch}(X, A). \tag{4.4.3.2}$$

If A is homotopically $\mathcal{O}_X$-flat at $\{x_s\}$, it can be rewritten as

$$C^{ch}(X, A, \{A^{\ell}_{x_s}\}) \xrightarrow{\sim} C^{ch}(X, A). \tag{4.4.3.3}$$

Proof. We will define a complex $\tilde{C}(A, \{\tilde{A}_s\}) \in C\mathcal{M}(X^{\mathcal{S}})$ acyclic off $X^{\mathcal{S}}_{(x_s)}$ and morphisms

$$C(A, \{\tilde{A}_s\}) \xleftarrow{\alpha} \tilde{C}(A, \{\tilde{A}_s\}) \xrightarrow{\beta} C(A) \tag{4.4.3.4}$$

such that α is a quasi-isomorphism and the morphism $\beta_S : \tilde{C}(A, \{\tilde{A}_s\}) \to C(A)_{(x_s)}$ defined by β is also a quasi-isomorphism. This defines (4.4.3.1).

Set $P := \underset{s\in S}{\oplus} \mathcal{C}one(\mathcal{O}_X \to j_{s*}\mathcal{O}_{U_s})[-1] \in C\mathcal{M}^{\ell}(X)$. The commutative $\mathcal{D}_X$-algebra $\mathrm{Sym}\, P$ is naturally $\mathbb{Z}^S$-graded, so the chiral algebra $A \otimes P$ is also $\mathbb{Z}^S$-graded. The projection $P \to \mathcal{O}_X^S \to \mathcal{O}_X$, where the right arrow is $(f_s) \mapsto \Sigma f_s$, yields a morphism of $\mathcal{D}_X$-algebras $\mathrm{Sym}\, P \to \mathcal{O}_X$, hence a morphism of chiral algebras $\tilde{\beta} : A \otimes \mathrm{Sym}\, P \to A$. We also have a morphism of chiral algebras $\tilde{\alpha} : A \otimes \mathrm{Sym}\, P \to A \otimes (\mathrm{Sym}\, P/\mathrm{Sym}^{\geq 2} P) = A^{\{\tilde{A}_s\}}$ compatible with the $\mathbb{Z}^S$-gradings (see 4.2.19 for notation).

Our $\tilde{C}(A, \{\tilde{A}_s\})$ is the component of degree 1^S of $C(A \otimes \mathrm{Sym}\, P)$, $\alpha : \tilde{C}(A, \{\tilde{A}_s\}) \to C(A, \{\tilde{A}_s\})$ is the morphism defined by $\tilde{\alpha}$,[48] and $\beta : \tilde{C}(A, \{\tilde{A}_s\}) \to C(A)$ is the morphism defined by $\tilde{\beta}$.

The projection $\mathrm{Sym}\, P \to \mathrm{Sym}\, P/\mathrm{Sym}^{\geq 2} P$ induces a quasi-isomorphism between the components of degree $\leq 1^S$. Therefore $\tilde{\alpha} : C(A \otimes \mathrm{Sym}\, P) \to C(A^{\{\tilde{A}_s\}})$ is a quasi-isomorphism on the components of degree $\leq 1^S$; hence α is a quasi-isomorphism. Since P is acyclic off S, our $\tilde{C}(A, \{\tilde{A}_s\})$ is acyclic off $X^{\mathcal{S}}_{(x_s)}$.

It remains to check that $\beta_S : \tilde{C}(A, \{\tilde{A}_s\}) \to C(A)_{(x_s)}$ is a quasi-isomorphism. We are playing with admissible complexes, so it suffices to check this on $U^{(I)} \subset X^I$ for any $I \in \mathcal{S}$. There the complex $\tilde{C}(A, \{\tilde{A}_s\})_{U^{(I)}}$ is naturally $\mathbb{Z}^{S\times I}$-graded. The intersection of $X^{\mathcal{S}}_{(x_s)}$ with $U^{(I)}$ is a disjoint union of components $((x_s) \times X^{I \smallsetminus S}) \cap U^{(I)}$ with respect to all embeddings $S \hookrightarrow I$. Our problem is local, so it suffices to check that β_S is a quasi-isomorphism on the complement in $U^{(I)}$ to all the above components but one. Here each of the $\mathbb{Z}^{S\times I}$-components of $\tilde{C}(A, \{\tilde{A}_s\})_{U^{(I)}}$ is acyclic except $(\boxtimes \tilde{A}_s) \boxtimes (A[1])^{\boxtimes I \smallsetminus S}$. On the other hand, $C(A)_{U^{(I)}} = (A[1])^{\boxtimes I}_{U^{(I)}}$, and the restriction of β to the above component is the morphism $(\boxtimes \tilde{A}_s) \boxtimes (A[1])^{\boxtimes I \smallsetminus S} \to (A[1])^{\boxtimes I}$ equal to the tensor product of the projections $\tilde{A}_s \to A[1]$ and the identity map for $(A[1])^{\boxtimes I \smallsetminus S}$. It evidently identifies $\tilde{C}(A, \{\tilde{A}_s\})$ with $C(A)_S$, and we are done. □

Remarks. (i) The above subject generalizes in an evident manner to the case of the chiral homology with coefficients (see 4.2.19). We leave the exact formulation of the general statement to the reader. Let us consider a particular situation when

[48] Recall (see 4.2.19) that $C(A, \{\tilde{A}_s\})$ is the degree 1^S component of $C(A^{\{\tilde{A}_s\}})$.

A is homotopically $\mathcal{O}_X$-flat at $\{x_s\}$, and in addition we have $\{x_t\} \subset X \smallsetminus \{x_s\}$ and A-modules M_t supported at x_t. Then there is a canonical quasi-isomorphism

$$C^{ch}(X, A, \{A^\ell_{x_s}, h(M_t)\}) \xrightarrow{\sim} C^{ch}(X, A, \{h(M_t)\}). \tag{4.4.3.5}$$

(ii) Suppose that A is a plain chiral algebra. Then the morphism $C^{ch}(X, A) \xrightarrow{\sim} C^{ch}(X, A, \{\tilde{A}_s\}) \to C^{ch}(X, A, \{A^\ell_{x_s}\})$ always yields an isomorphism $H^{ch}_0(X, A) \xrightarrow{\sim} H^{ch}_0(X, A, \{A^\ell_{x_s}\})$ (regardless of whether A is $\mathcal{O}_X$-flat at $\{x_s\}$ or not).

4.4.4. H^{ch}_0 as coinvariants. Let A be a not necessary unital plain chiral algebra. Suppose we have a finite non-empty subset $\{x_s\} \subset X$ and for each $s \in S$ a plain A-module M_s supported at x_s; let $j_S : U_S \hookrightarrow X$ be the complement to $\{x_s\}$. The spectral sequence (4.2.19.6) shows then that $H^{ch}_{<0}(X, A, \{M_s\}) = 0$ and $H^{ch}_0(X, A, \{M_s\}) = \mathrm{Coker}(d^1_{1,0} : (\otimes h(M_s)) \otimes H^0_{DR}(U_S, A) \to \otimes h(M_s))$. The latter differential can be described as follows. By 3.6.3 M_s is a $j_{S*}j^*_S A$-module, so $h(M_s)$ is a module over the Lie algebra $h(U_S, A) := \Gamma(U_S, h(A^{Lie}))$. By the lemma in 2.1.7 the projection $H^0_{DR}(U_S, A) \to h(U_S, A)$ is surjective, and $d^1_{1,0}$ is the composition of this projection and the action of $h(U_S, A)$ on the tensor product $\otimes h(M_s)$. Thus H^{ch}_0 equals the coinvariants of this action:

$$H^{ch}_0(X, A, \{M_s\}) = (\otimes h(M_s))_{h(U_S, A)}. \tag{4.4.4.1}$$

Remark. Suppose A is unital. Then the correlator map $\langle\ \rangle : \otimes A^\ell_{x_s} \to \langle A \rangle$ from 4.4.1 coincides, via the identifications $\langle A \rangle = H^{ch}_0(X, \{A^\ell_{x_s}\}) = (\otimes A^\ell_{x_s})_{h(U_S, A)}$ (see Remark (ii) in 4.4.3) and (4.4.4.1), with the obvious projection $\otimes A^\ell_{x_s} \to (\otimes A^\ell_{x_s})_{h(U_S, A)}$.

4.4.5. One usually computes $H^{ch}_0(X, A, \{M_s\})$ in a more practical way:

Proposition. *Let $P_{U_S} \subset A_{U_S}$ be any $\mathcal{D}$-submodule which generates A_{U_S} as a (non-unital) chiral algebra. Then the coinvariants for the actions of $h(U_S, P) := \Gamma(U_S, h(P_{U_S}))$ and $h(U_S, A)$ on $\otimes h(M_s)$ coincide. So, by (4.4.4.1), one has*

$$H^{ch}_0(X, A, \{M_s\}) = (\otimes h(M_s))_{h(U_S, P)}. \tag{4.4.5.1}$$

In the unital setting it suffices to assume that P generates A as a unital chiral algebra.

Proof. For a $\mathcal{D}$-submodule $Q_{U_S} \subset A_{U_S}$ denote by $\mathcal{I}(Q)$ the image of the composition $h(U_S, Q) \otimes (\otimes h(M_s)) \to h(U_S, A) \otimes (\otimes h(M_s)) \xrightarrow{\cdot} \otimes h(M_s)$ where $\cdot$ is the action map. We want to show that $\mathcal{I}(P) = \mathcal{I}(A)$.

Adding a unit to A, we can assume to be in the unital setting (see 3.3.3, 3.3.4). Since $h(U_S, \omega \cdot 1_A)$ acts trivially on $\otimes h(M_s)$, we can assume that $1_A \in P^\ell_{U_S}$.

(i) Suppose $|S| = 1$. Set $Q_{U_S} := \mu_A(j_* j^* P_{U_S} \boxtimes P_{U_S})$. It suffices to show that $\mathcal{I}(Q) \subset \mathcal{I}(P)$. The maps $\Gamma(U_S \times U_S, j_* j^* P_{U_S} \boxtimes P_{U_S}) \to \Gamma(U_S \times U_S, \Delta_* Q_{U_S}) \to h(U_S, Q)$ are surjective (the first one because U_S is affine and the second one since, by the lemma in 2.1.7, such is its restriction to $Q_{U_S} \subset \Delta_* Q_{U_S}$). So $\mathcal{I}(Q)$ is the image of the composition ξ of $\Gamma(U_S \times U_S, j_* j^* P_{U_S} \boxtimes P_{U_S}) \otimes h(M_s) \to h(U_S, Q) \otimes h(M_s) \xrightarrow{\cdot} h(M_s)$ where the first arrow is $\mu_A \otimes id_{h(M_s)}$. By the Jacobi identity, one has $\xi = \xi' - \xi''$ where ξ', ξ'' are the compositions $\Gamma(U_S \times U_S, j_* j^* P_{U_S} \boxtimes P_{U_S}) \otimes h(M_s) \to \Gamma(U_S, P_{U_S}) \otimes h(M_s) \xrightarrow{\cdot} h(M_s)$ and the first arrow is the chiral $j_{S*}j^*_S A$-action on

M_s along the second, resp. first, variable. The images of both ξ', ξ'' are contained in $\mathcal{I}(P)$, and we are done.

(ii) It suffices to consider the case $h(M_s) = A^\ell_{x_s}$. Indeed, both functors of the coinvariants commute with the inductive limits of the M_s, so we can assume that the M_s are finitely generated. The statement depends only on the restriction of A to U_S, so, modifying our A at S if necessary, we can assume that each M_s is a quotient of a sum of several copies of $i_{s*}A^\ell_{x_s}$ (see 3.6.6). If proven for some modules, our statement is automatically true for each their quotients, and also it is compatible with direct sums. So we can assume that $h(M_s) = A^\ell_{x_s}$.

Let us extend P_{U_S} to a $\mathcal{D}$-submodule $P \subset A$. Replacing A by its chiral subalgebra generated by P, we can assume that P generates A.

(iii) Suppose that $|S| > 1$. Let us show that $(\otimes A^\ell_{x_s})_{h(U_S,P)} \xrightarrow{\sim} (\otimes A^\ell_{x_s})_{h(U_S,A)}$.

We use induction by $|S|$. Pick $s_0 \in S$ and set $T := S \setminus \{s_0\}$; we denote the elements of T by t. Consider the map $\otimes A^\ell_{x_t} \to \otimes A^\ell_{x_s}$, $\otimes a_t \mapsto 1_{s_0} \otimes (\otimes a_t)$. We know that $H^{ch}_0(X, A) = H^{ch}_0(X, A, \{A^\ell_{x_t}\}) = H^{ch}_0(X, A, \{A^\ell_{x_s}\})$ (see Remark (ii) in 4.4.3). So, by (4.4.4.1),[49] our map induces an isomorphism $(\otimes A^\ell_{x_t})_{h(U_T,A)} \xrightarrow{\sim} (\otimes A^\ell_{x_s})_{h(U_S,A)}$. One has $(\otimes A^\ell_{x_t})_{h(U_T,P)} \xrightarrow{\sim} (\otimes A^\ell_{x_t})_{h(U_T,A)}$ by the induction assumption. Thus it suffices to prove that $(\otimes A^\ell_{x_t})_{h(U_T,P)} \to (\otimes A^\ell_{x_s})_{h(U_S,P)}$ is surjective. This follows immediately from the fact that the image of $h(U_S, P) \to A^{as}_{x_{s_0}}$ generates $A^{as}_{x_{s_0}}$ as a topological associative algebra.[50] $\square$

4.4.6. The material of 4.4.2–4.4.5 immediately generalizes to the situation where points x_s vary, i.e., to the relative situation over $U^{(S)}$. For example, the relative version of 4.4.5 looks as follows.

Suppose A is a plain chiral algebra and $P \subset A$ a $\mathcal{D}_X$-submodule that generates A as a chiral algebra. Let $\{M_s\}$ be an S-family of A-modules, $S \in \mathcal{S}$. We have the left $\mathcal{D}_{U^{(S)}}$-modules $P^\natural_{U^{(S)}} := j^{(S)*}P^\natural_{X^S}$ and $A^\natural_{U^{(S)}}$ (see (3.7.6.1)). The latter is a Lie algebra in the tensor category of left $\mathcal{D}_{U^{(S)}}$-modules which acts naturally on $j^{(S)*} \boxtimes M_s$. Consider the $\mathcal{D}_{U^{(S)}}$-modules of coinvariants $(j^{(S)*} \boxtimes M_s)_{A^\natural_{U^{(S)}}}$ and $(j^{(S)*} \boxtimes M_s)_{P^\natural_{U^{(S)}}} := \mathrm{Coker}(P^\natural_{U^{(S)}} \otimes j^{(S)*} \boxtimes M_s \to j^{(S)*} \boxtimes M_s)$. The relative version of 4.4.4 and 4.4.5 says that (see (4.2.19.7) for the notation)

$$H^0\mathcal{C}^{ch}(X, A, \{M_s\}) = j^{(S)}_*(j^{(S)*} \boxtimes M_s)_{A^\natural_{U^{(S)}}} = j^{(S)}_*(j^{(S)*} \boxtimes M_s)_{P^\natural_{U^{(S)}}}. \tag{4.4.6.1}$$

Example. Let A be the twisted enveloping algebra $U(\mathfrak{g}_{\mathcal{D}})^\kappa$ of the Kac-Moody extension $\mathfrak{g}^\kappa_{\mathcal{D}}$; see 2.5.9. Then one can take $P = \mathfrak{g}^\kappa_{\mathcal{D}}$, and the $\mathcal{D}_{U^{(S)}}$-module from (4.4.6.1) is called the *Knizhnik-Zamolodchikov (KZ) equation.*

4.4.7. Let A be a chiral algebra. Then for any $\mathcal{D}_X$-module M the tensor product $M \otimes A$ is naturally a chiral A-module (see Remark (i) in 3.3.4).

Suppose we have a finite non-empty collection $\{M_s\}$, $s \in S$, of $\mathcal{D}_X$-modules such that the M_s are Tor-independent from A (i.e., the supports of the $\mathcal{O}_X$-torsion of M_s are disjoint from that of A).

[49] See also Remark in 4.4.4 and (4.4.1.2).

[50] To see this, consider a filtration $A^\ell_{x_{s_0}n} := h(U_S, P)^n \cdot 1_{s_0}$ on $A^\ell_{x_{s_0}}$. Then the images of $A^\ell_{x_{s_0}n} \otimes (\otimes A^\ell_{x_t})$ form a constant filtration in $(\otimes A^\ell_{x_s})_{h(U_S,P)}$.

Proposition. *There is a canonical quasi-isomorphism*

$$C^{ch}(X,A)\otimes j_*^{(S)}j^{(S)*}\boxtimes M_s \xrightarrow{\sim} \mathcal{C}^{ch}(X,A,\{M_s\otimes A\}). \tag{4.4.7.1}$$

Proof. Notice that the image of the morphism $M\to M\otimes A$, $m\mapsto m\otimes 1_A$, consists of A-central sections (see 3.3.7). Therefore, by (a) in 4.3.2(iii), we have the product morphism $\cdot : C^{ch}(X,A)\otimes j_*^{(S)}j^{(S)*}\boxtimes M_s\to \mathcal{C}^{ch}(X,A,\{M_s\otimes A\})$.

To check that it is a quasi-isomorphism, we use the relative version of 4.4.3. When $(x_s)\in U^{(S)}$ varies, the complexes $C^{ch}(X,A,\{\tilde{A}_s\})$ form a complex C of left $\mathcal{D}$-modules on $U^{(S)}$, and the quasi-isomorphism (4.4.3.1) (coming from (4.4.3.4)) identifies it with the constant $\mathcal{D}$-module $C^{ch}(X,A)\otimes \mathcal{O}_{U^{(S)}}$. Tensoring our $\mathcal{D}$-modules by $(\boxtimes M_s)|_{U^{(S)}}$, we get $\alpha : j_*^{(S)}C\otimes(\boxtimes M_s)\xrightarrow{\sim} C^{ch}(X,A)\otimes j_*^{(S)}j^{(S)*}\boxtimes M_s$. On the other hand, the projections $\tilde{A}_s \xrightarrow{\sim} i_{s*}A^{\ell}_{x_s}$ yield $\beta : j_*^{(S)}C\otimes(\boxtimes M_s)\xrightarrow{\sim} \mathcal{C}^{ch}(X,A,\{M_s\otimes A\})$. One checks immediately that $\cdot=\beta\alpha^{-1}$, and we are done. □

For $A=\omega$ the functor $\mathcal{M}(X)\to\mathcal{M}(X,\omega)$, $M\mapsto M\otimes\omega$, is an equivalence of categories (see Example in 3.3.4). Combining the proposition with 4.3.3(i), we get

Corollary. *For any finite collection $\{M_s\}$, $s\in S$, of $\mathcal{D}_X$-modules one has*

$$j_*^{(S)}j^{(S)*}\boxtimes M_s \xrightarrow{\sim} \mathcal{C}^{ch}(X,\omega,\{M_s\}). \tag{4.4.7.2}$$

4.4.8. Proposition. *Let A be a non-unital chiral algebra, $A^+ := A\oplus\omega$ the corresponding unital algebra (see 3.3.3). Then the embeddings $A,\omega\hookrightarrow A^+$ yield a quasi-isomorphism*

$$C^{ch}(X,A)\oplus k \xrightarrow{\sim} C^{ch}(X,A^+). \tag{4.4.8.1}$$

Proof. Notice that $C^{ch}(X,A^+)_{\mathcal{PQ}} = C^{ch}(X,A)^+_{\mathcal{PQ}}\oplus C^{ch}(X,\omega)_{\mathcal{PQ}}$ where the subcomplex $C^{ch}(X,A)^+_{\mathcal{PQ}}$ is generated by all chains $f\boxtimes a_i\in\Gamma(U^{(I)},(A^+_{\mathcal{PQ}})^{\boxtimes I})$, $f\in\mathcal{O}_{U^{(I)}}$, $a_i\in A^+_{\mathcal{PQ}}$ such that at least one of the a_i's belongs to $A\subset A^+$. By 4.3.3(i) it suffices to show that $C^{ch}(X,A)_{\mathcal{PQ}}\hookrightarrow C^{ch}(X,A)^+_{\mathcal{PQ}}$ is a quasi-isomorphism.

Both complexes are filtered: the first one by the Cousin filtration and the second one by the number of a_i's from A as above. The embedding is compatible with filtrations. It is a filtered quasi-isomorphism: indeed, $\mathrm{gr}_n C^{ch}(X,A)_{\mathcal{PQ}} = \Gamma(U^{(n)},h((A_{\mathcal{PQ}}[1])^{\boxtimes n}))_{\Sigma_n} \xrightarrow{\sim} R\Gamma_{DR}(U^{(n)},A^{\boxtimes n})_{\Sigma_n}$, and also $\mathrm{gr}_n C^{ch}(X,A)^+_{\mathcal{PQ}} = C^{ch}(X,\omega,\{A\}_{n\,\text{copies}})_{\Sigma_n}$, so we are done by (4.4.7.2). □

Remark. Suppose that A is equipped with a commutative filtration; it extends in the obvious manner to A^+. We have $C^{ch}(X,A_\cdot)\in\mathcal{H}o\mathcal{BV}$ and $C^{ch}(X,A^+_\cdot)\in\mathcal{H}o\mathcal{BV}_u$ (see 4.3.2(iv) and 4.3.4). The above lemma (together with 4.1.7 and 4.1.15) shows that $C^{ch}(X,A^+_\cdot)$ comes from $C^{ch}(X,A_\cdot)$ by adding the unit. The same is true in the setting of commutative chiral algebras.

4.4.9. We return to the unital setting.

For a chiral subalgebra $B\subset A$ one defines the *relative Chevalley-Cousin* complex $C(B,A)_{\mathcal{PQ}}\in C\mathcal{M}(\mathcal{R}(X))$ as follows. Consider A as a B-module. For each $I\in\mathcal{S}$ we have a complex of $\mathcal{D}_{X^I}$-modules $\mathcal{C}^{ch}(X,B,A[1]_I)_{\mathcal{PQ}}$ (see 4.2.19). As a mere graded $\mathcal{D}_{X^I}$-module, our complex $C(B,A)_{\mathcal{PQ}\,X^I}$ equals

$$C(B,A)^{\cdot}_{\mathcal{PQ}\,X^I} := \bigoplus_{T\in Q(I)} \Delta_*^{(I/T)}\mathcal{C}^{ch}(X,B,A[1]_T)_{\mathcal{PQ}}. \tag{4.4.9.1}$$

Its differential is the sum of the differentials of $\Delta_*^{(I/T)}\mathcal{C}^{ch}(X,B,A[1]_T)_{\mathcal{PQ}}$ and the morphisms $\Delta_*^{(I/T)}\mathcal{C}^{ch}(X,B,A[1]_T)_{\mathcal{PQ}}[-1] \to \Delta_*^{(I/S)}\mathcal{C}^{ch}(X,B,A[1]_S)_{\mathcal{PQ}}$ for $S \in Q(T,|T|-1)$ coming from the binary chiral operation μ_A (cf. 3.4.11). For any $J \twoheadrightarrow I$ one has an evident embedding $\Delta_*^{(J/I)}C(B,A)_{\mathcal{PQ}\,X^I} \hookrightarrow C(B,A)_{\mathcal{PQ}\,X^J}$. We have defined the right $\mathcal{D}$-complex $C(B,A)_{\mathcal{PQ}}$ on $X^{\mathcal{S}}$; it is clearly admissible.

Set $C^{ch}(X,B,A)_{\mathcal{PQ}} := \Gamma(X^{\mathcal{S}}, h(C(B,A)_{\mathcal{PQ}}))$. As a mere graded module, our complex equals $\sum\limits_{m\geq 0, n>0} \Gamma(U^{(m+n)}, h((B_{\mathcal{PQ}}[1])^{\boxtimes m}\boxtimes(A_{\mathcal{PQ}}[1])^{\boxtimes n}))$. As in the proof of the proposition in 4.2.12 we see that $C^{ch}(X,B,A)_{\mathcal{PQ}} \xrightarrow{\sim} R\Gamma_{DR}(\mathcal{R}(X), C(B,A)_{\mathcal{PQ}})$. We have the Cousin spectral sequence (see 4.2.3)

$$E^1_{p,q} = H^{ch}_{p+q}(X,B,A[1]_{[1,p]})_{\Sigma_p} \tag{4.4.9.2}$$

converging to $H^{-p-q}C^{ch}(X,B,A)_{\mathcal{PQ}}$.

Consider an evident embedding of complexes $C^{ch}(X,A)_{\mathcal{PQ}} \hookrightarrow C^{ch}(X,B.A)_{\mathcal{PQ}}$. It has a left inverse $C^{ch}(X,B,A)_{\mathcal{PQ}} \twoheadrightarrow C^{ch}(X,A)_{\mathcal{PQ}}$ formed by the morphisms $\Gamma(U^{(m+n)}, h((B_{\mathcal{PQ}}[1])^{\boxtimes m}\boxtimes(A_{\mathcal{PQ}}[1])^{\boxtimes n})) \to \Gamma(U^{(m+n)}, h((A_{\mathcal{PQ}}[1])^{\boxtimes m+n}))$.

Proposition. *The embedding $C^{ch}(X,A)_{\mathcal{PQ}} \hookrightarrow C^{ch}(X,B.A)_{\mathcal{PQ}}$ is a quasi-isomorphism. Therefore the spectral sequence (4.4.9.2) converges to $H^{ch}_{p+q}(X,A)$.*

Proof. The complex $C = C^{ch}(X,B,A)_{\mathcal{PQ}}$ carries an increasing filtration C_m that corresponds to the first grading (by m), and $C_0 = C^{ch}(X,A)_{\mathcal{PQ}}$. It suffices to check that $\mathrm{gr}_m C$ is acyclic for $m>0$.

Consider the embedding $\tilde{j}: U^{(1+m)} \hookrightarrow X\times U^{(m)}$. Set $A^{(m)} := \tilde{j}_*\tilde{j}^* A\boxtimes \mathcal{O}_{U^{(m)}}$; this is a $U^{(m)}$-family of chiral algebras on X. Let $\mathcal{C}^{(m)}$ be the relative version of the chiral cochain complex for $A^{(m)}$; this is a complex of $\mathcal{O}_{U^{(m)}}$-modules. It is acyclic by the relative version of the lemma from 4.3.4. Now $\mathcal{C}^{(m)}$ is a left $\mathcal{D}_{U^{(m)}}$-module (since $A^{(m)}$ is), and $\mathrm{gr}_m C = R\Gamma_{DR}(U^{(m)}, B^{\boxtimes m}\otimes \mathcal{C}^{(m)}) = 0$; q.e.d. □

4.5. Rigidity and flat projective connections

Suppose one has a Z-family of curves $X=\{X_z\}$ equipped with (not necessary unital) chiral algebras A_z. The fiberwise chiral homologies $H^{ch}_{\cdot}(X_z,A_z)$ form quasi-coherent sheaves on Z. In this section we discuss an X-local structure on A which provides a flat projective connection ∇ on the chiral homology sheaves.

An example of such a structure is an extension of the $\mathcal{D}_{X/Z}$-action on A to a $\mathcal{D}_X$-action compatible with the chiral product: then the $C(A_z)$ form complexes of $\mathcal{D}$-modules on the fibration of Ran's spaces $\mathcal{R}(X_z)$, and ∇ is the corresponding Gauss-Manin connection. Such a simple picture occurs quite rarely though. Usually the action of "horizontal" vector fields on A is well defined only up to the adjoint action of A^{Lie}; i.e., more precisely, A carries an action of an extension of the Lie algebra of horizontal vector fields by the (relative) de Rham complex of A^{Lie} (which is naturally a homotopy Lie algebra). This suffices for the definition of ∇ due to a key rigidity property of chiral homology: the action of the homotopy Lie algebra $R\Gamma_{DR}(X_z, A_z^{Lie})$ on $C^{ch}(X_z,A_z)$ is canonically homotopically trivialized. A weaker structure, when A^{Lie} is replaced by $A^{Lie}/\omega_{X/Z}1_A$, leads to a projective connection.

In practice, one always considers instead of the whole of A^{Lie} its smaller Lie* subalgebra determined by the geometry of the situation. For example, suppose that our family of chiral algebras comes from a universal setting (i.e., from a vertex algebra). Then a flat projective on the chiral homology is produced by a Virasoro vector (see 3.7.25). For a family of Kac-Moody algebras, the Sugawara tensor, and the 0th chiral homology, we get the Knizhnik-Zamolodchikov connection. Another example: suppose we have a chiral algebra A on a curve X equipped with an action of the group of G-valued functions, G is an algebraic group. We get a family of twisted chiral algebras parametrized by the moduli space Bun_G of G-bundles on X (see 3.4.17). Then any Kac-Moody tensor in A provides a flat projective connection on the corresponding sheaves of chiral homology on Bun_G. Connections of this type on the 0th chiral homology are treated in Chapters 16–17 of [**FBZ**].

In 4.5.1 we explain why the de Rham complex of a Lie* algebra is a homotopy Lie algebra. In 4.5.2 the above-mentioned rigidity property of chiral homology is established; the key tool here is the BV algebra structure on the chiral chain complex. Some variants of the rigidity property, in the format needed for the construction of the connection, are discussed in 4.5.3. The input package for the construction of the connection on chiral homology is defined in 4.5.4; the corresponding connection is constructed in 4.5.5. The twisted seting, leading to a projective connection, is discussed in 4.5.6. Section 4.5.7 contains a streamlined construction of the $\mathcal{O}$-extension of the Lie algebra of vector fields on Z acting on chiral homology, and also compatibility with tensor products. Section 4.5.8 considers the case of chiral homology with coefficients, 4.5.9 compares the two settings, 4.5.10 gives a different construction of the connection in the case when the coefficient sheaves are supported at points (for a convenient explicit formula, see 4.5.12), and 4.5.11 compares the two constructions. In 4.5.13 we discuss the above-mentioned examples in more detail.

As always, we deal with differential graded super objects, so "Lie algebra" means "DG super Lie algebra", etc.

4.5.1. Let L be a Lie* algebra on X. Then the de Rham complex $DR(L)$ is naturally a homotopy Lie algebra. This means that there is a canonical object in the homotopy category of sheaves of Lie algebras identified with $DR(L)$ in the derived category of sheaves. Similarly, $R\Gamma_{DR}(X, L)$ is naturally a homotopy Lie algebra; i.e., there is a canonical object in the homotopy category of Lie algebras identified with $R\Gamma_{DR}(X, L)$ as a mere object of $D(k)$. We denote these homotopy Lie algebras by $DR(L)$, $R\Gamma_{DR}(X, L)$ by abuse of notation. One constructs them as follows.

Take $\mathcal{P}$, $\mathcal{Q}$ as in 4.2.12 and write $L_{\mathcal{P}} := L \otimes \mathcal{P}$, $L_{\mathcal{P}\mathcal{Q}} := L \otimes \mathcal{P} \otimes \mathcal{Q}$, etc. We have the quasi-isomorphisms of Lie* algebras $L \leftarrow L_{\mathcal{P}} \rightarrow L_{\mathcal{P}\mathcal{Q}}$ which yield quasi-isomorphisms of Lie algebras $h(L_{\mathcal{P}}) \rightarrow h(L_{\mathcal{P}\mathcal{Q}})$. They are canonically identified with $DR(L)$ in $DSh(X)$ (see 2.2.10), so we have defined the homotopy Lie algebra structure on $DR(L)$. The Lie algebra $\Gamma(X, h(L_{\mathcal{P}\mathcal{Q}}))$ is identified with $R\Gamma_{DR}(X, L)$ in $D(k)$;[51] it provides the homotopy Lie algebra structure on $R\Gamma_{DR}(X, L)$. The independence of the auxiliary choice of $\mathcal{P}$, $\mathcal{Q}$ follows from the lemma in 2.2.10 (or rather Remark after it) and the second lemma in 4.1.3.

Suppose that L acts on a not necessary unital chiral algebra A (see 3.3.3). Then $L_{\mathcal{P}\mathcal{Q}}$ acts on $A_{\mathcal{P}\mathcal{Q}}$, so the Lie algebra $\Gamma(X, h(L_{\mathcal{P}\mathcal{Q}}))$ acts on $A_{\mathcal{P}\mathcal{Q}}$ by derivations. Therefore $\Gamma(X, h(L_{\mathcal{P}\mathcal{Q}}))$ acts on $C^{ch}(X, A)_{\mathcal{P}\mathcal{Q}}$ by transport of structure. In particular, this action is compatible with the BV structure (see 4.3.1).

We see that $C^{ch}(X, A)$ is naturally an $R\Gamma_{DR}(X, L)$-module.[52]

[51] This follows by an argument from the proof of the proposition in 4.2.12.

[52] The independence of this construction from the auxiliary choice of $\mathcal{P}$, $\mathcal{Q}$ can be seen as above.

4.5.2. Rigidity. Suppose that the L-action on A comes from a morphism of Lie* algebras $\iota : L \to A^{Lie}$ and the adjoint action of A.

LEMMA. *The homotopy action of $R\Gamma_{DR}(X, L)$ on $C^{ch}(X, A)$ is canonically homotopically trivialized.*

Proof. Let us show that the action of $\Gamma(X, h(L_{\mathcal{PQ}}))$ extends naturally to an action on $C^{ch}(X, A)_{\mathcal{PQ}}$ of the contractible Lie algebra $\Gamma(X, h(L_{\mathcal{PQ}}))_\dagger$ (see 1.1.16). This action comes from the BV structure on $C^{ch}(X, A)_{\mathcal{PQ}}$ (see 4.3.1). Indeed, we have morphisms $\Gamma(X, h(L_{\mathcal{PQ}})) \xrightarrow{\iota} \Gamma(X, h(A^{Lie}_{\mathcal{PQ}})) \hookrightarrow C^{ch}(X, A)_{\mathcal{PQ}}[-1]$ of Lie algebras, so $\Gamma(X, h(L_{\mathcal{PQ}}))_\dagger$ acts via the canonical action of $C^{ch}(X, A)_{\mathcal{PQ}}[-1]_\dagger$ defined by the BV structure (see 4.1.6). $\square$

4.5.3. VARIANTS. *The material of this section will not be used until 4.5.5; the reader can skip it at the moment.*

Below, our L is a Lie* algebra which is homotopically quasi-induced as a $\mathcal{D}_X$-complex (see 2.1.11), so the natural projections $DR(L_{\mathcal{Q}}) \to h(L_{\mathcal{Q}})$, $\Gamma_{DR}(X, L_{\mathcal{Q}}) \to \Gamma(X, h(L_{\mathcal{Q}}))$ are quasi-isomorphisms (see the lemma in 4.1.4). The corresponding projection $\Gamma(X, h(L_{\mathcal{PQ}})) \to \Gamma(X, h(L_{\mathcal{Q}}))$ is a quasi-isomorphism of Lie algebras.

If L acts on a (not necessary unital) chiral algebra A, then $L_{\mathcal{Q}}$ acts on $A_{\mathcal{Q}}$ and $A_{\mathcal{PQ}}$ and so $\Gamma(X, h(L_{\mathcal{Q}}))$ acts on $C^{ch}(X, A)_{\mathcal{PQ}}$.

(i) Suppose that we are in the situation of 4.5.2. Let us show that the action of $\Gamma(X, h(L_{\mathcal{Q}}))$ on $C^{ch}(X, A)_{\mathcal{PQ}}$ is canonically homotopically trivialized.

One proceeds by providing a homotopy between this action and the one considered in the proof in 4.5.2[53] and then applying 4.5.2; for technical reasons we have to pass to a larger chiral chain complex $C^{ch}(X, A)_{\tilde{\mathcal{P}}\mathcal{Q}}$. Here is a precise construction.

Let $\mathcal{P}^+$ be the unital $\mathcal{D}_X$-algebra corresponding to $\mathcal{P}$, so $\mathcal{P} = \mathcal{P} \oplus \mathcal{O}_X$ as an $\mathcal{D}_X$-module, $\epsilon^+_{\mathcal{P}} : \mathcal{P}^+ \to \mathcal{O}_X$ the morphism of unital algebras defined by $\epsilon_{\mathcal{P}}$. Set $\mathcal{I} := \mathrm{Ker}\epsilon_{\mathcal{P}}$, $\mathcal{I}^+ := \mathrm{Ker}\epsilon^+_{\mathcal{P}}$. Then $\mathcal{P}$, $\mathcal{I}$, $\mathcal{I}^+$ are ideals in $\mathcal{P}^+$; one has $\mathcal{I}^+ \cdot \mathcal{P} \subset \mathcal{I}$, and $\mathcal{I}$ is contractible and $\mathcal{D}_X$-flat.[54]

Set $\tilde{\mathcal{P}} := \mathcal{C}one(\mathcal{I} \to \mathcal{P})$, $\tilde{\mathcal{P}}^+ := \mathcal{C}one(\mathcal{I}^+ \to \mathcal{P}^+)$. Then $\tilde{\mathcal{P}}^+$ is a unital $\mathcal{D}_X$-algebra, so that $\mathcal{P}^+ \hookrightarrow \tilde{\mathcal{P}}^+$ is an embedding of algebras, and $\tilde{\mathcal{P}}$ is an ideal in $\tilde{\mathcal{P}}^+$. Our $\epsilon^+_{\mathcal{P}}$ extends to a quasi-isomorphism of $\mathcal{O}_X$-algebras $\tilde{\mathcal{P}}^+ \to \mathcal{O}_X$. Its restriction $\epsilon_{\tilde{\mathcal{P}}} : \tilde{\mathcal{P}} \to \mathcal{O}_X$ is also a quasi-isomorphism, so $(\tilde{\mathcal{P}}, \epsilon_{\tilde{\mathcal{P}}})$ satisfies the same conditions as $(\mathcal{P}, \epsilon_{\mathcal{P}})$. Set $\tilde{\mathcal{P}}^+_\ddagger := \mathcal{C}one(\tilde{\mathcal{P}} \to \tilde{\mathcal{P}}^+)$; this is a contractible $\mathcal{D}_X$-algebra.

Consider the chiral algebra $A_{\tilde{\mathcal{P}}\mathcal{Q}} := A \otimes \tilde{\mathcal{P}} \otimes \mathcal{Q}$. The Lie* algebra $L_{\tilde{\mathcal{P}}+\mathcal{Q}} := L \otimes \tilde{\mathcal{P}}^+ \otimes \mathcal{Q}$ acts on it. The action of the normal Lie* subalgebra $L_{\tilde{\mathcal{P}}\mathcal{Q}} \subset L_{\tilde{\mathcal{P}}+\mathcal{Q}}$ coincides with the adjoint action via $\iota_{\tilde{\mathcal{P}}\mathcal{Q}} : L_{\tilde{\mathcal{P}}\mathcal{Q}} \to A^{Lie}_{\tilde{\mathcal{P}}\mathcal{Q}}$, and $L_{\mathcal{Q}} \subset L_{\tilde{\mathcal{P}}+\mathcal{Q}}$ acts via $\iota_{\mathcal{Q}}$ by the adjoint action of $A_{\mathcal{Q}}$ and the trivial action on $\tilde{\mathcal{P}}$.

Now the contractible Lie algebra $\Gamma(X, h(L_{\tilde{\mathcal{P}}+\mathcal{Q}}))_\ddagger := \Gamma(X, h(L_{\tilde{\mathcal{P}}^+_\ddagger\mathcal{Q}}))$ acts naturally on $C^{ch}(X, A)_{\tilde{\mathcal{P}}\mathcal{Q}}$. Namely, the subalgebra $\Gamma(X, h(L_{\tilde{\mathcal{P}}+\mathcal{Q}})) \subset \Gamma(X, h(L_{\tilde{\mathcal{P}}+\mathcal{Q}}))_\ddagger$ acts via the above action on $A_{\tilde{\mathcal{P}}\mathcal{Q}}$, and the ideal $\Gamma(X, h(L_{\tilde{\mathcal{P}}\mathcal{Q}}))_\dagger \subset \Gamma(X, h(L_{\tilde{\mathcal{P}}+\mathcal{Q}}))_\ddagger$ acts as in the proof in 4.5.2 (with $\mathcal{P}$ replaced by $\tilde{\mathcal{P}}$).

This action provides the homotopical trivialization of the $\Gamma(X, h(L_{\mathcal{Q}}))$-action we promised.

[53]Note that the $L_{\mathcal{Q}}$-action cannot be realized directly as a part of the $L_{\mathcal{PQ}}$-action used in 4.5.2: since $\mathcal{P}$ is non-unital, there is no embedding $L_{\mathcal{Q}} \to L_{\mathcal{PQ}}$, and the projection $L_{\mathcal{PQ}} \to L_{\mathcal{Q}}$ is *not* compatible with the actions.

[54]The Tor-dimension of $\mathcal{D}_X$ equals 1 since $\dim X = 1$.

(ii) Suppose that A is unital and the L-action on A comes from a morphism of Lie* algebras $\bar{\iota} : L \to A^{Lie}/\omega 1_A$ and the adjoint action of $A^{Lie}/\omega 1_A$ on A. Let us show that the $\Gamma(X, h(L_{\mathcal{Q}}))$-action on $C^{ch}(X,A)_{\tilde{\mathcal{P}}\mathcal{Q}}$ is homotopically equivalent to the multiplication by a character. Below $\tilde{\mathcal{P}}$, etc., are as in (i).

Denote by $L^\flat$ an ω-extension of L defined as the pull-back of the ω-extension A^{Lie} of $A^{Lie}/\omega 1_A$ by $\bar{\iota}$. So we have a morphism of Lie* algebras $\iota^\flat : L^\flat \to A^{Lie}$ with $i^\flat 1^\flat = 1_A$. Set $L^\flat_{\tilde{\mathcal{P}}\mathcal{Q}} := L^\flat \otimes \tilde{\mathcal{P}} \otimes \mathcal{Q}$, etc.

As in (i), we have the Lie* algebra $L_{\tilde{\mathcal{P}}+\mathcal{Q}}$ which acts naturally on $A_{\tilde{\mathcal{P}}\mathcal{Q}}$, and a contractible Lie* algebra $L_{\tilde{\mathcal{P}}\underset{\ddagger}{+}\mathcal{Q}}$. Set $L^\diamondsuit_{\tilde{\mathcal{P}}\underset{\ddagger}{+}\mathcal{Q}} := \mathcal{C}one(L^\flat_{\tilde{\mathcal{P}}\mathcal{Q}} \to L_{\tilde{\mathcal{P}}+\mathcal{Q}})$; this is a Lie* algebra[55] which is a central $\omega_{\tilde{\mathcal{P}}\mathcal{Q}}[1]$-extension of $L_{\tilde{\mathcal{P}}\underset{\ddagger}{+}\mathcal{Q}}$.

The Lie algebra $\Gamma(X, h(L^\diamondsuit_{\tilde{\mathcal{P}}\underset{\ddagger}{+}\mathcal{Q}}))$ is a central extension of the contractible Lie algebra $\Gamma(X, h(L_{\tilde{\mathcal{P}}+\mathcal{Q}}))_\ddagger := \Gamma(X, h(L_{\tilde{\mathcal{P}}\underset{\ddagger}{+}\mathcal{Q}}))$ from (i) by $\Gamma(X, h(\omega_{\tilde{\mathcal{P}}\mathcal{Q}}))[1]$. Denote by $\Gamma(X, h(L_{\tilde{\mathcal{P}}+\mathcal{Q}}))^\flat_\ddagger$ its push-out by the embedding $\Gamma(X, h(\omega_{\tilde{\mathcal{P}}\mathcal{Q}}))[1] \hookrightarrow C^{ch}(X,\omega)_{\tilde{\mathcal{P}}\mathcal{Q}}$.

Now the Lie algebra $\Gamma(X, h(L_{\tilde{\mathcal{P}}+\mathcal{Q}}))^\flat_\ddagger$ acts naturally on $C^{ch}(X,A)_{\tilde{\mathcal{P}}\mathcal{Q}}$ in a way that the central subalgebra $C^{ch}(X,\omega)_{\tilde{\mathcal{P}}\mathcal{Q}} \subset \Gamma(X, h(L_{\tilde{\mathcal{P}}+\mathcal{Q}}))^\flat_\ddagger$ acts by homotheties according to the $C^{ch}(X,\omega)_{\tilde{\mathcal{P}}\mathcal{Q}}$-module structure on $C^{ch}(X,A)_{\tilde{\mathcal{P}}\mathcal{Q}}$. This action is determined by the property that $\Gamma(X, h(L_{\tilde{\mathcal{P}}+\mathcal{Q}})) \subset \Gamma(X, h(L_{\tilde{\mathcal{P}}+\mathcal{Q}}))^\flat_\ddagger$ acts according to the $L_{\tilde{\mathcal{P}}+\mathcal{Q}}$-action on $A_{\tilde{\mathcal{P}}\mathcal{Q}}$, and the image of $\Gamma(X, h(L^\flat_{\tilde{\mathcal{P}}\mathcal{Q}}))_\dagger$ in $\Gamma(X, h(L^\diamondsuit_{\tilde{\mathcal{P}}\underset{\ddagger}{+}\mathcal{Q}})) \subset \Gamma(X, h(L_{\tilde{\mathcal{P}}+\mathcal{Q}}))^\flat_\ddagger$ acts as in the proof in 4.5.2 (with L, ι, $\mathcal{P}$ replaced by $L^\flat$, $\iota^\flat$, $\tilde{\mathcal{P}}$).

Our $\Gamma(X, h(L_{\tilde{\mathcal{P}}+\mathcal{Q}}))^\flat_\ddagger$ is homotopically equivalent to k acting on $C^{ch}(X,A)_{\tilde{\mathcal{P}}\mathcal{Q}}$ by homotheties (see 4.3.3). Since the $\Gamma(X, h(L_{\mathcal{Q}}))$-action on $C^{ch}(X,A)_{\tilde{\mathcal{P}}\mathcal{Q}}$ is a part of the $\Gamma(X, h(L_{\tilde{\mathcal{P}}+\mathcal{Q}}))^\flat_\ddagger$-action, it is homotopically multiplication by a character, as was promised.

REMARK. As follows from the above, the Lie algebra $H^0_{DR}(X,L)$ acts on $H^{ch}_{\cdot}(X,A)$ according to the character $H^0_{DR}(X,L) \to H^1_{DR}(X,\omega) \xrightarrow{\mathrm{tr}} k$ where the first arrow is minus the boundary map for the extension $L^\flat$.

4.5.4. Concocting a connection. Let Z be an affine k-scheme. Let $\pi : X \to Z$ be a smooth proper Z-family of curves;[56] for a point $z \in Z$ the corresponding curve is denoted by X_z. The notions we dealt with have an obvious relative version: we consider $\mathcal{D}_{X/Z}$-modules, and it is clear what chiral algebras on X/Z (= Z-families of chiral algebras) are, Lie* algebras on X/Z, etc. A Lie* algebra L on X/Z yields a sheaf $h_{/Z}(L) := L/(L\Theta_{X/Z})$ of Lie $\pi^{-1}\mathcal{O}_Z$-algebras.

So let A be a (not necessary unital) chiral algebra on X/Z which we assume to be $\mathcal{O}_Z$-flat. It defines a complex of quasi-coherent $\mathcal{O}_Z$-modules $\pi^{ch}_{\cdot}(X/Z,A)$ which is a relative version of the complex $\Gamma^{ch}(X,A)$ from 4.2.11. Replacing A by $A \otimes \mathcal{Q}$, where $\mathcal{Q}$ is a Dolbeault $\mathcal{D}_{X/Z}$-algebra, we get a chiral chain complex $R\pi^{ch}_{\cdot}(X/Z,A)$, which is an object of the derived category $D(Z,\mathcal{O}_Z)$ of quasi-coherent $\mathcal{O}_Z$-modules ($=\Gamma(Z,\mathcal{O}_Z)$-modules). Also choosing $\mathcal{P}$ as in 4.2.12, we can represent $R\pi^{ch}_{\cdot}(X/Z,A)$ by a complex $C^{ch}(X/Z,A)_{\mathcal{P}\mathcal{Q}}$ (see (4.2.12.2)).

We are going to describe a certain structure of X-local origin which yields an integrable connection on $R\pi^{ch}_{\cdot}(X/Z,A)$. In fact, we consider a slightly more general

[55] Since $L_{\tilde{\mathcal{P}}+\mathcal{Q}}$ acts on $L^\flat_{\tilde{\mathcal{P}}\mathcal{Q}}$ and the arrow is compatible with the $L_{\tilde{\mathcal{P}}+\mathcal{Q}}$-actions.

[56] We are sorry for the abuse of notation: before X meant an individual curve.

situation starting with a given Lie algebroid $\mathcal{L}$ on Z (see 2.9.1); the output is a homotopy $\mathcal{L}$-action on $R\pi_{\cdot}^{ch}(X/Z, A)$. The case of the connection corresponds to $\mathcal{L} = \Theta_Z$ (for a smooth Z).

So let $\mathcal{L}$ be a Lie algebroid on Z. It yields a Lie $\pi^{-1}\mathcal{O}_X$-algebroid $\pi^\sharp\mathcal{L}$ acting on $\mathcal{O}_X$ which is an extension of $\pi^{-1}\mathcal{L}$ by $\Theta_{X/Z}$ (see 2.9.5).[57]

Suppose we have the following package:

(a) a Lie* algebra L on X/Z,

(b) a Lie $\pi^{-1}\mathcal{O}_Z$-algebroid extension $\mathcal{K}$ of $\pi^\sharp\mathcal{L}$ by $h_{/Z}(L)$ and a section $\mathfrak{s}$: $\Theta_{X/Z} \to \mathcal{K}$,

(c) an action of $\mathcal{K}$ on L,

(d) an action of $\mathcal{K}$ on A and a morphism of Lie* algebras $\iota : L \to A^{Lie}$.

The following properties should hold:

(i) The $\mathcal{K}$-action on L and A is compatible with the $\mathcal{D}_{X/Z}$-module structure on them,[58] the Lie* bracket on L, and the chiral product on A. It is $\pi^{-1}\mathcal{O}_Z$-linear with respect to the $\mathcal{K}$-variable.[59] The morphism $\iota : L \to A$ commutes with the $\mathcal{K}$-actions.

(ii) $\mathfrak{s}(\Theta_{X/Z}) \subset \mathcal{K}$ is a normal Lie $\pi^{-1}\mathcal{O}_Z$-subalgebra.

Therefore $\mathcal{K}$ is an extension of $\pi^{-1}\mathcal{L}$ by $\mathcal{K}_0 := h_{/Z}(L) \times \Theta_{X/Z}$.

(iii) The Lie subalgebra $\Theta_{X/Z} \subset \mathcal{K}_0$ acts on A and L via the $\mathcal{D}_{X/Z}$-module structures, and $h_{/Z}(L) \subset \mathcal{K}_0$ acts on L via the adjoint action and on A via ι and the adjoint action. The adjoint action of $\mathcal{K}$ on $h_{/Z}(L) \subset \mathcal{K}_0$ coincides with the $\mathcal{K}$-action on $h_{/Z}(L)$ coming from the $\mathcal{K}$-action on L.

We call such a package an *$\mathcal{L}$-action on A governed by* $\iota : L \to A^{Lie}$.

4.5.5. PROPOSITION. *Suppose that as a mere $\mathcal{D}_{X/Z}$-complex our L is homotopically quasi-induced. Then our package yields a homotopy left $\mathcal{L}$-module structure on the complex $R\pi_{\cdot}^{ch}(X/Z, A)$. In particular, if $\mathcal{L}$ is a plain Lie $\mathcal{O}_Z$-algebroid, then the $R^i\pi_{\cdot}^{ch}(X/Z, A)$ are left $\mathcal{L}$-modules.*

Proof. We get the homotopy $\mathcal{L}$-action from the obvious $R\pi.\mathcal{K}$-action trivializing homotopically the action of $R\pi.\mathcal{K}_0$ by means of (i) in 4.5.3. Here is the precise construction.

(i) We use C^{ch} chiral chain complexes, so one has to make some auxiliary choices of resolutions:

(a) Notice that our datum is contravariantly functorial with respect to morphisms of $\mathcal{L}$. Replacing $\mathcal{L}$ by its appropriate left resolution, we can assume that $\mathcal{L}$ is homotopically flat (or even semi-free) as a complex of $\mathcal{O}_Z$-modules.

(b) Choose a Dolbeault $\mathcal{O}_X$-algebra $\mathcal{Q}$ equipped with a left $\pi^\dagger\mathcal{L}$-action.

To construct $\mathcal{Q}$ one can essentially repeat the construction from the proof of the first lemma in 4.1.3. Namely, we pick a Jouanolou map $p : Y \to X$ which yields a Dolbeault $\mathcal{O}_X$-algebra $\mathcal{P} := p.\Omega_{Y/X}$. Let $\mathcal{Q}$ be a DG $\mathcal{O}_X$-algebra equipped with a left $\pi^\dagger\mathcal{L}$-action (see 2.9.5) and a morphism of $\mathcal{O}_X$-algebras $\mathcal{P} \to \mathcal{Q}$ which is universal with respect to this structure. Our $\mathcal{Q}$ is a Dolbeault $\mathcal{O}_X$-algebra.[60]

[57] We apologize for the discrepancy of notation: the smooth map $\pi : Y \to X$ from 2.9.5 is now $\pi : X \to Z$.

[58] Our $\mathcal{K}$ acts on $\mathcal{O}_X$ (via $\mathcal{K} \to \pi^\dagger\mathcal{L}$) preserving $\pi^{-1}\mathcal{O}_Z \subset \mathcal{O}_X$; hence it acts on $\mathcal{D}_{X/Z}$.

[59] I.e., A and L are left $\mathcal{K}$-modules with respect to the Lie $\pi^{-1}\mathcal{O}_Z$-algebroid structure on $\mathcal{K}$.

[60] Let us check that $\mathcal{Q}$ is a homotopically flat resolution of $\mathcal{O}_X$. Since $\pi^\dagger\mathcal{L}$ is $\mathcal{O}_X$-flat, the enveloping algebra $U(\pi^\dagger\mathcal{L})$ is also homotopically $\mathcal{O}_X$-flat (see 2.9.2). Locally on X our $\mathcal{P}$ is

(c) Choose a non-unital $\mathcal{O}_X$-algebra resolution $\epsilon_{\mathcal{P}} : \mathcal{P} \to \mathcal{O}_X$ equipped with a left $\pi^\dagger\mathcal{L}$-action such that $\mathcal{P}^{>0} = 0$ and each $\mathcal{P}^a$ is $\mathcal{D}_{X/Z}$-flat. Here we consider $\mathcal{P}$ as a $\mathcal{D}_{X/Z}$-module via the standard embedding $\Theta_{X/Z} \hookrightarrow \pi^\dagger\mathcal{L}$.

One can construct $\mathcal{P}$ by an obvious modification of Example in 4.2.12. Set $\mathcal{P}^0 := \mathrm{Sym}^{>0}U(\pi^\dagger\mathcal{L})$, where $U(\pi^\dagger\mathcal{L})$ is considered as a left $\pi^\dagger\mathcal{L}$-module. Let $\epsilon_{\mathcal{P}} : \mathcal{P}^0 \to \mathcal{O}_X$ be the morphism of algebras with $\pi^\dagger\mathcal{L}$-action defined by the morphism of left $\pi^\dagger\mathcal{L}$-modules $U(\pi^\dagger\mathcal{L}) \to \mathcal{O}_X$, $1_{U(\pi^\dagger\mathcal{L})} \mapsto 1_{\mathcal{O}_X}$. Finally set $\mathcal{P}^{-1} := \mathrm{Ker}\,\epsilon_{\mathcal{P}}$, and $\mathcal{P}^a = 0$ for $a \neq 0, -1$.

(ii) Since $\Theta_{X/Z} \subset \pi^\dagger\mathcal{L}_Z$, our $\mathcal{P}$, $\mathcal{Q}$ are automatically $\mathcal{D}_{X/Z}$-algebras. The Lie $\pi^{-1}\mathcal{O}_Z$-algebroid $\mathcal{K}$ acts on them via $\mathcal{K} \to \pi^\dagger\mathcal{L}$.

Set $\mathcal{K}_{\mathcal{Q}0} := h_{/Z}(L_{\mathcal{Q}}) \times (\mathcal{Q} \otimes \Theta_{X/Z})$; let $\mathcal{K}_{\mathcal{Q}}$ be the push-out of $\mathcal{K}$ by the obvious morphism of Lie $\pi^{-1}\mathcal{O}_Z$-algebras $\mathcal{K}_0 \to \mathcal{K}_{\mathcal{Q}0}$. Since $\mathcal{K}$ acts on the target of this morphism, our $\mathcal{K}_{\mathcal{Q}}$ is a Lie $\pi^{-1}\mathcal{O}_Z$-algebroid extension of $\pi^{-1}\mathcal{L}$ by $\mathcal{K}_{\mathcal{Q}0}$. Set $\bar{\mathcal{K}}_{\mathcal{Q}} := \mathcal{K}_{\mathcal{Q}}/\mathcal{Q} \otimes \Theta_{X/Z}$; this is a Lie $\pi^{-1}\mathcal{O}_Z$-algebroid which is an extension of $\pi^{-1}\mathcal{L}$ by $h_{/Z}(L_{\mathcal{Q}})$.

(iii) As in (i) in 4.5.3, our $\mathcal{P}$ yields modified algebras $\tilde{\mathcal{P}}$ (which satisfies the same conditions as $(\mathcal{P}, \epsilon_{\mathcal{P}})$, see (i)(c) above), $\tilde{\mathcal{P}}^+$, and $\tilde{\mathcal{P}}^+_\ddagger$. Each of them is naturally $\pi^\dagger\mathcal{L}$-equivariant. We have the chiral algebra $A_{\tilde{\mathcal{P}}\mathcal{Q}}$ together with the action of the Lie* algebra $L_{\tilde{\mathcal{P}}+\mathcal{Q}}$ on it (see (i) in 4.5.3).

The Lie $\pi^{-1}\mathcal{O}_Z$-algebroid $\mathcal{K}_{\mathcal{Q}}$ acts naturally on $A_{\tilde{\mathcal{P}}\mathcal{Q}}$. The restriction of this action to $h_{/Z}(L_{\mathcal{Q}}) \subset \mathcal{K}_{\mathcal{Q}0}$ coincides with the action via the embedding $L_{\mathcal{Q}} \hookrightarrow L_{\tilde{\mathcal{P}}+\mathcal{Q}}$. Our algebroid acts also on $L_{\tilde{\mathcal{P}}+\mathcal{Q}}$, and the above constructions are compatible with this action.

(iv) Consider the complex of $\mathcal{O}_Z$-modules $C^{ch}(X, A)_{\tilde{\mathcal{P}}\mathcal{Q}}$ which represents the object of the derived category $R\pi^{ch}_\cdot(X/Z, A)$. It carries an action of the $\mathcal{O}_Z$-algebra $\pi.h_{/Z}(L_{\tilde{\mathcal{P}}+\mathcal{Q}})_\ddagger := \pi.h_{/Z}(L_{\tilde{\mathcal{P}}^+_\ddagger\mathcal{Q}})$, defined in (i) in 4.5.3, and the Lie $\mathcal{O}_Z$-algebroid $\pi.\mathcal{K}_{\mathcal{Q}}$. Notice that $\pi.h_{/Z}(L_{\tilde{\mathcal{P}}+\mathcal{Q}})_\ddagger$ is contractible, as follows from the relative version of the lemma in 4.1.4 applied to $L_{\mathcal{Q}}$.[61] The action of $\pi.\mathcal{K}_{\mathcal{Q}}$ vanishes on $\pi.(\mathcal{Q}\otimes\Theta_{X/Z})$, so it factors through the quotient $\pi.\bar{\mathcal{K}}_{\mathcal{Q}}$ which is an extension of $\mathcal{L}$ by $\pi.h_{/Z}(L_{\mathcal{Q}})$.

Let $\tilde{\mathcal{L}}$ be the push-out of $\pi.\bar{\mathcal{K}}_{\mathcal{Q}}$ by the morphism $\pi.h_{/Z}(L_{\mathcal{Q}}) \hookrightarrow \pi.h_{/Z}(L_{\tilde{\mathcal{P}}+\mathcal{Q}})_\ddagger$. This is a Lie $\mathcal{O}_Z$-algebroid[62] which is an extension of $\mathcal{L}$ by a contractible ideal $\pi.h_{/Z}(L_{\tilde{\mathcal{P}}+\mathcal{Q}})_\ddagger$. The above actions form an $\tilde{\mathcal{L}}$-action on $C^{ch}(X, A)_{\tilde{\mathcal{P}}\mathcal{Q}}$.

The homotopy category of left $\tilde{\mathcal{L}}$-modules is naturally equivalent to that of $\mathcal{L}$-modules, so $C^{ch}(X, A)_{\tilde{\mathcal{P}}\mathcal{Q}}$ defines an object of the latter category. Its independence from the auxiliary choices from (i) above is left to the reader. This is the promised homotopy left $\mathcal{L}$-module structure on $R\pi^{ch}_\cdot(X/Z, A)$. □

Remark. The construction of the Lie algebroid $\tilde{\mathcal{L}}$ used only data (a)–(c) of 4.5.4.

4.5.6. A twisted version. In practice, one usually finds a weaker variant of the package from 4.5.4 which leads to a twisted $\mathcal{L}$-action (i.e., an action of a central extension of $\mathcal{L}$) on chiral homology. If $\mathcal{L} = \Theta_Z$, we get a flat projective connection.

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isomorphic to Sym$V$ where V is an acyclic complex of free $\mathcal{O}_X$-modules, hence $\mathcal{Q}$ is isomorphic to $\mathrm{Sym}(U(\pi^\dagger\mathcal{L}) \otimes V)$.

[61]The relative version of the lemma in 4.1.4, together with its proof, remains true because the functor $R\pi.$ has homological dimension 1 (since π is proper of relative dimension 1).

[62]Since $\pi.\mathcal{K}_{\mathcal{Q}0} \to \pi.h_{/Z}(L_{\tilde{\mathcal{P}}+\mathcal{Q}})_\ddagger$ is a $\pi.\mathcal{K}_{\mathcal{Q}}$-equivariant morphism of Lie $\mathcal{O}_Z$-algebras.

Below we assume that our chiral algebra A is unital. So suppose we have $\mathcal{L}$ and data (a)–(c) as in 4.5.4, and a version of (d) that looks as follows:

($\bar{\text{d}}$) an action of $\mathcal{K}$ on A and a morphism of Lie* algebras $\bar{\iota}: L \to A^{Lie}/\omega 1_A$.

We demand properties (i)–(iii) of 4.5.4 with ι replaced with $\bar{\iota}$.

Our $\bar{\iota}$ defines, as in (ii) in 4.5.3, an ω-extension $L^\flat$ of L and a morphism of Lie* algebras $\iota^\flat : L^\flat \to A^{Lie}$ which lifts $\bar{\iota}$ and such that $\iota^\flat 1^\flat = 1_A$. The $\mathcal{K}$-action on L lifts to $L^\flat$ so that $\iota^\flat$ commutes with the $\mathcal{K}$-actions.

The ω-extension $L^\flat$ is important for the reasons explained in the remark after the proposition below.[63] Our package is called a *twisted $\mathcal{L}$-action on A governed by $\iota^\flat : L^\flat \to A^{Lie}$*.

PROPOSITION. *Suppose that L is a homotopically quasi-induced $\mathcal{D}_{X/Z}$-complex. Then the above package yields a homotopy $\mathcal{O}_Z$-extension $\mathcal{L}^\flat$ of $\mathcal{L}$ and a left unital homotopy $\mathcal{L}^\flat$-action on the chiral complex $R\pi^{ch}_{\cdot}(X/Z, A)$.*

REMARK. The extension $\mathcal{L}^\flat$ depends only on data (a)–(c), $L^\flat$, and the $\mathcal{K}$-action on $L^\flat$. It does not depend on A and $\iota^\flat$.

Proof. Let us repeat steps (i)–(iii) of the proof in 4.5.5. So we have the complex $C^{ch}(X/Z, A)_{\tilde{\mathcal{P}}\mathcal{Q}}$ which represents $R\pi^{ch}_{\cdot}(X/Z, A)$. It is a module over the commutative non-unital algebra $C^{ch}(X/Z,\omega)_{\tilde{\mathcal{P}}\mathcal{Q}}$ (where $\omega := \omega_{X/Z}$) which is a homotopy unit $\mathcal{O}_Z$-algebra (see 4.3.4). We will define a $C^{ch}(X/Z,\omega)_{\tilde{\mathcal{P}}\mathcal{Q}}$-extension $\tilde{\mathcal{L}}^\flat$ of the Lie $\mathcal{O}_Z$-algebroid $\tilde{\mathcal{L}}$ from step (iv) of the proof in 4.5.5 and an action of $\tilde{\mathcal{L}}^\flat$ on $C^{ch}(X/Z, A)_{\tilde{\mathcal{P}}\mathcal{Q}}$ such that $C^{ch}(X/Z,\omega)_{\tilde{\mathcal{P}}\mathcal{Q}} \subset \tilde{\mathcal{L}}^\flat$ acts according to the $C^{ch}(X/Z,\omega)_{\tilde{\mathcal{P}}\mathcal{Q}}$-module structure.

Let $\pi.h_{/Z}(L_{\tilde{\mathcal{P}}+\mathcal{Q}})^\flat_\ddagger$ be the relative version of the Lie algebra $\Gamma(X, h(L_{\tilde{\mathcal{P}}+\mathcal{Q}}))^\flat_\ddagger$ defined in (ii) in 4.5.3. This is a Lie $\mathcal{O}_Z$-algebra which is a central extension of the contractible Lie algebra $\pi.h_{/Z}(L_{\tilde{\mathcal{P}}+\mathcal{Q}})_\ddagger$ by $C^{ch}(X/Z,\omega)_{\tilde{\mathcal{P}}\mathcal{Q}}$.

Now $C^{ch}(X/Z, A)_{\tilde{\mathcal{P}}\mathcal{Q}}$ carries a natural action of $\pi.h_{/Z}(L_{\tilde{\mathcal{P}}+\mathcal{Q}})^\flat_\ddagger$ (see (ii) in 4.5.3) and of the Lie $\mathcal{O}_Z$-algebroid $\pi.\bar{\mathcal{K}}_\mathcal{Q}$ (see steps (ii) and (iv) in 4.5.5). The latter algebroid is an extension of $\mathcal{L}$ by $\pi.h_{/Z}(L_\mathcal{Q})$. Let $\tilde{\mathcal{L}}^\flat$ be the push-out of $\pi.\bar{\mathcal{K}}_\mathcal{Q}$ by the morphism $\pi.h_{/Z}(L_\mathcal{Q}) \hookrightarrow \pi.h_{/Z}(L_{\tilde{\mathcal{P}}+\mathcal{Q}})^\flat_\ddagger$ (see (II) in 4.5.3). This is a Lie $\mathcal{O}_Z$-algebroid[64] which is a $C^{ch}(X/Z,\omega)_{\tilde{\mathcal{P}}\mathcal{Q}}$-extension of $\tilde{\mathcal{L}}$. Our actions define the promised action of $\tilde{\mathcal{L}}^\flat$ on $C^{ch}(X/Z, A)_{\tilde{\mathcal{P}}\mathcal{Q}}$. $\square$

REMARK. One can rephrase slightly the definition of $\tilde{\mathcal{L}}^\flat$. Consider the Lie* algebra $L^\diamondsuit_{\tilde{\mathcal{P}}^+_\ddagger\mathcal{Q}}$ from (ii) in 4.5.3. Let $\bar{\mathcal{K}}^\diamondsuit_\ddagger$ be an extension of $\pi^{-1}\mathcal{L}$ by $h_{/Z}(L^\diamondsuit_{\tilde{\mathcal{P}}^+_\ddagger\mathcal{Q}})$ defined as the push-out of $\bar{\mathcal{K}}_\mathcal{Q}$ by the morphism $h_{/Z}(L_\mathcal{Q}) \to h_{/Z}(L^\diamondsuit_{\tilde{\mathcal{P}}^+_\ddagger\mathcal{Q}})$. This is naturally a Lie $\pi^{-1}\mathcal{O}_Z$-algebroid. Then $\pi.\bar{\mathcal{K}}_\mathcal{Q}$ is an extension of $\tilde{\mathcal{L}}$ by $\pi.h_{/Z}(\omega_{\tilde{\mathcal{P}}\mathcal{Q}})[1]$. It is clear that $\tilde{\mathcal{L}}^\flat$ is the push-out of this extension by $\pi.h_{/Z}(\omega_{\tilde{\mathcal{P}}\mathcal{Q}})[1] \hookrightarrow C^{ch}(X/Z, A)_{\tilde{\mathcal{P}}\mathcal{Q}}$.

4.5.7. REMARKS. (i) Suppose we are in the situation of 4.5.6 and $\mathcal{L}$ is a plain Lie algebroid. Then $H^{\neq 0}\tilde{\mathcal{L}}^\flat = 0$, and the definition of $\mathcal{L}^\flat = H^0\tilde{\mathcal{L}}^\flat$ can be streamlined as follows.

Set $\bar{\mathcal{K}} := \mathcal{K}/\Theta_{X/Z}$; this is a Lie $\pi^{-1}\mathcal{O}_Z$-algebroid which is an extension of $\pi^{-1}\mathcal{L}$ by $h_{/Z}(L)$. Since L is a quasi-induced $\mathcal{D}_{X/Z}$-module, $h_{/Z}(L^\flat)$ is a central

[63]The package can be easily reformulated so that $L^\flat$ and $\iota^\flat$ become entry data.

[64]Since $\pi.h_{/Z}(L_\mathcal{Q}) \hookrightarrow \pi.h_{/Z}(L_{\tilde{\mathcal{P}}+\mathcal{Q}})^\flat_\ddagger$ is a $\pi.\bar{\mathcal{K}}_\mathcal{Q}$-equivariant morphism of Lie $\mathcal{O}_Z$-algebras.

extension of $h_{/Z}(L)$ by $h_{/Z}(\omega)$. Set $\mathcal{N} := \mathcal{C}one(h_{/Z}(L^\flat) \to \bar{\mathcal{K}})$; this is a complex with cohomology $H^0\mathcal{N} = \mathcal{L}$, $H^{-1}\mathcal{N} = h_{/Z}(\omega)$. The $\mathcal{K}$-action on $h_{/Z}(L^\flat)$ factors through $\bar{\mathcal{K}}$, so $\mathcal{N}$ is naturally a Lie $\pi^{-1}\mathcal{O}_Z$-algebroid. Thus $R^0\pi_\cdot\mathcal{N}$ is a Lie $\mathcal{O}_Z$-algebroid which is an extension of $\mathcal{L}$ by $R^1\pi_\cdot h_{/Z}(\omega) = \mathcal{O}_Z$.

Now there is a canonical isomorphism of the Lie algebroid $\mathcal{O}_Z$-extensions

$$\mathcal{L}^\flat \xrightarrow{\sim} H^0 R\pi_\cdot\mathcal{N}. \tag{4.5.7.1}$$

To see this, we use the description of $\tilde{\mathcal{L}}^\flat$ from the remark at the end of 4.5.6. Notice that as a morphism in the derived category $\pi_\cdot h_{/Z}(\omega_{\tilde{\mathcal{P}}\mathcal{Q}})[1] \to C^{ch}(X/Z, A)_{\tilde{\mathcal{P}}\mathcal{Q}}$ amounts to the trace morphism $\pi_\cdot h_{/Z}(\omega_{\tilde{\mathcal{P}}\mathcal{Q}})[1] \to \tau_{\geq 0}(\pi_\cdot h_{/Z}(\omega_{\tilde{\mathcal{P}}\mathcal{Q}})[1]) \xrightarrow{\sim} \mathcal{O}_Z$. Therefore $\mathcal{L}^\flat = H^0\pi_\cdot\bar{\mathcal{K}}_\mathcal{Q}$.

Now the standard morphisms $\tilde{\mathcal{P}}^+ \to \mathcal{O}_X$, $\mathcal{O}_X \to \mathcal{Q}$ yield morphisms of Lie $\pi^{-1}\mathcal{O}_Z$-algebroids $\bar{\mathcal{K}}^\diamond_\ddagger \to \mathcal{N}_\mathcal{Q} \leftarrow \mathcal{N}$ where $\mathcal{N}_\mathcal{Q} := \mathcal{C}one(h_{/Z}(L^\flat_\mathcal{Q}) \to \bar{\mathcal{K}}_\mathcal{Q})$. The second arrow is a quasi-isomorphism; the first one induces an isomorphism of cohomology in degrees ≥ -1. Therefore we get $H^0\pi_\cdot\bar{\mathcal{K}}_\mathcal{Q} \xrightarrow{\sim} H^0R\pi_\cdot\mathcal{N}$ which is (4.5.7.1).

(ii) Suppose that for a given $\mathcal{L}$ we have a finite family of chiral algebras A_α together with twisted $\mathcal{L}$-actions on each A_α. Then one has a natural twisted $\mathcal{L}$-action on $\otimes A_\alpha$ such that the corresponding $\mathcal{O}_X$-extension of $\mathcal{L}$ is (homotopically equivalent to) the Baer sum of the corresponding extensions $\mathcal{L}^{\flat_\alpha}$. Namely, we take $L = \Pi L_\alpha$, $\mathcal{K}$ the fibered product of $\mathcal{K}_\alpha$ over $\pi^\natural\mathcal{L}$, and $\iota = \Sigma\iota_\alpha$. The product map (4.3.5.1) is compatible with the (twisted) $\mathcal{L}$-actions.

4.5.8. Suppose we are in the situation of 4.5.6; let us show that the construction of loc. cit. generalizes immediately to the case of chiral homology with coefficients.

Let $\{M_s\}_{s\in S}$ be a finite non-empty family of $\mathcal{O}_Z$-flat chiral A-modules. By 4.2.19, we have the corresponding chiral homology complex $R\pi_\cdot^{ch}(X/Z, A, \{M_s\})$ of $\mathcal{O}_Z$-modules.

Now suppose that the Lie $\pi^{-1}\mathcal{O}_S$-algebroid $\mathcal{K}$ acts on each M_s. We assume that this action is compatible with the $\mathcal{D}_{X/Z}$-module and the chiral A-module structure on M_s, and is $\pi^{-1}\mathcal{O}_S$-linear with respect to $\mathcal{K}$, that the Lie subalgebra $h_{/Z}(L) \subset \mathcal{K}$ acts via $\bar{\iota}$ and the $h_{/Z}(A^{Lie}/1_A\omega)$-action on M_s, and that $\Theta_{X/Z} \subset \mathcal{K}$ acts according to the $\mathcal{D}_{X/Z}$-module structure on M_s.

Such a package yields then a left unital homotopy action of $\mathcal{L}^\flat$ on the complex $R\pi_\cdot^{ch}(X/Z, A, \{M_s\})$. Namely, it defines in the obvious manner a twisted $\mathcal{L}$-action governed by $\iota^\flat$ on the chiral algebra $A^{\{M_s\}}$ (see 4.2.19), so the Lie algebroid $\tilde{\mathcal{L}}^\flat$ from the proof in 4.5.6 acts on $C^{ch}(X/Z, A, \{M_s\})_{\tilde{\mathcal{P}}\mathcal{Q}}$ as on the direct summand of $C^{ch}(X/Z, A^{\{M_s\}})_{\tilde{\mathcal{P}}\mathcal{Q}}$. This action is homotopy unital, i.e., $C^{ch}(X/Z, A, \{M_s\})_{\tilde{\mathcal{P}}\mathcal{Q}}$ is naturally a homotopy unital $C^{ch}(X/Z, \omega)_{\tilde{\mathcal{P}}\mathcal{Q}}$-module, and $C^{ch}(X/Z, \omega)_{\tilde{\mathcal{P}}\mathcal{Q}} \subset \tilde{\mathcal{L}}^\flat$ acts on $C^{ch}(X/Z, A, \{M_s\})_{\tilde{\mathcal{P}}\mathcal{Q}}$ according to this module structure.

4.5.9. Suppose we have a finite non-empty set S and for each $s \in S$ a section $x_s : Z \to X$ whose images for different s do not intersect. As in 4.4.3 we get the A-modules $\tilde{A}_s$. The $\mathcal{K}$-action on A yields one on $\tilde{A}_s$ which evidently satisfies the above compatibilities, so $\mathcal{L}^\flat$ acts on $R\pi_\cdot^{ch}(X/Z, A, \{\tilde{A}_s\})$.

LEMMA. *The identification $R\pi_\cdot^{ch}(X/Z, A, \{\tilde{A}_s\}) \xrightarrow{\sim} R\pi_\cdot^{ch}(X/Z, A)$ of (4.4.3.2) is compatible with the $\mathcal{L}^\flat$-actions.*

Proof. We want to show that the above canonical isomorphism in the derived category of $\mathcal{O}_Z$-modules lifts naturally to an isomorphism between the objects in the derived category of $\mathcal{O}_Z$-modules equipped with homotopically unital left $\tilde{\mathcal{L}}^\flat$-actions.

Recall (see the proof in 4.4.3) that our isomorphism comes from diagram (4.4.3.4) which, in turn, arises from the morphisms of chiral algebras $A^{\{\tilde{A}_s\}} \overset{\tilde{\alpha}}{\leftarrow} A \otimes \operatorname{Sym} P \overset{\tilde{\beta}}{\longrightarrow} A$. The twisted $\mathcal{L}$-action governed by $\iota^\flat$ on A defines one on each of the above chiral algebras; the morphisms $\tilde{\alpha}$, $\tilde{\beta}$ are compatible with this action, and it preserves the $\mathbb{Z}^S$-gradings on $A^{\{\tilde{A}_s\}}$ and $A \otimes \operatorname{Sym} P$. Passing to the corresponding $C^{ch}_{\tilde{\mathcal{P}}\mathcal{Q}}$-complexes, we get quasi-isomorphisms[65] $C^{ch}(X/Z, A, \{\tilde{A}_s\})_{\tilde{\mathcal{P}}\mathcal{Q}} \overset{\alpha}{\leftarrow} \tilde{C}^{ch}(X/Z, A, \{\tilde{A}_s\})_{\tilde{\mathcal{P}}\mathcal{Q}} \overset{\beta}{\to} C^{ch}(X/Z, A)_{\tilde{\mathcal{P}}\mathcal{Q}}$ of left homotopically unital $\tilde{\mathcal{L}}^\flat$-modules, which form the promised lifting. □

4.5.10. Suppose we are in the situation of 4.5.8 and each M_s is supported at the image of a section $x_s : Z \to X$; assume that the images of x_s for different s do not intersect. Then one can compute $R\pi^{ch}_{\cdot}(X/Z, A, \{M_s\})$ by means of an economic complex $C^{ch}(X/Z, A, \{M_s\})_{\tilde{\mathcal{P}}}$ that does not use $\mathcal{Q}$; see (4.2.19.3). Let us describe a variant $\hat{\mathcal{L}}^\flat$ of the extension $\tilde{\mathcal{L}}^\flat$ that acts naturally on $C^{ch}(X/Z, A, \{M_s\})_{\tilde{\mathcal{P}}}$. This construction will be compared with the general one from 4.5.8 in 4.5.11.

Below, $j_S : U_S \hookrightarrow X$ is the complement to $\bigcup x_s(Z)$; $\mathcal{P}$, $\tilde{\mathcal{P}}$, etc., are as in (iii) in 4.5.5 and in 4.5.2.

(i) Consider the Lie $\pi^{-1}\mathcal{O}_Z$-algebroid $\bar{\mathcal{K}} := \mathcal{K}/\mathfrak{s}(\Theta_{X/Z})$ which is an extension of $\pi^{-1}\mathcal{L}$ by $h_{/Z}(L)$, and the Lie* algebra $L^{\diamondsuit}_{\tilde{\mathcal{P}}^+_\ddagger} := \mathcal{C}one(L^\flat_{\tilde{\mathcal{P}}} \to L_{\tilde{\mathcal{P}}+})$ (cf. (ii) in 4.5.3). Let Φ, $\tilde{\Phi}$ be the push-outs of $\bar{\mathcal{K}}$ by, respectively, $h_{/Z}(L) \to h_{/Z}(j_{S*}j_S^* L_{\tilde{\mathcal{P}}+})$ and $h_{/Z}(L) \to h_{/Z}(j_{S*}j_S^* L^{\diamond}_{\tilde{\mathcal{P}}^+_\ddagger})$. Thus $\Phi \subset \tilde{\Phi}$ are naturally Lie $\pi^{-1}\mathcal{O}_Z$-algebroids, so $\pi_\cdot\Phi \subset \pi_\cdot\tilde{\Phi}$ are Lie $\mathcal{O}_Z$-algebroids. We have an embedding $1^\flat : \pi_\cdot h_{/Z}(j_{S*}j_S^*\omega_{\tilde{\mathcal{P}}})[1] \hookrightarrow \pi_\cdot\tilde{\Phi}$ whose cokernel maps quasi-isomorphically onto $\mathcal{L}$.[66] Let $\hat{\mathcal{L}}$ be the push-out of $\pi_\cdot\tilde{\Phi}$ by $\pi_\cdot h_{/Z}(j_{S*}j_S^*\omega_{\tilde{\mathcal{P}}})[1] \hookrightarrow C^{ch}(X/Z, j_{S*}j_S^*\omega)_{\tilde{\mathcal{P}}}$. The latter complex (which is the relative version of $\Gamma(X^{\mathcal{S}}, h(j_{S*}j_S^*\omega_{\tilde{\mathcal{P}}})))$ is acyclic, so $\hat{\mathcal{L}} \to \mathcal{L}$ is a quasi-isomorphism of Lie algebroids.

We will define an $\mathcal{O}_Z$-extension $\hat{\mathcal{L}}^\flat$ of $\hat{\mathcal{L}}$ that acts on $C^{ch}(X/Z, A, \{M_s\})_{\tilde{\mathcal{P}}}$.

(ii) For $s \in S$ set $S_s := S \smallsetminus \{s\}$ and consider a Lie* algebra $j_{S*}j_S^*L^\flat/1^\flat j_{S_s*}j_{S_s}^*\omega$ which is a central $x_{s*}\mathcal{O}_Z$-extension of $j_{S*}j_S^*L$. Applying $h_{/Z}$, we get a central $x_{s\cdot}\mathcal{O}_Z$-extension $h_{/Z}(j_{S*}j_S^*L)^{\flat_s}$ of $h_{/Z}(j_{S*}j_S^*L)$. Pulling this extension back to $h_{/Z}(j_{S*}j_S^*L_{\tilde{\mathcal{P}}+})$, we get a central $x_{s\cdot}\mathcal{O}_Z$-extension $h_{/Z}(j_{S*}j_S^*L_{\tilde{\mathcal{P}}+})^{\flat_s}$ of the latter Lie algebra.

Notice that the morphisms $h_{/Z}(j_{S*}j_S^*L^\flat) \to h_{/Z}(j_{S*}j_S^*L) \leftarrow h_{/Z}(L)$ lift naturally to $h_{/Z}(j_{S*}j_S^*L)^{\flat_s}$. Denote by $h_{/Z}(j_{S*}j_S^*L^{\diamond}_{\tilde{\mathcal{P}}^+_\ddagger})^{\flat_s}$ the cone of the morphism $h_{/Z}(j_{S*}j_S^*L^\flat) \to h_{/Z}(j_{S*}j_S^*L_{\tilde{\mathcal{P}}+})^{\flat_s}$; this is a Lie algebra which is a central $s_\cdot\mathcal{O}_Z$-extension of $h_{/Z}(j_{S*}j_S^*L^{\diamond}_{\tilde{\mathcal{P}}^+_\ddagger})$. Let $\Phi^{\flat_s} \subset \tilde{\Phi}^{\flat_s}$ be the push-outs of $\bar{\mathcal{K}}$ by, respectively, the morphisms $h_{/Z}(L) \to h_{/Z}(j_{S*}j_S^*L_{\tilde{\mathcal{P}}+})^{\flat_s}$, $h_{/Z}(L) \to h_{/Z}(j_{S*}j_S^*L^{\diamond}_{\tilde{\mathcal{P}}^+_\ddagger})^{\flat_s}$. These are Lie $\pi^{-1}\mathcal{O}_Z$-algebroid $s_\cdot\mathcal{O}_Z$-extensions of Φ and $\tilde{\Phi}$.

[65] Here $\tilde{C}^{ch}(X/Z, A, \{\tilde{A}_s\})_{\tilde{\mathcal{P}}\mathcal{Q}}$ is the component of $C^{ch}(X/Z, A \otimes \operatorname{Sym} P)_{\tilde{\mathcal{P}}\mathcal{Q}}$ of degree 1^S.

[66] We use here, as in the proof of 4.5.5, a relative version of the lemma from 4.1.4.

(iii) So for each $s \in S$ we have an $\mathcal{O}_Z$-extension $\pi.\tilde{\Phi}^{\flat_s}$ of the Lie $\mathcal{O}_Z$-algebroid $\pi.\tilde{\Phi}$; denote by $\pi.\tilde{\Phi}^\flat$ the Baer sum of these extensions. Define $\pi.\Phi^\flat \subset \pi.\tilde{\Phi}^\flat$ in a similar way. By construction, $\pi.\tilde{\Phi}^\flat$ contains $\pi.h_{/Z}(j_{S*}j_S^* L^\flat_{\tilde{\mathcal{P}}})[1]$ as a submodule. By the sum of residues formula $\pi.h_{/Z}(j_{S*}j_S^*\omega_{\tilde{\mathcal{P}}})[1] \subset \pi.h_{/Z}(j_{S*}j_S^* L^\flat_{\tilde{\mathcal{P}}})[1]$ is a *subcomplex* in $\pi.\tilde{\Phi}^\flat$. Let $\hat{\mathcal{L}}^\flat$ be the push-out of $\pi.\tilde{\Phi}^\flat$ by the morphism $\pi.h_{/Z}(j_{S*}j_S^*\omega_{\tilde{\mathcal{P}}})[1] \hookrightarrow C^{ch}(X/Z, j_{S*}j_S^*\omega)_{\tilde{\mathcal{P}}}$. Our $\hat{\mathcal{L}}^\flat$ is naturally a Lie $\mathcal{O}_Z$-algebroid which is an $\mathcal{O}_Z$-extension of $\hat{\mathcal{L}}$.

(iv) Let us define the $\hat{\mathcal{L}}^\flat$-action on $C^{ch}(X/Z, A, \{M_s\})_{\tilde{\mathcal{P}}}$. Our $\hat{\mathcal{L}}^\flat$ is the sum of the ideals $C^{ch}(X/Z, j_{S*}j_S^*\omega)_{\tilde{\mathcal{P}}}$, $\pi.h_{/Z}(j_{S*}j_S^* L^\flat_{\tilde{\mathcal{P}}})[1]$, and the subalgebroid $\pi.\Phi^\flat$. We define the action on these submodules separately, leaving it to the reader to check the compatibilities. Take any chain $c = (\otimes m_s) \otimes c_A \in C_n^{ch}(X/Z, A, \{M_s\})_{\tilde{\mathcal{P}}} := (\otimes h_{/Z}(M_s)) \otimes \Gamma(U_S^{(n)}, h_{/Z}((A_{\tilde{\mathcal{P}}}[1])^{\boxtimes n}))_{\Sigma_n}$. For any $\alpha \in C^{ch}(X/Z, j_{S*}j_S^*\omega)_{\tilde{\mathcal{P}}}$ and $\beta \in \pi.h_{/Z}(j_{S*}j_S^* L^\flat_{\tilde{\mathcal{P}}})[1]$ one has $\alpha(c) := c \boxtimes 1_A(\alpha)$, $\beta(c) := c \boxtimes \iota^\flat(\beta)$. A section $\phi^\flat \in (\pi.\Phi)^\flat$ can be represented by a collection $(\phi, \{\ell_s^\flat\})$ where $\phi \in \pi.\Phi$ and $\ell_s^\flat \in \pi.h_{/Z}(j_{S*}j_S^* L^\flat)$ are such that for any $s \in S$ the image of $\phi - \ell_s$ in $\Gamma(U_S, \bar{\mathcal{K}})$ is regular at $x_s(Z)$; i.e., it belongs to $\Gamma(U_{S_s}, \bar{\mathcal{K}})$. Now

$$\phi^\flat(c) := (\otimes m_s) \otimes \phi(c_A) + \sum_{s \in S} (\underset{s' \neq s}{\otimes} m_{s'}) \otimes ((\phi - \ell_s)m_s + \ell_s^\flat m_s) \otimes c_A. \tag{4.5.10.1}$$

4.5.11. Let us show that the construction from 4.5.10 is naturally homotopically equivalent to the general construction from 4.5.8. Consider the quasi-isomorphisms

$$C^{ch}(X, A, \{M_s\})_{\tilde{\mathcal{P}}_{\mathcal{Q}}} \to C^{ch}(X, A_{\mathcal{Q}}, \{M_{s\mathcal{Q}}\})_{\tilde{\mathcal{P}}} \leftarrow C^{ch}(X, A, \{M_s\})_{\tilde{\mathcal{P}}}, \tag{4.5.11.1}$$

comparing the chiral chain complexes we consider (see (4.2.19.5)). The left complex carries the action of the extension $\tilde{\mathcal{L}}^\flat$ from 4.5.8 and 4.5.6; the right one carries the action of $\hat{\mathcal{L}}^\flat$ from 4.5.10. The middle complex carries an action of an extension $\hat{\mathcal{L}}'^\flat$ defined in the same way as $\hat{\mathcal{L}}^\flat$ with L, $L^\flat$ replaced by $L_{\mathcal{Q}} := L \otimes \mathcal{Q}$, $L^\flat_{\mathcal{Q}} := L^\flat \otimes \mathcal{Q}$ and the $h_{/Z}(L)$-extension $\bar{\mathcal{K}}$ of $\pi^{-1}\mathcal{L}$ replaced by its push-out $\bar{\mathcal{K}}_{\mathcal{Q}}$ by $h_{/Z}(L) \to h_{/Z}(L_{\mathcal{Q}})$. Thus $\hat{\mathcal{L}}'^\flat$ is an extension of $\hat{\mathcal{L}}'$ by $\otimes \mathcal{Q}_s$; the details of the construction are left to the reader. There are evident quasi-isomorphisms

$$\tilde{\mathcal{L}}^\flat \to \hat{\mathcal{L}}'^\flat \leftarrow \hat{\mathcal{L}}^\flat \tag{4.5.11.2}$$

of Lie algebroid extensions of $\mathcal{L}$, and (4.5.11.1) is equivariant with respect to these morphisms. This establishes the promised homotopy equivalence.

4.5.12. Suppose we are in the situation of 4.5.10 and $\mathcal{L}$ is a plain Lie algebroid, so $\mathcal{L}^\flat \xrightarrow{\sim} H^0\mathcal{L}^\flat$. Let us extract from 4.5.10 a description of the action of $\mathcal{L}^\flat$ on $C^{ch}(X/Z, A, \{M_s\})_{\tilde{\mathcal{P}}}$ as on a mere object of the derived category of $\mathcal{O}_Z$-modules.

First, as in the end of 4.5.10, for $\gamma \in \mathcal{L}$ its lifting $\gamma^\flat \in \mathcal{L}^\flat$ is given by a collection $(\phi, \{\ell_s^\flat\})$ where ϕ is a section of $\bar{\mathcal{K}}$ over U_S that lifts γ and the $\ell_s^\flat \in h_{/Z}(j_{S*}j_S^* L^\flat)$ are such that the sections $\phi - \ell_s \in \Gamma(U_S, \bar{\mathcal{K}})$ lie in $\Gamma(U_{S_s}, \bar{\mathcal{K}})$, i.e., they are regular at $x_s(Z)$. Now $\gamma^\flat$ acts on the chiral chain complex as the endomorphism given by formula (4.5.10.1).

In particular, suppose that A is a plain chiral algebra and the M_s are plain A-modules. Then $H_0^{ch}(X/Z, A, \{M_s\}) = \langle A \rangle_{/Z} = (\otimes h_{/Z}(M_s))_{h_{/Z}(U_S, A)}$ (see (4.4.4.1)), and $\mathcal{L}^\flat$ acts on it as

$$\gamma^\flat(\otimes m_s) = \sum_{s\in S}(\underset{s'\neq s}{\otimes} m_{s'}) \otimes ((\phi - \ell_s)m_s + \ell_s^\flat m_s). \tag{4.5.12.1}$$

4.5.13. Examples. Below we assume for simplicity that Z is smooth.

(i) Let X be a Z-family of curves as in 4.5.4. Consider the following canonical package (a)–(c) from 4.5.4:

Take $\mathcal{L} = \Theta_Z$, so $\pi^\dagger\mathcal{L} = \Theta_X$ and $\pi^\sharp\mathcal{L}$ is the algebra of vector fields on X preserving π. Let $L = \Theta_{X/Z\mathcal{D}} = \Theta_{X/Z} \otimes \mathcal{D}_{X/Z}$ be the Lie* algebra corresponding to the Lie algebra of vertical vector fields (see Example (i) in 2.5.6(b)), so $h_{/Z}(L) = \Theta_{X/Z}$. Then $\pi^\sharp\mathcal{L}$ acts on L by transport of structure. Let $\mathcal{K}$ be the push-out of $\pi^\sharp\mathcal{L}$ by the diagonal embedding morphism $\Theta_{X/Z} \hookrightarrow \Theta_{X/Z} \times h_{/Z}(L)$, and let $\mathfrak{s} : \Theta_{X/Z} \hookrightarrow \mathcal{K}$ be the embedding of the first multiple. Our $\mathcal{K}$ is a Lie $\pi^{-1}\mathcal{O}_Z$-algebroid in the obvious manner. It acts naturally on L so that $\pi^\sharp\mathcal{L} \subset \mathcal{K}$ acts by transport of structure, $h_{/Z}(L)$ by the adjoint action, and $\Theta_{X/Z} \overset{\mathfrak{s}}{\hookrightarrow} \mathcal{K}$ via the $\mathcal{D}_{X/Z}$-module structure on L. Our datum satisfies the properties (i)–(iii) in 4.5.4.

Now let A be a (not necessary unital) chiral algebra on X equipped with an action of $\pi^\sharp\mathcal{L}$. For example, such is any universal chiral algebra in the sense of 3.3.14. The $\pi^\sharp\mathcal{L}$-action extends then to a $\mathcal{K}$-action so that $\Theta_{X/Z} \overset{\mathfrak{s}}{\hookrightarrow} \mathcal{K}$ acts via the $\mathcal{D}_{X/Z}$-module structure on A. To complete datum (d) of 4.5.4, we need a Virasoro vector, i.e., a morphism of Lie* algebras $\iota : L \to A^{Lie}$. The conditions from 4.5.4 mean that ι commutes with the $\mathcal{L}^\sharp$-actions and $\Theta_{X/Z} \subset \mathcal{L}^\sharp$ acts on A according to the Lie action defined by ι (see (3.7.25.1)); i.e., for any vertical field $\tau \in \Theta_{X/Z}$ its $\mathcal{L}^\sharp$-action on A is $a \mapsto ad_{\iota(\tau)}a - a\cdot\tau$ where the first term is the adjoint action and the second term is the right $\mathcal{D}_{X/Z}$-module action.

According to 4.5.5, such datum provides a flat connection on the chiral homology sheaves. Usually one finds a twisted version of the above situation, when A is unital and ι is replaced by $\bar{\iota} : L \to A^{Lie}/\omega 1_A$. It leads, by 4.5.6, to a flat projective connection. The corresponding $\iota^\flat : L^\flat \to A$ is then a Virasoro vector of a certain central charge c (see 3.7.25).

(ii) Suppose we are in the situation of 3.4.17, so X is a single curve and we have a group $\mathcal{D}_X$-scheme G acting on a chiral algebra A. Assume that G is smooth; let $L = \mathrm{Lie}(G)$ be its Lie* algebra (see (iv) in 2.5.7) which is a vector $\mathcal{D}_X$-bundle. Suppose we have a G-equivariant morphism of Lie* algebras $\iota : L \to A^{Lie}$; here G acts on L by the adjoint action.

Let $P = P_Z$ be a Z-family of $\mathcal{D}_X$-scheme G-torsors on X, which is the same as a G-torsor on $X \times Z$[67] equipped with a relative connection[68] compatible with the G-action. By 3.4.17, our P yields then a Z-family of twisted chiral algebras $A(P)$ on $X \times Z$.

Let us show that our datum defines then a package (a)–(d) from 4.5.4 for $A(P)$ and the Lie $\mathcal{O}_Z$-algebroid $\mathcal{L} = \Theta_Z$. The Lie* algebra from (a) is $L(P)$:= the P-twist of L with respect to the adjoint action. Then $h_{/Z}(L(P))$ identifies in the usual way with the Lie algebra of vertical vector fields on P which commute with the G-action and are compatible with the connection.

[67] I.e., a torsor with respect to the pull-back of G by the projection $X \times Z \to X$.

[68] I.e., a connection along X-directions.

Our family of curves is constant, so $\mathcal{L}^\sharp$ is a semi-direct product of $\pi^{-1}\mathcal{L}$ (acting by vector fields along X-directions) and $\Theta_{X/Z}$. Let $\bar{\mathcal{K}}$ be the sheaf of pairs (τ, τ^P) where $\tau \in \pi^{-1}\mathcal{L} \subset \Theta_{X\times Z}$ and τ^P is its lifting to a vector field on P which commutes with the G-action and is compatible with the connection. This is naturally a Lie $\pi^{-1}\mathcal{O}_Z$-algebroid which is an extension of $\pi^{-1}\mathcal{L}$ by $h_{/Z}(L(P))$.[69] Set $\mathcal{K} := \bar{\mathcal{K}} \ltimes \Theta_{X/Z}$. Then $\mathcal{K}$ acts in an evident manner on $L(P)$, so we have defined data (b) and (c) from 4.5.4. The morphism from (d) is $\iota(P) : L(P) \to A(P)^{Lie}$, the P-twist of ι.

Properties (i)–(iii) of 4.5.4 are immediate, so 4.5.5 defines a flat connection on the chiral cohomology of the family $A(P)$. One finds more often a twisted version of the above situation, when instead of ι one has a G-equivariant morphism of Lie* algebras $\bar{\iota} : L \to A^{Lie}$. Then we get a twisted package from 4.5.6, hence a flat projective connection on the chiral homology.

The most interesting particular case of the above situation occurs when G is the jet scheme of an algebraic group H (see Remark (iv) in 3.4.17) and $L^\flat$ corresponding to $\bar{\iota}$ is a Kac-Moody extension (see 2.5.9). Then $\mathcal{D}_X$-scheme G-torsors are the same as G-torsors on X (see (3.4.17.2)), so the chiral homology of $A(P)$ forms a twisted $\mathcal{D}$-module on the moduli space Bun_H of H-bundles on X.

4.6. The case of commutative $\mathcal{D}_X$-algebras

For a commutative chiral algebra its chiral homology is naturally a homotopy commutative algebra (see 4.3.1 and 4.3.4). So we have a functor from the homotopy category of commutative $\mathcal{D}_X$-algebras to that of commutative DG algebras. The theorem in 4.6.1 says that it is equal to the left derived functor of the functor $R^\ell \mapsto \langle R\rangle$ from 2.4.1. The key point is the computation of the chiral homology of a polynomial $\mathcal{D}_X$-algebra done in 4.6.2. Its relative version is presented in 4.6.4 after some needed preliminaries on semi-free modules (see 4.6.3). The linearized version of 4.6.1 (that deals with the chiral homology of $R^\ell[\mathcal{D}_X]$-modules) is considered in 4.6.5. It implies that the chiral homology commutes with the cotangent complex functor (see 4.6.6). We show also that chiral homology functor preserves perfect complexes (see 4.6.7) and commutes with their duality (see 4.6.8). By the proposition in 4.1.17, these facts imply that for a (very) smooth $\mathcal{D}_X$-algebra its chiral homology algebra is essentially a complete intersection (see 4.6.9).

As in 4.3.7, "$\mathcal{D}_X$-algebra" means "commutative unital DG super $\mathcal{D}_X$-algebra" and "plain $\mathcal{D}_X$-algebra" means "$\mathcal{D}_X$-algebra supported in degree 0".

4.6.1. The chiral homology as a left derived functor. Let F^ℓ be a $\mathcal{D}_X$-algebra. We say that F is *semi-free* if it is $\mathcal{O}_X$-semi-free in the sense of 4.3.7 and is *convenient* if it is semi-free and one can find $F_0 \subset F_1 \subset \cdots \subset F$ as in loc. cit. such that the corresponding V_i have the property $\Gamma(X, h(V_i)) = 0$. We call $F.$ a *convenient filtration* on F and $\{V_i\} \subset F$ *convenient generators*.

By Remark (b) in (the proof of) 4.3.7 every $\mathcal{D}_X$-algebra admits a convenient left resolution.

Consider the morphism $\phi_F : C^{ch}(X, F) \to \langle F\rangle$ from (4.2.17.1); it is naturally a morphism in the homotopy category of commutative algebras (see 4.3.2(i)).

[69] The projection $\bar{\mathcal{K}} \to \pi^{-1}\mathcal{L}$ is surjective since G is smooth as a $\mathcal{D}_X$-scheme.

THEOREM. *If F is convenient, then ϕ_F is a quasi-isomorphism:*

$$\phi_F : C^{ch}(X, F) \xrightarrow{\sim} \langle F \rangle. \tag{4.6.1.1}$$

Therefore on the category of commutative unital chiral algebras, C^{ch} coincides with the left derived functor of $\langle\ \rangle$.

Proof. (a) Since both parts of (4.6.1.1) are compatible with inductive limits, our problem reduces to the following statement:

Let $R \to F$ be an elementary morphism (see 4.3.7) such that the corresponding $V \subset F$ has the property $\Gamma(X, h(V)) = 0$. Suppose that $\phi_R : C^{ch}(X, R) \to \langle R \rangle$ is a quasi-isomorphism. Then $\phi_F : C^{ch}(X, F) \to \langle F \rangle$ is also a quasi-isomorphism.

(b) Let us show that the above statement follows from its particular case when $R^\ell = \mathcal{O}_X$ and the differential of F vanishes.

Recall that as a mere $\mathbb{Z}$-graded $\mathcal{D}_X$-algebra, F equals $R \otimes \operatorname{Sym} V$, and the ring filtration $F_a := R \otimes \operatorname{Sym}^{\leq a} V$ is compatible with the differential. The functor $\langle\ \rangle$ commutes with tensor products, so as a mere $\mathbb{Z}$-graded algebra, $\langle F \rangle$ equals $\langle R \rangle \otimes \langle \operatorname{Sym} V \rangle$. Notice that $\langle \operatorname{Sym} V \rangle = \operatorname{Sym} K$ where $K = H^0_{DR}(X, V[1])$ since, by 2.1.12, the maximal constant quotient of V^ℓ equals $K \otimes \mathcal{O}_X$. The filtration F_a defines a filtration on $C^{ch}(X, F)$ (see 4.2.18); we also have a filtration $\langle R \rangle \otimes \operatorname{Sym}^{\leq a} K$ on $\langle F \rangle$ compatible with the differential. Our morphism is compatible with filtrations, so, by (4.2.18.1), it suffices to check that $\operatorname{gr} \phi_F$ is a quasi-isomorphism. Now $\operatorname{gr} \phi_F = \phi_{\operatorname{gr} F}$, and $\operatorname{gr} F$ equals $R \otimes \operatorname{Sym} V$ as a DG algebra (the differential kills V). By 4.3.6 we are reduced to the situation of $R^\ell = \mathcal{O}_X$; i.e., $F = \operatorname{Sym} V$.

(c) Now $F = \operatorname{Sym} V$, $d_F = 0$, so $\langle F \rangle = \operatorname{Sym} K$ where $K := H^0_{DR}(X, V[1])$. We want to show that $\phi_F : C^{ch}(X, \operatorname{Sym}^{>0} V) \to \operatorname{Sym}^{>0} K$ is a quasi-isomorphism.

4.6.2. We will deduce this fact from the next proposition which is valid under less restrictive assumptions on V. Take any $V \in C\mathcal{M}(X)$; set $F := \operatorname{Sym} V$. Let us represent $C^{ch}(X, F)$ by the commutative algebra $C^{ch}(X, F)_{\mathcal{PQ}}$ (see 4.3.2) and $R\Gamma_{DR}(X, V)$ by the complex $\Gamma(X, h(V_{\mathcal{PQ}}))$. The embedding $\Gamma(X, h(V_{\mathcal{PQ}}))[1] \subset C^{ch}_1(X, F) \hookrightarrow C^{ch}(X, F)$ (see 4.2.12) yields a morphism of commutative algebras

$$\operatorname{Sym}(\Gamma(X, h(V_{\mathcal{PQ}}))[1]) \to C^{ch}(X, F). \tag{4.6.2.1}$$

PROPOSITION. *If V is homotopically $\mathcal{O}_X$-flat, then this is a quasi-isomorphism.*

End of the proof of the theorem. Consider V as in part (c) of the proof in 4.6.1. We know that $\Gamma(X, h(V_{\mathcal{PQ}}))[1] \xrightarrow{\sim} R\Gamma_{DR}(X, V[1]) \xrightarrow{\sim} H^0_{DR}(X, V[1]) = K$, and it is clear that (4.6.2.1) is right inverse to ϕ_F. So the proposition implies that ϕ_F is a quasi-isomorphism, and we are done.

Proof of Proposition. (i) Set $F^{>0} := \operatorname{Sym}^{>0} V$. By 4.4.8 it suffices to show that

$$\operatorname{Sym}^{>0}(\Gamma(X, h(V_{\mathcal{PQ}}))[1]) \to C^{ch}(X, F^{>0}) \tag{4.6.2.2}$$

is a quasi-isomorphism.

(ii) Let us show first that (4.6.2.2) comes from a morphism of certain $\mathcal{D}$-complexes on $X^{\mathcal{S}}$. Recall that the Chevalley-Cousin complex $C(F^{>0})$ is a commutative algebra in $C\mathcal{M}(X^{\mathcal{S}})^*$. Set $P := \operatorname{Sym}^{>0}_*(\Delta^{(\mathcal{S})}_* V[1])$ (see 3.4.10); this is again a commutative algebra in $C\mathcal{M}(X^{\mathcal{S}})^*$. The embeddings $\Delta^{(\mathcal{S})}_* V \hookrightarrow \Delta^{(\mathcal{S})}_* F^{>0} \hookrightarrow$

$C(F^{>0})$ define a morphism of commutative algebras $P \to C(F^{>0})$. Passing to the de Rham cohomology, we get

$$R\Gamma_{DR}(X^{\mathcal{S}}, P) \to R\Gamma_{DR}(X^{\mathcal{S}}, C(F^{>0})) = C^{ch}(X, F^{>0}). \tag{4.6.2.3}$$

Now there is a canonical quasi-isomorphism

$$\mathrm{Sym}^{>0} R\Gamma_{DR}(X, V[1]) \xrightarrow{\sim} R\Gamma_{DR}(X^{\mathcal{S}}, P). \tag{4.6.2.4}$$

Namely, we know that $R\Gamma_{DR}(X, V[1]) \xrightarrow{\sim} R\Gamma_{DR}(X^{\mathcal{S}}, \Delta_*^{(\mathcal{S})} V[1])$, and (4.6.2.3) is the symmetric power of this quasi-isomorphism via (4.2.8.6).

It is immediate that (4.6.2.2) is the composition of (4.6.2.3) and (4.6.2.4). So to prove the proposition, one needs to check that (4.6.2.3) is a quasi-isomorphism.

(iii) Consider a filtration $W.$ on $C(F^{>0})$ defined as follows. On each $C(F^{>0})_{X^I}$ our filtration is compatible with the $Q(I)$-grading (see (3.4.11.1)), and for $T \in Q(I)$ one has $W_n \Delta_*^{(I/T)} j_*^{(T)} j^{(T)*}(F^{>0}[1])^{\boxtimes T} := \Delta_*^{(I/T)}((F^{>0}[1])^{\boxtimes T} \cdot W_n j_*^{(T)} \mathcal{O}_{U^{(T)}})$; see 3.1.6 and 3.1.7. Our $C(F^{>0})$ also carries a $\mathbb{Z}_{>0}$-grading defined by the action of homotheties on V; denote the components by $C(F)^{(n)}$. Subcomplex P and filtration $W.$ are compatible with the grading. One has $W_{-1}C(F^{>0}) = C(F^{>0})$, and $W_{-n-1}C(F)^{(n)} = 0$, $W_{-n}C(F^{>0})^{(n)} = P^{(n)}$. To prove the proposition, it suffices to show that

$$R\Gamma_{DR}(X^{\mathcal{S}}, P^{(n)}) \xrightarrow{\sim} R\Gamma_{DR}(X^{\mathcal{S}}, \mathrm{gr}^W_{-n} C(F^{>0})). \tag{4.6.2.5}$$

One has $\mathrm{gr}^W_{-n} C(F^{>0})_{X^I} = \underset{S \in Q(I)}{\oplus} \underset{T \in Q(S,n)}{\oplus} \Delta_*^{(I/T)}(\boxtimes_T (F^{>0}[1])^{\otimes S_t}) \otimes \mathcal{L}ie^*_{S/T}$; see (3.1.10.1) (we forget about the differential). Therefore

$$\mathrm{gr}^W_{-1} C(F^{>0})_{X^I} = \underset{S \in Q(I)}{\oplus} \Delta_*^{(I)}(F^{>0}[1])^{\otimes S} \otimes \mathcal{L}ie^*_S, \tag{4.6.2.6}$$

and there is an obvious identification

$$\mathrm{gr}^W_{-n} C(F^{>0}) = \mathrm{Sym}^n_* \mathrm{gr}^W_{-1} C(F^{>0}). \tag{4.6.2.7}$$

Here Sym^n_* is nth symmetric power with respect to the $\otimes^*$ tensor structure (see 3.4.10). Now (4.6.2.7) is compatible with the differentials; i.e., it is an isomorphism of complexes. Since $P^{(n)} = \mathrm{Sym}^n_* P^{(1)}$, (4.6.2.5) for arbitrary n follows from the case $n = 1$ by (4.2.8.6).

(iv) The complex $\mathrm{gr}^W_{-1} C(F^{>0})$ is supported on the diagonal $X \subset X^{\mathcal{S}}$, so we can consider it as an $\mathcal{S}^\circ$-diagram Φ of complexes of $\mathcal{D}$-modules on X. The subcomplex $P^{(1)}$ identifies with a constant subdiagram $V \subset \Phi$. Now $R\Gamma_{DR}(X^{\mathcal{S}}, \mathrm{gr}^W_{-1} C(F^{>0}))$ is the de Rham cohomology of X with coefficients in the homotopy direct limit $C(\mathcal{S}^\circ, \Phi)$ (see 4.1.1(iv)). Similarly, $R\Gamma_{DR}(X^{\mathcal{S}}, P^{(1)}) = R\Gamma_{DR}(X, C(\mathcal{S}^\circ, V))$, and $C(\mathcal{S}^\circ, V) \xrightarrow{\sim} V$. We will show that $V \to C(\mathcal{S}^\circ, \Phi)$ is a quasi-isomorphism. This implies (4.6.2.5) for $n = 1$, hence finishes the proof.

Define a grading $\Phi = \oplus \Phi_m$ so that the subdiagram Φ_m collects all summands in (4.6.2.6) with $|S| = m$. Then $d(\Phi_m) \subset \Phi_{m-1}$, and every Φ_m is the induced $\mathcal{S}^\circ$-diagram that corresponds to a representation of the symmetric group Σ_m on $(F^{>0}[1])^{\otimes n} \otimes \mathcal{L}ie^*_n$ (see Exercise in 4.1.1(iv)). Therefore, by loc. cit., the projection $C(\mathcal{S}^\circ, \Phi) \to \varinjlim \Phi$ is a quasi-isomorphism, and $\varinjlim \Phi$ is a complex with terms $((F^{>0}[1])^{\otimes m} \otimes \mathcal{L}ie^*_m)_{\Sigma_m}$. In other words, $\varinjlim \Phi^\ell$, as a mere graded $\mathcal{D}$-module, coincides with the cofree Lie coalgebra generated by the left $\mathcal{D}$-module $(\mathrm{Sym}^{>0} V^\ell)[1]$.

The differential equals the canonical differential defined by the commutative algebra structure on $F^{>0\ell}$. Therefore $V \to \varinjlim \Phi$ is a quasi-isomorphism (see theorem 7.5 in [**Q1**]),[70] and we are done. $\square$

4.6.3. Semi-free modules. We need to fix some terminology. Let R^ℓ be a $\mathcal{D}_X$-algebra. The DG category of $R^\ell[\mathcal{D}_X]$-modules (which are the same as central chiral R-modules) is denoted by $\mathcal{M}(X, R^\ell)$ and its derived category by $D\mathcal{M}(X, R^\ell)$.

An $R^\ell[\mathcal{D}_X]$-module M is said to be *semi-free* if it admits a filtration $M_{-1} = 0 \subset M_0 \subset M_1 \subset \cdots$, $\bigcup M_i = M$, such that for each i there exists a $\mathbb{Z}$-graded $\mathcal{D}_X$-submodule $N_i \subset M_i$ which is a locally projective $\mathcal{D}_X$-module, $d(N_i) \subset M_{i-1}$, and the morphism $R^\ell \otimes N_i \to \mathrm{gr}_i M$ is an isomorphism. We refer to the N_i as *semi-free generators* of M. We say that the generators N_i are *convenient* if $\Gamma(X, h(N_i)) = 0$; in this situation M is said to be *convenient*. A semi-free module is automatically R^ℓ-flat.

A linearized version of the proof of part (i) of the lemma in 4.3.7 (which is an immediate modification of the usual construction of [**Sp**]) shows that every $R^\ell[\mathcal{D}_X]$-module admits a semi-free resolution. Moreover, one can choose it so that N_i is isomorphic to a direct sum of (shifts of) $\mathcal{D}_X$-modules $\mathcal{L}_{\mathcal{D}}$ where $\mathcal{L}$ is a line bundle on X. If wanted, we can assume that the $\mathcal{L}$ are of sufficiently negative degree (cf. Remark (b) in 4.3.7), so the resolution is convenient.

A morphism of $\mathcal{D}_X$-algebras $f : R \to F$ yields an evident exact DG functor $f_\cdot : \mathcal{M}(X, F^\ell) \to \mathcal{M}(X, R^\ell)$ and its left adjoint $f^* : \mathcal{M}(X, R^\ell) \to \mathcal{M}(X, F^\ell)$, $f^*M := F^\ell \underset{R^\ell}{\otimes} M$.

Lemma. *The left derived functor $Lf^* : D\mathcal{M}(X, R^\ell) \to D\mathcal{M}(X, F^\ell)$ is well defined and is left adjoint to $f_\cdot : D\mathcal{M}(X, F^\ell) \to D\mathcal{M}(X, R^\ell)$. If f is a quasi-isomorphism, then*

$$D\mathcal{M}(X, R^\ell) \underset{f_\cdot}{\overset{Lf^*}{\rightleftarrows}} D\mathcal{M}(X, F^\ell) \tag{4.6.3.1}$$

are mutually inverse equivalences.

Proof. Use semi-free resolutions of $R^\ell[\mathcal{D}_X]$-modules. $\square$

4.6.4. The proposition in 4.6.2 admits a version with parameters. Let R^ℓ be a $\mathcal{D}_X$-algebra, V an $R^\ell[\mathcal{D}_X]$-module. Consider the symmetric R^ℓ-algebra $F^\ell := \mathrm{Sym}_{R^\ell} V^\ell$.

Fix auxiliary $\mathcal{P}$, $\mathcal{Q}$ and set $C^{ch}(\cdot) := C^{ch}(X, \cdot)_{\mathcal{PQ}}$ (see 4.2.12). We have a commutative homotopy unital $C^{ch}(R)$-algebra $C^{ch}(F)$ and a homotopy unital $C^{ch}(R)$-module $C^{ch}(R, V) := C^{ch}(X, R, \{V\})_{\mathcal{PQ}}$ (see 4.2.19). Let $\mathrm{Sym}^L_{C^{ch}(R)} C^{ch}(R, V)$ be the symmetric algebra of a homotopically $C^{ch}(R)$-flat resolution of $C^{ch}(R, V)$. The obvious morphism of $C^{ch}(R)$-modules $C^{ch}(R, V) \to C^{ch}(F)$ yields a morphism of the homotopy unital $C^{ch}(R)$-algebras

$$\mathrm{Sym}^L_{C^{ch}(R)} C^{ch}(R, V) \to C^{ch}(F). \tag{4.6.4.1}$$

[70] Strictly speaking, [**Q1**] considers the setting of complexes over a field of characteristic 0 subject to some boundedness condition. From the modern point of view, the statement is a consequence of the Koszul duality of the operads $\mathcal{C}om$ and $\mathcal{L}ie$ (in characteristic 0); see [**GK**].

THEOREM. *If V is homotopically R-flat, then (4.6.4.1) is a quasi-isomorphism.*

Proof. Choose a semi-free resolution $W \to V$. Since V is homotopically R-flat, the morphism $\mathrm{Sym}_R W \to \mathrm{Sym} V$ is a quasi-isomorphism. Replacing V by W, we can assume that V is semi-free (we use the fact that $\mathrm{Sym}^L_{C^{ch}(R)}$ preserves quasi-isomorphisms). The corresponding filtration on V makes (4.6.4.1) a morphism of filtered $C^{ch}(R)$-algebras. Using (4.2.18.3), one can replace V by $\mathrm{gr}\, V$; hence we are reduced to the case when $V = R^\ell \otimes N$ where N is a locally projective $\mathbb{Z}$-graded $\mathcal{D}_X$-module (considered as a complex with zero differential). So $F = R \otimes \mathrm{Sym}\, N^\ell$.

By (4.4.7.1), one has $C^{ch}(X,R) \otimes R\Gamma_{DR}(X,N) \xrightarrow{\sim} C^{ch}(X,R,V)$; by 4.3.6, $C^{ch}(X,R) \otimes C^{ch}(X, \mathrm{Sym}\, N) \xrightarrow{\sim} C^{ch}(X,F)$. This identifies (4.6.4.1) with (4.6.2.1) (for $V = P$) tensored by $C^{ch}(X,R)$, so we are done by the proposition in 4.6.2. □

4.6.5. Here is a version of 4.6.1 for the chiral homology with coefficients.

Let R^ℓ be a $\mathcal{D}_X$-algebra. Recall that the constant $\mathcal{D}_X$-algebra $\langle R\rangle \otimes \mathcal{O}_X$ is a quotient of R^ℓ. Let M be a (DG) $R^\ell[\mathcal{D}_X]$-module (= the central chiral R-module). Consider $M_{\langle R\rangle} := (\langle R\rangle \otimes \mathcal{O}_X) \underset{R^\ell}{\otimes} M$; this is the maximal quotient of M which is a $\langle R\rangle \otimes \mathcal{D}_X$-module.

The functor $M \to M_{\langle R\rangle}$ is right exact. Let $M \mapsto M^L_{\langle R\rangle}$ be its left derived functor $D\mathcal{M}(X,R^\ell) \to D(X, \langle R\rangle \otimes \mathcal{O}_X)$. So if M is homotopically R^ℓ-flat, then $M^L_{\langle R\rangle} \xrightarrow{\sim} M_{\langle R\rangle}$.

Passing to the de Rham cohomology, we get a triangulated functor $D\mathcal{M}(X,R^\ell) \to D(\langle R\rangle)$, $M \mapsto R\Gamma_{DR}(X, M^L_{\langle R\rangle})$; here $D(\langle R\rangle)$ is the derived category of unital $\langle R\rangle$-modules.

Now let $\langle M\rangle_R$ be the maximal constant quotient of $M^\ell_{\langle R\rangle}$.[71] This is naturally an $\langle R\rangle$-module. The functor $M \mapsto \langle M\rangle_R$ is right exact. As is clear from the proof of the next lemma, it admits the left derived functor $M \mapsto \langle M\rangle^L_R$.

LEMMA. *One has a natural isomorphism*

$$R\Gamma_{DR}(X, M^L_{\langle R\rangle}) \xrightarrow{\sim} \langle M\rangle^L_R[-1]. \tag{4.6.5.1}$$

Proof. One has a natural morphism $R\Gamma_{DR}(X, M^L_{\langle R\rangle}) \to \langle M\rangle_R[-1]$ coming from the canonical morphisms $M^L_{\langle R\rangle} \to M_{\langle R\rangle} \to \langle M\rangle_R \otimes \omega$ and the trace map $R\Gamma_{DR}(X, \langle M\rangle_R \otimes \omega) = \langle M\rangle_R \otimes R\Gamma_{DR}(X,\omega) \xrightarrow{tr} \langle M\rangle_R[-1]$. For convenient M it is a quasi-isomorphism (see 4.6.3), and we are done. □

Suppose now that R is convenient (see 3.6.1), so $\langle R\rangle = C^{ch}(X,R)$. By 4.3.4, we have a triangulated functor $D\mathcal{M}(X,R^\ell) \to D(\langle R\rangle)$, $M \mapsto C^{ch}(X,R,M)$.

PROPOSITION. *One has a natural isomorphism*

$$C^{ch}(X,R,M) \xrightarrow{\sim} \langle M\rangle^L_R[-1]. \tag{4.6.5.2}$$

Proof. Let us define a natural morphism

$$C^{ch}(X,R,M) \to \langle M\rangle_R[-1]. \tag{4.6.5.3}$$

[71] So $\langle M\rangle_R$ is obtained from the complex $M_{\langle R\rangle}$ by term-by-term application of the functor $H_0^{DR}(X,\cdot)$ (see 2.1.12).

We have the $\mathbb{Z}$-graded $\mathcal{D}_X$-algebra $R^{\{M\}\ell}$ with components $R^{\{M\}0} = R$, $R^{\{M\}1} = M[-1]$, and $C^{ch}(X, R, M)$ is the degree 1 component of $C^{ch}(X, R^{\{M\}})$ (see 4.2.19). The $\mathbb{Z}$-graded algebra $\langle R^{\{M\}}\rangle$ has components $\langle R^{\{M\}}\rangle^0 = \langle R\rangle$, $\langle R^{\{M\}}\rangle^1 = \langle M\rangle_R[-1]$. The morphism $\phi_{R^{\{M\}}} : C^{ch}(X, R^{\{M\}}) \to \langle R^{\{M\}}\rangle$ of commutative algebras (see 4.2.17 and (i) in 4.3.2) then yields (4.6.5.3).

Let us check that (4.6.5.3) is a quasi-isomorphism for convenient M (see 4.6.3). This establishes (4.6.5.2).

The filtration M_i induces on $\langle M\rangle_R$ a filtration such that $\mathrm{gr}_i\langle M\rangle_R = \langle \mathrm{gr}_i M\rangle$. The same is true for $C^{ch}(X, R, M)$, so it suffices to check our statement for M replaced by $\mathrm{gr}\, M$. Thus we can assume that $M = R^\ell \otimes N$ where N is a complex of locally projective $\mathcal{D}_X$-modules with zero differential such that $\Gamma(X, h(N)) = 0$.

Consider $S^\ell := \mathrm{Sym}_{R^\ell}(M^\ell[-1]) = R^\ell \otimes \mathrm{Sym}(N^\ell[-1])$. One has an evident projection $S \to R^{\{M\}}$ compatible with the $\mathbb{Z}$-gradings on both algebras. It yields an isomorphism between the degree 1 components of the chiral homology, so one has $C^{ch}(X, R, M) = C^{ch}(X, S)^1$. We also have $\langle S\rangle = \mathrm{Sym}_{\langle R\rangle}(\langle M\rangle_R)$, so our projection yields an isomorphism of the degree 1 components $\langle S\rangle^1 \xrightarrow{\sim} \langle R^{\{M\}}\rangle^1 = \langle M\rangle_R[-1]$. Since S is convenient, we have $C^{ch}(X, S) = \langle S\rangle$ (see 4.6.1). Therefore (4.6.5.3) is a quasi-isomorphism; q.e.d. □

Corollary. *The functor $C^{ch}(X, R, \cdot) : D\mathcal{M}(X, R^\ell) \to D(\langle R\rangle)$ is left adjoint to the functor $D(\langle R\rangle) \to D\mathcal{M}(X, R^\ell)$, $P \mapsto P \otimes \omega_X[1]$.* □

Remark. Here is a version of the above statements for several $R^\ell[\mathcal{D}_X]$-modules M_s (we will not use it). One has an $\langle R\rangle \otimes \mathcal{D}_{X^S}$-module $\underset{\langle R\rangle}{\boxtimes} M_{s\langle R\rangle}$ which yields $R\Gamma_{DR}(U^{(S)}, \underset{\langle R\rangle}{\boxtimes} M_{s\langle R\rangle}) \in D(\langle R\rangle)$. Now for R convenient and M_s homotopically R-flat there is a canonical isomorphism

$$C^{ch}(X, R, \{M_s\}) \xrightarrow{\sim} R\Gamma_{DR}(U^{(S)}, \underset{\langle R\rangle}{\boxtimes} M_{s\langle R\rangle}). \tag{4.6.5.4}$$

4.6.6. The cotangent complex. The definition of the cotangent complex (we recalled it in 4.1.5) renders itself immediately into the setting of $\mathcal{D}_X$-algebras. Namely, for a commutative $\mathcal{D}_X$-algebra R^ℓ we have its $R^\ell[\mathcal{D}_X]$-module of differentials $\Omega_R := \Omega_{R^\ell/X}$. Now ${}^L\Omega_R$ is an object of $D\mathcal{M}(X, R^\ell)$ equipped with a morphism ${}^L\Omega_R \to \Omega_R$ defined as follows. If R^ℓ is semi-free, then ${}^L\Omega_R = \Omega_R$. One checks that if $\phi : R^\ell \to F^\ell$ is a quasi-isomorphism of semi-free algebras, then $d\phi : L\phi^*\Omega_R \to \Omega_F$ is a quasi-isomorphism. If R is arbitrary, then one chooses a semi-free resolution $\tilde{R} \to R$ and defines ${}^L\Omega_R$ as the image of $\Omega_{\tilde{R}}$ by the equivalence $D\mathcal{M}(X, \tilde{R}^\ell) \xrightarrow{\sim} D\mathcal{M}(X, R^\ell)$; its independence of the choice of $\tilde{R}$ follows from the exercise in 4.3.7. Notice that the image of ${}^L\Omega_R$ in the derived category of DG R^ℓ-modules (forgetting the $\mathcal{D}_X$-action) equals the usual cotangent complex of R^ℓ relative to $\mathcal{O}_X$.

Proposition. *There is a canonical quasi-isomorphism*

$$ {}^L\Omega_{C^{ch}(X,R)} \xrightarrow{\sim} C^{ch}(X, R, {}^L\Omega_R)[1]. \tag{4.6.6.1}$$

Proof. Let $\Omega_{C^{ch}(X,R)_{\mathcal{PQ}}}$ be the module of $C^{ch}(X, R)_{\mathcal{PQ}}$-differentials relative to $C^{ch}(X, \omega)_{\mathcal{PQ}}$. The canonical odd derivation of the algebra $R^{\{\Omega_R\}}$ (see 4.2.19 for notation) yields a derivation $C^{ch}(X,R)_{\mathcal{PQ}} \to C^{ch}(X,R,\Omega_R)_{\mathcal{PQ}}[1]$ by transport of structure. So we have a morphism of $C^{ch}(X,R)_{\mathcal{PQ}}$-modules

$$\Omega_{C^{ch}(X,R)_{\mathcal{PQ}}} \to C^{ch}(X,R,\Omega_R)_{\mathcal{PQ}}[1]. \tag{4.6.6.2}$$

Its composition with the canonical morphism ${}^L\Omega_{C^{ch}(X,R)} \to \Omega_{C^{ch}(X,R)_{\mathcal{PQ}}}$ is a morphism ${}^L\Omega_{C^{ch}(X,R)} \to C^{ch}(X,R,\Omega_R)[1]$ in the derived category of unital $C^{ch}(X,R)$-modules. Replacing R by a semi-free resolution, we get an arrow ${}^L\Omega_{C^{ch}(X,R)} \to C^{ch}(X,R,{}^L\Omega_R)[1]$. It remains to show that this is a quasi-isomorphism.

We need a lemma. For arbitrary R consider the $\langle R\rangle$-module $\langle\Omega_R\rangle_R$ (see 4.6.5). The universal derivation $R^\ell \to \Omega_R$ yields a derivation $\langle R\rangle \to \langle\Omega_R\rangle_R$ between the constant quotients, hence a morphism of DG $\langle R\rangle$-modules

$$\Omega_{\langle R\rangle} \to \langle\Omega_R\rangle_R. \tag{4.6.6.3}$$

Lemma. *Suppose that, as a mere graded $\mathcal{D}_X$-algebra, R is freely generated by some $\mathcal{D}_X$-module. Then (4.6.6.3) is an isomorphism.*

Proof of Lemma. Our statement has nothing to do with the differential on R, so we can forget about it. Suppose that $R^\ell = \operatorname{Sym} V^\ell$. Denote by $\langle V\rangle$ the maximal constant quotient of V^ℓ. Then $\langle R\rangle = \operatorname{Sym}\langle V\rangle$; hence $\Omega_{\langle R\rangle} = \langle R\rangle \otimes \langle V\rangle$ (we identify $\langle V\rangle$ with its image in $\Omega_{\langle R\rangle}$ by the universal derivation). Similarly, $\Omega_R = R^\ell \otimes V$, so $\langle\Omega_R\rangle_R = \langle R\rangle \otimes \langle V\rangle$. Now (4.6.6.3) identifies the generators of the free $\langle R\rangle$-module, and we are done. □

Let us finish the proof of the proposition. We can assume that R is convenient. Then, by the theorem in 4.6.1, one has $C^{ch}(X,R) \xrightarrow{\sim} \langle R\rangle$. The arrow $C^{ch}(X,R,\Omega_R)[1] \to \langle\Omega_R\rangle_R$ from (4.6.5.3) is also a quasi-isomorphism. Indeed, Ω_R is convenient (since the images of convenient generators of R by the universal derivation are convenient generators of Ω_R), and our assertion was checked in the proof of the proposition in 4.6.5. The above isomorphisms identify our morphism ${}^L\Omega_{C^{ch}(X,R)} \to C^{ch}(X,R,{}^L\Omega_R)[1]$ with (4.6.6.3). We are done by the lemma. □

4.6.7. Perfect $R^\ell[\mathcal{D}_X]$-complexes. Let us check that the chiral homology preserves perfect complexes.

Let R^ℓ be any commutative $\mathcal{D}_X$-algebra. We have the derived category of $R^\ell[\mathcal{D}_X]$-modules $D\mathcal{M}(X,R^\ell)$ and its subcategory $D\mathcal{M}(X,R^\ell)_{perf}$ of perfect complexes (see 4.1.16).

Proposition. (i) *Perfectness is a local property for the Zariski topology of X.*

(ii) *The functor $C^{ch}(X,R,\cdot) : D\mathcal{M}(X,R^\ell) \to D(C^{ch}(X,R))$ preserves perfect complexes.*

Proof. (i) Suppose we have $P \in D\mathcal{M}(X,R^\ell)$ and a finite covering $\{U_\alpha\}$ of X such that the P_{U_α} are perfect. For an $R^\ell[\mathcal{D}_X]$-module M let $M \to C(M)$ be the Čech resolution of M with respect to this covering.

If $j : U \hookrightarrow X$ is an intersection of some U_α's, then $P|_U$ is perfect; hence the functor $M \mapsto \operatorname{Hom}(P, j_*j^*M) = \operatorname{Hom}(P|_U, M_U)$ commutes with direct sums. Thus $M \mapsto \operatorname{Hom}(P,M) = \operatorname{Hom}(P, C(M))$ also commutes with direct sums, so P is perfect; q.e.d.

(ii) Follows from the fact that the right adjoint functor to $C^{ch}(X,R,\cdot)$, as described in the corollary in 4.6.5, commutes with direct sums. □

Suppose that R has non-positive degrees, so $D\mathcal{M}(X, R^\ell)$ is a t-category. We define the span of its objects as in 4.1.16; every perfect complex has finite span.

LEMMA. (i) *If an object $M \in D\mathcal{M}(X, R^\ell)$ has span in $[b, a]$ locally on X, then the span of M lies in $[b-1, a]$.*

(ii) *If $M \in D\mathcal{M}(X, R^\ell)$ has span $[b, a]$, then $C^{ch}(X, R, M) \in D(C^{ch}(X, R))$ has span in $[b+1, a+1]$.*

Proof. (i) Clear since the cohomological dimension of X equals 1.

(ii) Use the corollary from 4.6.5. □

COROLLARY. *If R^ℓ is a plain $\mathcal{D}_X$-algebra and M a finitely generated locally projective $R^\ell[\mathcal{D}_X]$-module, then $C^{ch}(X, R, M)$ is a perfect $C^{ch}(X, R)$-module of span $[0, 1]$.* □

4.6.8. Compatibility with duality. Suppose that R has non-positive degrees.

PROPOSITION. (i) *An object $P \in D\mathcal{M}(X, R^\ell)$ is perfect of span $[b, a]$ if and only if it can be represented as a retract in $D\mathcal{M}(X, R^\ell)$ of a semi-free $R^\ell[\mathcal{D}_X]$-module with finitely many generators whose degrees are in $[b, a]$.*

(ii) *For a perfect P its dual is perfect, and the functor $C^{ch}(X, R, \cdot)$ commutes with duality.* □

4.6.9. THEOREM. *For a plain very smooth $\mathcal{D}_X$-algebra R^ℓ (see 2.3.15) the algebra $C^{ch}(X, R)$ is perfect of span ≤ 1 (see 4.1.17). In particular, if the higher chiral homology of R vanishes, then $\langle R \rangle$ is a complete intersection.*

REMARK. Probably, the theorem remains true if X is a projective variety of arbitrary dimension n, and one defines $C^{ch}(X, R)$ as the left derived functor of $R \mapsto \langle R \rangle$ (see 3.4); the estimate for the span is $\leq n$.

Proof of Theorem. We know that $\langle R \rangle = H^0 C^{ch}(X, R)$ is finitely generated (see (ii) in the proposition in 2.4.2), so, by 4.1.17, it suffices to check that the cotangent complex ${}^L\Omega_{C^{ch}(X,R)}$ is a perfect $C^{ch}(X, R)$-module of span in $[-1, 0]$.

Since R is very smooth, one has ${}^L\Omega_R \xrightarrow{\sim} \Omega_R$. Since Ω_R is a locally projective $\mathcal{D}_X$-module, $C^{ch}(X, R, \Omega_R)$ is a perfect $C^{ch}(X, R)$-module of span in $[0, 1]$ (see the corollary in 4.6.7). By 4.6.6, it equals ${}^L\Omega_{C^{ch}(X,R)}[-1]$, and we are done. □

4.7. Chiral homology of the de Rham-Chevalley algebras

In this section we consider the chiral homology of (the DG version of) the orbit space of a Lie* R-algebroid. In particular, we interpret the homotopy 1-Poisson structure on the chiral homology of a coisson algebra R. Namely, under appropriate regularity condition it defines a Lagrangian embedding of $\mathrm{Spec}\langle R \rangle$ into a formal symplectic tube. If R is the Gelfand-Dikii coisson algebra (see 2.6.8), then this is the formal neighbourhood of the space of global opers on X in the symplectic space of all local systems.

4.7.1. The orbit space of a Lie* algebroid. Let R^ℓ be a plain commutative $\mathcal{D}_X$-algebra and $\mathcal{L}$ a Lie* R-algebroid which is a vector $\mathcal{D}_X$-bundle on $\mathrm{Spec} R^\ell$. Let us show that the DG version of the orbit space functor commutes with the chiral homology.

The dual $\mathcal{D}_X$-bundle $\mathcal{L}^\circ$ is a Lie R-coalgebroid, so we have the corresponding de Rham-Chevalley DG $\mathcal{D}_X$-algebra $\mathcal{C}_R(\mathcal{L}^\circ) =: \mathcal{C}_R(\mathcal{L})$ (see (ii) and (iii) in 1.4.14). It carries a decreasing filtration by DG ideals $F^n := \mathcal{C}_R(\mathcal{L})^{\geq n}$ with $\mathrm{gr}_F \mathcal{C}_R(\mathcal{L}) = \mathrm{Sym}_R(\mathcal{L}^\circ[-1])$,[72] and we consider $\mathcal{C}_R(\mathcal{L}) = \varprojlim \mathcal{C}_R(\mathcal{L})/F^n$ as a filtered topological DG $\mathcal{D}_X$-algebra. Thus $\mathcal{C}_R(\mathcal{L})_{\mathcal{P}\mathcal{Q}} := \varprojlim (\mathcal{C}_R(\mathcal{L})/F^n)_{\mathcal{P}\mathcal{Q}}$ is also a filtered topological algebra, $\mathrm{gr}_F \mathcal{C}_R(\mathcal{L})_{\mathcal{P}\mathcal{Q}} = \mathrm{Sym}_R(\mathcal{L}^\circ[-1])_{\mathcal{P}\mathcal{Q}}$. We get a filtered topological homotopy unital commutative DG algebra $C^{ch}(X, \mathcal{C}_R(\mathcal{L}))_{\mathcal{P}\mathcal{Q}} := \varprojlim C^{ch}(X, \mathcal{C}_R(\mathcal{L})/F^n)_{\mathcal{P}\mathcal{Q}}$; one has $\mathrm{gr}_F C^{ch}(X, \mathcal{C}_R(\mathcal{L}))_{\mathcal{P}\mathcal{Q}} = C^{ch}(X, \mathrm{Sym}_R(\mathcal{L}^\circ[-1]))_{\mathcal{P}\mathcal{Q}}$. As an object of the corresponding homotopy category $\mathcal{H}o\mathcal{C}omu\mathcal{T}$,[73] our algebra does not depend on the auxiliary choice of $\mathcal{P}$, $\mathcal{Q}$, so we write simply $C^{ch}(X, \mathcal{C}_R(\mathcal{L}))$.

Consider now a homotopy unital commutative algebra $C^{ch}(X, R)_{\mathcal{P}\mathcal{Q}}$ and a homotopy unital Lie $C^{ch}(X, R)_{\mathcal{P}\mathcal{Q}}$-algebroid $C^{ch}(X, R, \mathcal{L})_{\mathcal{P}\mathcal{Q}}$ (see (v) in 4.3.2).[74] Choose any left resolution (which is a quasi-isomorphism of homotopy unital Lie $C^{ch}(X, R)$-algebroids) $P \to C^{ch}(X, R, \mathcal{L})_{\mathcal{P}\mathcal{Q}}$ which is homotopically $C^{ch}(X, R)_{\mathcal{P}\mathcal{Q}}$-projective. Consider the de Rham-Chevalley complex $\mathcal{C}(P) := \mathcal{C}_{C^{ch}(X,R)_{\mathcal{P}\mathcal{Q}}}(P)$ which is a filtered homotopy unital topological DG algebra. In the unital setting it was defined in 2.9.1, and in the present homotopy unital setting the definition is similar. Namely, as a mere topological graded module our $\mathcal{C}(P)$ is equal to $C^{ch}(X, R)_{\mathcal{P}\mathcal{Q}} \oplus \mathrm{Hom}_{C^{ch}(X,R)_{\mathcal{P}\mathcal{Q}}}(\underset{i>0}{\oplus} \mathrm{Sym}^i_{C^{ch}(X,R)_{\mathcal{P}\mathcal{Q}}}(P[1]), C^{ch}(X, R)_{\mathcal{P}\mathcal{Q}})$, the algebra structure on $\mathcal{C}(P)$ comes since $\mathrm{Sym}^{>0}(P[1])$ is naturally a $C^{ch}(X, R)_{\mathcal{P}\mathcal{Q}}$-coalgebra (the coproduct comes from the diagonal map $P[1] \to P[1] \times P[1]$). As an object of $\mathcal{H}o\mathcal{C}omu\mathcal{T}$, $\mathcal{C}(P)$ does not depend on the auxiliary choice of P, $\mathcal{P}$, $\mathcal{Q}$, so we denote it simply by $\mathcal{C}_{C^{ch}(X,R)}(C^{ch}(X, R, \mathcal{L}))$.

PROPOSITION. *There is a canonical isomorphism in* $\mathcal{H}o\mathcal{C}omu\mathcal{T}$

$$C^{ch}(X, \mathcal{C}_R(\mathcal{L})) \xrightarrow{\sim} \mathcal{C}_{C^{ch}(X,R)}(C^{ch}(X, R, \mathcal{L})). \tag{4.7.1.1}$$

Proof. We will define a natural morphism of homotopy unital filtered topological DG algebras

$$C^{ch}(X, \mathcal{C}_R(\mathcal{L}))_{\mathcal{P}\mathcal{Q}} \to \mathcal{C}(P) \tag{4.7.1.2}$$

which induces a quasi-isomorphism of the associated graded algebras.

The action of $\mathcal{L}$ on R extends naturally to an action of the contractible Lie* algebra $\mathcal{L}_\dagger$ on $\mathcal{C}_R(\mathcal{L})$ (see 1.4.14). Thus $C^{ch}(X, R, \mathcal{L})_\dagger$, hence $P_\dagger$, acts on $C^{ch}(X, \mathcal{C}_R(\mathcal{L}))$.

If we forget about the differential, then the filtrations $F^\cdot$ naturally split, so both terms of (4.7.1.2) acquire an extra $\mathbb{Z}_{\geq 0}$-grading, not merely a filtration. We define (4.7.1.2) as a unique $C^{ch}(X, R)_{\mathcal{P}\mathcal{Q}}$-linear morphism which is compatible with the grading, commutes with the action of the $P[1]$-part of $P_\dagger$, and is the identity map on the degree 0 component $C^{ch}(X, R)_{\mathcal{P}\mathcal{Q}}$. We leave it to the reader to check that it is actually a morphism of DG algebras. It is a quasi-isomorphism on gr^F due to 4.6.8 and 4.6.4. □

[72]This is *not* a filtration in the sense of 3.3.12 and 4.2.18, so the corresponding spectral sequence need not converge.

[73]It is the same as the homotopy category of commutative unital DG algebras A equipped with a filtration by DG ideals $A = F^0 \supset F^1 \supset \cdots$ with homotopy equivalences being morphisms f such that $\mathrm{gr}_F f$ is a quasi-isomorphism.

[74]Here "homotopy unital Lie algebroid" means that we have a Lie algebroid which is a homotopy unital $C^{ch}(X, R)_{\mathcal{P}\mathcal{Q}}$-module and whose action kills the homotopy unit in $C^{ch}(X, R)_{\mathcal{P}\mathcal{Q}}$.

4.7.2. A description of H_0^{ch}. We are in the situation of 4.7.1. Suppose, in addition, that the top cohomology $H^1C^{ch}(X,R,\mathcal{L}^\circ)=H^1_{DR}(X,\mathcal{L}^\circ_{\langle R\rangle})$ vanishes (see 2.4.7 and 4.6.5 for notation). By 2.2.17 (or 4.6.8 and 4.6.7), this amounts to the fact that $H^0_{DR}(X,\mathcal{L}_{\langle R\rangle})=0$ and $H^1_{DR}(X,\mathcal{L}_{\langle R\rangle})$ is a projective $\langle R\rangle$-module. Then $H^0C^{ch}(X,R,\mathcal{L}^\circ)=H^0_{DR}(X,\mathcal{L}^\circ_{\langle R\rangle})$ and $H^1_{DR}(X,\mathcal{L}_{\langle R\rangle})$ are mutually dual projective $\langle R\rangle$-modules of finite rank.

By 4.7.1, the positive cohomology groups $H^{>0}C^{ch}(X,\mathcal{C}_R(\mathcal{L})/F^n)$ vanish for each n, so $\cdots\to H^0C^{ch}(X,\mathcal{C}_R(\mathcal{L})/F^{n+1})\to H^0C^{ch}(X,\mathcal{C}_R(\mathcal{L})/F^n)\to\cdots$ are surjective maps . The projective limit $H_0^{ch}(X,\mathcal{C}_R(\mathcal{L}))$ is a complete topological algebra, and $\langle R\rangle=H_0^{ch}(X,\mathcal{C}_R(\mathcal{L}))/I$ where I is an open ideal. As follows from 4.7.1, the topology on $H_0^{ch}(X,\mathcal{C}_R(\mathcal{L}))$ coincides with the I-adic topology, and one has an exact sequence $H_1^{ch}(X,R)\to H^0_{DR}(X,\mathcal{L}^\circ_{\langle R\rangle})\to I/I^2\to 0$.

We see that the formal scheme $\operatorname{Spf} H_0^{ch}(X,\mathcal{C}_R(\mathcal{L}))$ is a "formal tube" around $\langle\mathcal{Y}\rangle$ where $\mathcal{Y}:=\operatorname{Spec} R^\ell$. Let us describe it in geometric terms using the formal $\mathcal{D}_X$-scheme groupoid $\mathcal{G}$ on $\mathcal{Y}$ defined by $\mathcal{L}$ (see 1.4.15).

We have the sheaf of ind-affine ind-schemes $\langle\mathcal{Y}\rangle_X$ on $X_{\acute{e}t}$ and a formal groupoid $\langle\mathcal{G}\rangle_X$ on it (see 2.4.1),[75] hence for a test commutative algebra F the sheaf of F-points $\langle\mathcal{Y}\rangle_X(F)$ and the sheaf of groupoids $\langle\mathcal{G}\rangle_X(F)$ on it. Denote by $\langle\mathcal{Y}/\mathcal{G}\rangle(F)$ the stack of sections of the quotient stack $\langle\mathcal{Y}\rangle_X(F)/\langle\mathcal{G}\rangle_X(F)$. Its objects can be seen explicitly as pairs $(U_\cdot,f_\cdot)$ where $U_\cdot$ is an étale hypercovering of X and $f_\cdot:U_\cdot\times\operatorname{Spec}F\to\mathcal{G}_\cdot$ is a morphism of simplicial $\mathcal{D}_X$-spaces; here $\mathcal{G}_\cdot$, $\mathcal{G}_i=\mathcal{G}\underset{\mathcal{Y}}{\times}\cdots\underset{\mathcal{Y}}{\times}\mathcal{G}$ (i copies), is the classifying simplicial formal $\mathcal{D}_X$-scheme.[76] We leave the description of the morphisms in $\langle\mathcal{Y}/\mathcal{G}\rangle(F)$ to the reader.

The stack $\langle\mathcal{Y}/\mathcal{G}\rangle(F)$ depends on F in a functorial way.

PROPOSITION. (i) *Automorphisms of objects of $\langle\mathcal{Y}/\mathcal{G}\rangle(F)$ are all trivial, so $\langle\mathcal{Y}/\mathcal{G}\rangle$ is a set-valued functor on the category of commutative algebras.*

(ii) *There is a natural isomorphism*

$$\langle\mathcal{Y}/\mathcal{G}\rangle\xrightarrow{\sim}\operatorname{Spf} H_0^{ch}(X,\mathcal{C}_R(\mathcal{L})).\tag{4.7.2.1}$$

Proof. (i) Take any object $\phi\in\langle\mathcal{Y}/\mathcal{G}\rangle(F)$ and its automorphism ν. Since $\mathcal{G}$ is a formal groupoid, there are nilpotent ideals I and J of F such that ν mod$I$ is the identity and ϕ mod$J$ comes from some $\psi\in\langle\mathcal{Y}\rangle(F/J)$. It suffices to show that $\bar\nu:=\nu$ mod$I(I+J)$ also equals the identity. Consider ψ as a morphism of $\mathcal{D}_X$-algebras $R^\ell\to F/J\otimes\mathcal{O}_X$. One can view $\bar\nu$ as a section of $h(\mathcal{L}\underset{R^\ell}{\otimes}(I/(I+J)\otimes\mathcal{O}_X))$. This is a trivial vector space by our conditions on $\mathcal{L}$, and we are done.

(ii) Take any $\phi\in\langle\mathcal{Y}/\mathcal{G}\rangle(F)$. We will show that ϕ defines naturally a homotopy morphism $\phi_{\mathcal{C}}:\mathcal{C}_R(\mathcal{L})^\ell\to\mathcal{O}_X\otimes F$. Then (4.7.2.1) is the map $\phi\mapsto\tilde\phi:=H_0^{ch}(\phi_{\mathcal{C}})$.

Recall first the construction of $\mathcal{G}_\cdot$ from part (a) of the proof of the proposition in 1.4.15. Fix $n\geq 0$ and set $C_n:=\mathcal{C}_R(\mathcal{L})^\ell/F^{n+1}$. Choose a C_n-semi-free resolution $\Psi_n\to R^\ell$ (see 4.3.7); let $T_n^{(a)}$ be the C_n-tensor product of $a+1$ copies of Ψ_n.

[75] We extend functor $\langle\ \rangle$ to ind-affine $\mathcal{D}_X$-schemes in the obvious way.

[76] Of course, it suffices to consider $U_\cdot$ associated to a covering $\{U_\alpha\}$; then $f_\cdot$ amounts to a collection $(f_\alpha,g_{\alpha\beta})$ where $f_\alpha:U_\alpha\times\operatorname{Spec}F\to\mathcal{Y}_{U_\alpha}$ and $g_{\alpha\beta}:U_{\alpha\beta}\times\operatorname{Spec}F\to\mathcal{G}_{U_{\alpha\beta}}$ are morphisms of $\mathcal{D}_{U_\alpha}$- and $\mathcal{D}_{U_{\alpha\beta}}$-schemes (here $U_{\alpha\beta}:=U_\alpha\underset{X}{\times}U_\beta$) such that $g_{\alpha\beta}$ connects the restriction of f_α, f_β to $U_{\alpha\beta}$, and $g_{\alpha\beta}g_{\beta\gamma}=g_{\alpha\gamma}$ on $U_{\alpha\beta\gamma}$.

Then $T_n^{(\cdot)}$ is naturally a cosimplicial DG $\mathcal{D}_X$-algebra. One has $H^{>0}T_n^{(\cdot)} = 0$, so $\tau_{\leq 0}T_n^{(\cdot)} \hookrightarrow T_n^{(\cdot)}$ is a quasi-isomorphism. Set $E_n^{(\cdot)} := H^0T_n^{(\cdot)}$; then $\operatorname{Spec} E_n^{(\cdot)}$ is the nth infinitesimal neighborhood of $\operatorname{Spec} R^\ell \subset \mathcal{G}_\cdot$.

Write ϕ as a pair $(U_\cdot, f_\cdot)$ (see above). Choose n sufficiently large so that $f_\cdot$ takes values in $\operatorname{Spec} E_n^{(\cdot)} \subset \mathcal{G}_\cdot$. Consider the evident morphisms of cosimplicial DG $\mathcal{D}_X$-algebras[77]

$$\mathcal{C}_R(\mathcal{L})^\ell \twoheadrightarrow C_n \to T_n^{(\cdot)} \hookleftarrow \tau_{\leq 0}T_n^{(\cdot)} \to E_n^{(\cdot)} \xrightarrow{f_\cdot} \mathcal{O}_{U_\cdot} \otimes F \hookleftarrow \mathcal{O}_X \otimes F. \tag{4.7.2.2}$$

Choose a homotopy inverse functor which assigns to a cosimplicial DG $\mathcal{D}_X$-algebra a DG $\mathcal{D}_X$-algebra (e.g., the Thom-Sullivan construction from [**HS**] will do). Apply it to (4.7.2.2); we get a sequence morphisms of DG $\mathcal{D}_X$-algebras where every arrow directed to the left is a homotopy equivalence. Our $\phi_{\mathcal{C}}$ is the composition. As a morphism in the homotopy category, it does not depend on the auxiliary choices involved.

(iii) It remains to check that the morphism $\langle \mathcal{Y}/\mathcal{G}\rangle \to \operatorname{Spf} H_0^{ch}(X, \mathcal{C}_R(\mathcal{L}))$, $\phi \mapsto \tilde{\phi}$, is an isomorphism. Both spaces are formal neighborhoods of $\operatorname{Spec} H_0^{ch}(X, R)$, so it suffices to show that for any ϕ as above we have an isomorphism between the first infinitesimal neighborhoods of ϕ and $\tilde{\phi}$. Fix an extension F' of F by an ideal J of square 0, and consider the sets of liftings $\mathcal{E}$, $\tilde{\mathcal{E}}$ of ϕ, $\tilde{\phi}$ to F'. We want to show that the map $\mathcal{E} \to \tilde{\mathcal{E}}$, $\phi' \mapsto \tilde{\phi}'$, is bijective.

Set $K := {}^L\Omega_{\mathcal{C}_R(\mathcal{L})} \overset{L}{\underset{\phi_{\mathcal{C}}}{\otimes}} (\mathcal{O}_X \otimes F) \in D\mathcal{M}(X, \mathcal{O}_X \otimes F)$ where ${}^L\Omega_{\mathcal{C}_R(\mathcal{L})}$ is the cotangent complex (see 4.6.6). It follows from (4.6.6.1), (4.6.5.1), (4.6.5.2) that the pull-back of the cotangent complex ${}^L\Omega_{C^{ch}(X, \mathcal{C}_R(\mathcal{L}))}$ by the composition of morphisms $C^{ch}(X, \mathcal{C}_R(\mathcal{L})) \to H_0^{ch}(X, \mathcal{C}_R(\mathcal{L})) \xrightarrow{\phi_{\mathcal{C}}} F$ equals $R\Gamma_{DR}(X, K)[1] \in D(F)$. Then $\tilde{\mathcal{E}}$ is controlled by this complex in the usual way. Namely, there is a class $\tilde{c} \in \operatorname{Ext}^1_F(R\Gamma_{DR}(X, K)[1], J) = \operatorname{Ext}^1(K, \omega_X \otimes J)$ which vanishes if and only if $\tilde{\mathcal{E}} \neq \emptyset$; if $c = 0$, then $\tilde{\mathcal{E}}$ is a $\operatorname{Hom}_F(R\Gamma_{DR}(X, K)[1], J) = \operatorname{Hom}(K, \omega_X \otimes J)$-torsor.[78]

Locally on X our $\phi_{\mathcal{C}}$ factors through a morphism $f : R^\ell \to \mathcal{O}_X \otimes F$. Therefore the canonical exact triangle ${}^L\Omega_{\mathcal{C}_R(\mathcal{L})} \overset{L}{\underset{\mathcal{C}_R(\mathcal{L})}{\otimes}} R \to {}^L\Omega_R \to \mathcal{L}^\circ$ in $D\mathcal{M}(X, R^\ell)$ shows that $H^{>1}K = 0$ and c vanishes locally on X if and only if f lifts (locally) to $f' : R^\ell \to \mathcal{O}_X \otimes F'$. We can assume that the latter condition holds (otherwise both $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are empty). Then $\tilde{c} \in \operatorname{Ext}^1(\tau_{\geq 0}K, \omega_X \otimes J) \subset \operatorname{Ext}^1(K, \omega_X \otimes J)$;[79] if $\tilde{c} = 0$, then $\tilde{\mathcal{E}}$ is a $\operatorname{Hom}(\tau_{\geq 0}K, \omega_X \otimes J)$-torsor.

Let us describe $\tau_{\geq 0}K$. For any $a \geq 1$ and $i \in [0, a]$ the sheaf of relative 1-forms with respect to the ith boundary projection $\mathcal{G}_a \to \mathcal{G}_{a-1}$ identifies canonically with $p_i^*\mathcal{L}^\circ$ where $\pi_i : \mathcal{G}_a \to \mathcal{Y}$ is the ith structure projection (see the proof of the proposition in 1.4.15), so we have a canonical morphism $\chi_a : \Omega_{\mathcal{G}_a} \to \underset{i \in [o,a]}{\oplus} \pi_i^*\mathcal{L}^\circ$. Let $\chi_0 : \Omega_{\mathcal{Y}} \to \mathcal{L}^\circ$ be the coaction morphism. Then the $\chi_\cdot$ form a morphism of simplicial $\mathcal{D}_X$-modules on $\mathcal{G}_\cdot$; set $\Xi_\cdot := \mathcal{C}one(\chi_\cdot)[-1]$. This complex has the property that for every simplicial structure map $\partial : \mathcal{G}_a \to \mathcal{G}_b$ the corresponding morphism $\partial^*\Xi_b \to \Xi_a$ is a quasi-isomorphism.

[77] Here $\mathcal{O}_{U_\cdot}$ is the cosimplicial $\mathcal{O}_X$-algebra such that $\operatorname{Spec} \mathcal{O}_{U_\cdot} = U_\cdot$.

[78] We use the global duality for the de Rham cohomology.

[79] More precisely, $\tilde{c} \in \operatorname{Ker}(\operatorname{Ext}^1(\tau_{\geq 0}K, \omega_X \otimes J) \to \operatorname{Ext}^1(H^0K, \omega_X \otimes J))$.

Thus we get a complex of cosimplicial $(\mathcal{O}_{U_\cdot} \otimes F)[\mathcal{D}_X]$-modules $f_\cdot^*\Xi$. The total complex $\operatorname{tot} f_\cdot^*\Xi$ is a complex of $(\mathcal{O}_X \otimes F)[\mathcal{D}_X]$-modules. It follows from the construction of $\phi_{\mathcal{C}}$ that it identifies canonically with $\tau_{\geq 0}K$.

Now $\mathcal{E}$ is controlled by the Hom complex $\operatorname{Hom}(\operatorname{tot} f_\cdot^*\Xi, \operatorname{tot}\omega_{U_\cdot} \otimes J)$ (we assume that U_0 is affine). Since $\mathcal{L}^\circ$ is locally projective, one has $H^1\operatorname{Hom}(\operatorname{tot} f_\cdot^*\Xi, \operatorname{tot}\omega_{U_\cdot} \otimes J) \subset \operatorname{Ext}^1(\tau_{\geq 0}K, \omega_X \otimes J)$, $H^0\operatorname{Hom}(\operatorname{tot} f_\cdot^*\Xi, \operatorname{tot}\omega_{U_\cdot} \otimes J) \xrightarrow{\sim} \operatorname{Hom}(\tau_{\geq 0}K, \omega_X \otimes J)$. One checks that the embedding transforms the obstraction c to non-emptiness of $\mathcal{E}$ to $\tilde{c}$, and if $c = 0$, then our map $\mathcal{E} \to \tilde{\mathcal{E}}$ is a morphism of torsors with respect to the second isomorphism map, and we are done. □

Remark. Suppose in addition that R^ℓ is a smooth $\mathcal{D}_X$-algebra, $H^0_{DR}(X, \Omega_{\langle R\rangle}) = 0$, and $H^1_{DR}(X, \Omega_{\langle R\rangle})$ is a projective $\langle R\rangle$-module, so, by (ii) in the proposition in 2.4.7, $\langle \mathcal{Y}\rangle$ is smooth. Then $\langle \mathcal{Y}/\mathcal{G}\rangle$ is a smooth formal scheme, and the normal bundle to $\langle \operatorname{Spec} R^\ell\rangle \hookrightarrow \langle \mathcal{Y}/\mathcal{G}\rangle$ is equal to $H^1_{DR}(X, \mathcal{L}_{\langle R\rangle})$.

4.7.3. The case of a coisson algebra. Suppose that R is a plain very smooth coisson algebra, so $\Omega = \Omega_R$ is a Lie* R-algebroid. As in the end of 1.4.18, consider the filtered topological (-1)-coisson algebra $\mathcal{C}^{cois}(R)$. According to (iv) in the proposition in 4.3.1, $C^{ch}(X, \mathcal{C}^{cois}(R))_{\mathcal{PQ}} = C^{ch}(X, \mathcal{C}_R(\Omega))_{\mathcal{PQ}}$ is a homotopy unital topological filtered Poisson algebra (we follow the notation of 4.7.1). As an object of the corresponding homotopy category,[80] it does not depend on the auxiliary choices of $\mathcal{P}$, $\mathcal{Q}$, so we use the notation $C^{ch}(X, \mathcal{C}^{cois}(R))$.

On the other hand, by (iii) in the proposition in 4.3.1, $C^{ch}(X, R)_{\mathcal{PQ}}$ is a homotopy unital 1-Poisson algebra. Choose its semi-free resolution $\psi : \Phi \to C^{ch}(X, R)_{\mathcal{PQ}}$ as a 1-Poisson algebra. Then $\Omega_\Phi[-1]$ is a Lie Φ-algebroid. It is semi-free as a Φ-module and is perfect by 4.6.9. Consider the topological filtered Poisson algebra $\mathcal{C}^{pois}(\Phi)$; as a mere topological filtered algebra it equals the de Rham-Chevalley algebra of the Lie algebroid $\Omega_\Phi[-1]$ (see 2.9.1).[81] As an object of the homotopy category of topological filtered Poisson algebras, $\mathcal{C}^{pois}(\Phi)$ does not depend on the auxiliary choices of $\mathcal{P}$, $\mathcal{Q}$, Φ, so we denote it simply by $\mathcal{C}^{pois}(C^{ch}(X, R))$.

Proposition. *There is a canonical homotopy equivalence of topological filtered Poisson algebras*

$$C^{ch}(X, \mathcal{C}^{cois}(R)) \xrightarrow{\sim} \mathcal{C}^{pois}(C^{ch}(X, R)). \tag{4.7.3.1}$$

Proof. We can assume that the quasi-isomorphism $\psi : \Phi \to C^{ch}(X, R)_{\mathcal{PQ}}$ of 1-Poisson algebras is surjective. Then ψ yields a morphism $(\Phi, \Omega_\Phi[-1]) \to (C^{ch}(X, R)_{\mathcal{PQ}}, \Omega_{C^{ch}(X,R)_{\mathcal{PQ}}}[-1])$ in $\mathcal{L}ie\mathcal{A}lg$. The arrow (coming from (4.6.6.2)) $\Omega_{C^{ch}(X,R)_{\mathcal{PQ}}}[-1] \to C^{ch}(X, R, \Omega)_{\mathcal{PQ}}$ is a morphism of Lie $C^{ch}(X, R)_{\mathcal{PQ}}$-algebroids; the composition $\Omega_\Phi[-1] \to C^{ch}(X, R, \Omega)_{\mathcal{PQ}}$ is a quasi-isomorphism by (4.6.6.1). Set $P := C^{ch}(X, R)_{\mathcal{PQ}} \underset{\Phi}{\otimes} \Omega_\Phi[-1]$; the morphism $P \to C^{ch}(X, R, \Omega)_{\mathcal{PQ}}$ is a quasi-isomorphism of Lie $C^{ch}(X, R)_{\mathcal{PQ}}$-algebroids since Ω_Φ is a semi-free Φ-module. We

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[80] Which coincides with the homotopy category of (unital) filtered Poisson algebras, i.e., Poisson algebras A equipped with a filtration $A = A^0 \supset A^1 \supset \cdots$ by DG ideals such that $\{A^i, A^j\} \subset A^{i+j-1}$, and homotopy equivalences are morphisms that induce quasi-isomorphisms between gr's. One can assume that filtrations are complete, i.e., $A = \varprojlim A/A^i$.

[81] In 2.9.1 we considered the unital setting; in the present homotopy unital setting the definition of the de Rham-Chevalley complex should be modified as in 4.7.1, so $\mathcal{C}^{pois}(\Phi)$ equals $\Phi \oplus \operatorname{Hom}_\Phi(\operatorname{Sym}^{>0}\Omega_\Phi, \Phi)$ as a mere topological graded module.

get the filtered quasi-isomorphisms of DG algebras

$$C^{ch}(X, \mathcal{C}^{cois}(R))_{\mathcal{PQ}} \to \mathcal{C}_{C^{ch}(X,R)_{\mathcal{PQ}}}(P) \leftarrow \mathcal{C}^{pois}(\Phi) \tag{4.7.3.2}$$

where the left arrow is (4.7.1.2). Let $T \hookrightarrow C^{ch}(X, \mathcal{C}^{cois}(R))_{\mathcal{PQ}} \times \mathcal{C}^{pois}(\Phi)$ be the fibered product of these arrows. Our T carries the induced filtration, and both projections

$$C^{ch}(X, \mathcal{C}^{cois}(R))_{\mathcal{PQ}} \leftarrow T \to \mathcal{C}^{pois}(\Phi) \tag{4.7.3.3}$$

are filtered quasi-isomorphisms. One checks that T is a Poisson subalgebra of $C^{ch}(X, \mathcal{C}^{cois}(R))_{\mathcal{PQ}} \times \mathcal{C}^{pois}(\Phi)$, and we are done. □

4.7.4. The formal neighbourhood of global opers. We are in the situation of 4.7.3. Suppose in addition that $H^0_{DR}(X, \Omega_{\langle R\rangle}) = 0$ and $H^1_{DR}(X, \Omega_{\langle R\rangle})$ is a projective $\langle R\rangle$-module, so $C^{ch}(X, R) \xrightarrow{\sim} \langle R\rangle$ is a smooth algebra. Its homotopy 1-Poisson algebra structure yields a filtered topological Poisson algebra $\mathcal{C}^{pois}(\langle R\rangle)$ which is concentrated in degree 0. The corresponding reduced algebra equals $\langle R\rangle$, and $\mathrm{Spec}\langle R\rangle \hookrightarrow \mathrm{Spf}\, \mathcal{C}^{pois}(\langle R\rangle)$ is a Lagrangian embedding. According to 4.7.3 and (4.7.2.1), one has a canonical identification

$$\mathrm{Spf}\, \mathcal{C}^{pois}(\langle R\rangle) \xrightarrow{\sim} \langle \mathrm{Spec}\, R^{\ell}/\mathcal{G}\rangle \tag{4.7.4.1}$$

where $\mathcal{G}$ is the formal $\mathcal{D}_X$-scheme groupoid on $\mathrm{Spec}\, R^{\ell}$ defined by the Lie* algebroid Ω_R.

Here is an important example of this situation. Let R be the Gelfand-Dikii coisson algebra W^{κ}_c for a non-degenerate κ, so $\mathrm{Spec}\, R^{\ell} = \mathcal{O}p_{\mathfrak{g}}$ (see 2.6.8). If our curve X has genus > 1, then the above conditions are satisfied, so $\mathrm{Spec}\langle R\rangle = \mathcal{O}p_{\mathfrak{g}}(X) =$ the space of global $\mathfrak{g}$-opers on X has a canonical Lagrangian embedding into a symplectic formal tube $\mathrm{Spf}\, \mathcal{C}^{pois}(\langle R\rangle)$. It is known (see [**BD**]) that the forgetting-of-B-structure map $(\mathfrak{F}_B, \nabla) \mapsto (\mathfrak{F}_G, \nabla)$ is a closed embedding of $\mathcal{O}p_{\mathfrak{g}}(X)$ into the moduli space $\mathcal{L}oc\mathcal{S}ys_G$ of G-bundles with connection on X. It follows then from (4.7.4.1) and the second proposition in 2.6.8 that $\mathrm{Spf}\, \mathcal{C}^{pois}(\langle R\rangle)$ coincides with the formal neighbourhood of $\mathcal{O}p_{\mathfrak{g}}$ in $\mathcal{L}oc\mathcal{S}ys_G$.

Remark. One can show that the latter identification is compatible with symplectic structures where $\mathcal{L}oc\mathcal{S}ys_G$ is equipped with the usual symplectic structure defined by κ.

4.8. Chiral homology of chiral envelopes

We show that the chiral homology of the chiral enveloping algebra of a Lie* algebra L is equal to the homology of the homotopy Lie algebra $R\Gamma_{DR}(X, L)$. Similar facts hold for the chiral homology with coefficients, in the twisted setting, and in the chiral Lie algebroid setting. As an application, we show that the chiral homology of a cdo on a very smooth $\mathcal{D}_X$-scheme $\mathrm{Spec}\, R^{\ell}$ can be interpreted as a certain twisted de Rham homology of $\mathrm{Spec}\, C^{ch}(X, R^{\ell})$. In particular, it is finite-dimensional.

4.8.1. Enveloping algebras of Lie* algebras. Let L be a Lie* DG algebra on X. We have a homotopy Lie algebra $R\Gamma_{DR}(X, L)$ (see 4.5.1) which yields the Chevalley homology complex $C(R\Gamma_{DR}(X, L))$ canonically defined as an object of

$\mathcal{H}o\mathcal{B}\mathcal{V}_u$; similarly, we have the reduced complex $\bar{C}(R\Gamma_{DR}(X,L)) \in \mathcal{H}o\bar{\mathcal{B}\mathcal{V}}$ (see (a) in 4.1.8).

Consider the chiral envelope $U(L)$. The standard commutative filtration defines, by 4.3.4, a homotopy filtered unital BV algebra structure on $C^{ch}(X,U(L))$; i.e., we have $C^{ch}(X,U(L)) \in \mathcal{H}o\mathcal{B}\mathcal{V}_u$.

Theorem. *There is a canonical morphism in* $\mathcal{H}o\mathcal{B}\mathcal{V}_u$

$$C(R\Gamma_{DR}(X,L)) \to C^{ch}(X,U(L)) \tag{4.8.1.1}$$

which is an isomorphism if L is $\mathcal{O}_X$-flat. Therefore $H^{ch}_\cdot(X,U(L))$ is the homology of the homotopy Lie algebra $R\Gamma_{DR}(X,L)$.

Proof. Denote by $U(L)^{>0}$ the kernel of the augmentation morphism $U(L) \to \omega$ (which sends L to 0). This is a non-unital chiral algebra, and $U(L) = (U(L)^{>0})^+$. Thus, by Remark in 4.4.8, $C^{ch}(X,U(L)) \in \mathcal{H}o\mathcal{B}\mathcal{V}_u$ is obtained from the non-unital algebra $C^{ch}(X,U(L)^{>0}) \in \mathcal{H}o\bar{\mathcal{B}\mathcal{V}}$ by adding the unit. We will define a canonical morphism in $\mathcal{H}o\bar{\mathcal{B}\mathcal{V}}$

$$\bar{C}(R\Gamma_{DR}(X,L)) \to C^{ch}(X,U(L)^{>0}) \tag{4.8.1.2}$$

which is an isomorphism when L is $\mathcal{O}_X$-flat. Adding the unit, one gets (4.8.1.1).

Let us represent $C^{ch}(X,U(L)^{>0})$ by a filtered BV algebra $C^{ch}(X,U(L)^{>0})_{\mathcal{P}\mathcal{Q}}$ (see (iv) in 4.3.2 and 4.2.18) and $R\Gamma_{DR}(X,L)$ by a Lie algebra $\Gamma(X,h(L_{\mathcal{P}\mathcal{Q}}))$ (see 4.5.1). The embeddings of the Lie algebras $\Gamma(X,h(L_{\mathcal{P}\mathcal{Q}})) \hookrightarrow \Gamma(X,h(U(L)^{>0}_{\mathcal{P}\mathcal{Q}})) \hookrightarrow C^{ch}(X,U(L)^{>0})_{\mathcal{P}\mathcal{Q}}[-1]$ identify $\Gamma(X,h(L_{\mathcal{P}\mathcal{Q}}))$ with the first term of the filtration on $C^{ch}(X,U(L)^{>0})_{\mathcal{P}\mathcal{Q}}[-1]$. By adjunction (see (a) in 4.1.8), one has a morphism $\alpha : \bar{C}(\Gamma(X,h(L_{\mathcal{P}\mathcal{Q}}))) \to C^{ch}(X,U(L)^{>0})_{\mathcal{P}\mathcal{Q}}$ of filtered BV algebras which is our (4.8.1.2).

Consider the morphism $\mathrm{Sym}^{>0}(\Gamma(X,h(L_{\mathcal{P}\mathcal{Q}}))[1]) \xrightarrow{\mathrm{gr}\,\alpha} \mathrm{gr}.C^{ch}(X,U(L)^{>0})_{\mathcal{P}\mathcal{Q}} = C^{ch}(X,\mathrm{gr}\,U(L)^{>0})_{\mathcal{P}\mathcal{Q}}$ (see (4.2.18.3)). It equals the composition of the map (4.6.2.2) for $V = L$ and the map $C^{ch}(X,\mathrm{Sym}^{>0}L) \to C^{ch}(X,\mathrm{gr}\,U(L)^{>0})$ coming from the canonical morphism $\mathrm{Sym}^{>0}L \to \mathrm{gr}U(L)^{>0}$. If L is $\mathcal{O}_X$-flat, then the latter map is a quasi-isomorphism by the Poincaré-Birkhoff-Witt (see 3.7.14), and the former one is a quasi-isomorphism by the proposition in 4.6.2. We are done. □

Here are some variants of the above theorem:

4.8.2. Coefficients. Let $T \subset X$ be a finite non-empty subset and $j_{U_T} : U_T := X \smallsetminus T \hookrightarrow X$ its complement. Suppose we have a Lie* algebra L on U_T and for every $t \in T$ a chiral L-module M_t supported at t (see 3.7.16–3.7.19).

We have the enveloping chiral algebra $U(L)$ and the M_t are $U(L)$-modules, so one has the corresponding chiral homology complex $C^{ch}(X,U(L),\{M_s\})$; the standard filtration on $U(L)$ makes it a filtered complex. On the other hand, we have a homotopy Lie algebra $R\Gamma_{DR}(U_T,L)$ and each $h(M_t)$ is an $R\Gamma_{DR}(U_T,L)$-module, so we have the corresponding Lie algebra homology complex $C(R\Gamma_{DR}(U_T,L),\otimes h(M_t))$ filtered in the obvious way.

Proposition. *There is a canonical morphism of filtered complexes*

$$C(R\Gamma_{DR}(U_T,L),\otimes h(M_t)) \to C^{ch}(X,U(L),\{M_t\}), \tag{4.8.2.1}$$

which is a filtered quasi-isomorphism if L is $\mathcal{O}_X$-flat.

Proof. Let us represent $R\Gamma_{DR}(U_T, L)$ by a Lie algebra $\Gamma(U_T, h(L_{\mathcal{P}}))$, so the $h(M_t)$ are $\Gamma(U_T, h(L_{\mathcal{P}}))$-modules, and we have the corresponding filtered Chevalley complex $C(\Gamma(U_T, h(L_{\mathcal{P}})), \otimes h(M_t))$. Similarly, $C^{ch}(X, U(L), \{M_t\})$ comes as the filtered complex $C^{ch}(U_T, U(L), \{M_t\})_{\mathcal{P}}$ (see (4.2.19.3)). The morphism of Lie* algebras $L_{\mathcal{P}} \to U(L)_{\mathcal{P}}$ yields then in the obvious way a morphism of filtered complexes $C(\Gamma(U_T, h(L_{\mathcal{P}})), \otimes h(M_t)) \to C^{ch}(U_T, U(L), \{M_t\})_{\mathcal{P}}$ which is our (4.8.2.1).

Let us show that (4.8.2.1) is a filtered quasi-isomorphism if L is $\mathcal{O}_X$-flat. By (4.2.18.3) and the PBW theorem in 3.7.14, one can pass to gr reducing our statement to the case when L is commutative and its action on M_t is trivial. Then one has $C(\Gamma(U_T, h(L_{\mathcal{P}})), \otimes h(M_t)) = (\otimes h(M_t)) \otimes (k \oplus \mathrm{Sym}^{>0}(\Gamma(X, h(j_{T*}j_T^* L_{\mathcal{P}}))[1]))$ and $C^{ch}(U_T, U(L), \{M_t\})_{\mathcal{P}} = (\otimes h(M_t)) \otimes (k \oplus C^{ch}(X, j_{T*}j_T^*(\mathrm{Sym}^{>0}L)_{\mathcal{P}}))$. Our morphism is the tensor product of the identity map for $\otimes h(M_t)$ and the direct sum of id_k and the arrow (4.6.2.2) for $V = j_{T*}j_T^* L$. We are done by 4.6.2. $\square$

4.8.3. Twisted enveloping algebras. Let L be a Lie* DG algebra on X, $L^\flat$ its ω-extension, and $U(L)^\flat$ the corresponding twisted chiral envelope (see 3.7.20). The standard filtration $U(L)^\flat_\cdot$ defines a homotopy unital BV structure on the chiral complex, so we have $C^{ch}(X, U(L)^\flat_\cdot) \in \mathcal{H}o\mathcal{BV}_u$ (see 4.3.4). Explicitly, it is represented by the homotopy unital filtered BV algebra $C^{ch}(X, U(L)^\flat)_{\mathcal{P}\mathcal{Q}}$.

As in 4.5.1, we have a Lie algebra $\Gamma(X, h(L^\flat_{\mathcal{P}\mathcal{Q}}))$ which is a central extension of $\Gamma(X, h(L_{\mathcal{P}\mathcal{Q}}))$ by $\Gamma(X, h(\omega_{\mathcal{P}\mathcal{Q}}))$. The latter complex computes the de Rham homology of X (shifted by 1), so we have a trace map $\mathrm{tr} : \Gamma(X, h(\omega_{\mathcal{P}\mathcal{Q}})) \to k[-1]$ which is canonical up to a homotopy. Pushing out our central extension by tr, we get a central $k[-1]$-extension $\Gamma(X, h(L_{\mathcal{P}\mathcal{Q}}))^\flat$ of $\Gamma(X, h(L_{\mathcal{P}\mathcal{Q}}))$ by $k[-1]$. It yields the twisted Chevalley complex $C(\Gamma(X, h(L_{\mathcal{P}\mathcal{Q}})))^\flat$ which is a filtered unital BV algebra (see (c) in 4.1.8). As an object of the homotopy category $\mathcal{H}o\mathcal{BV}_u$, it does not depend on the auxiliary choices; we denote it by $C(R\Gamma_{DR}(X, L))^\flat \in \mathcal{H}o\mathcal{BV}_u$.

PROPOSITION. *For an $\mathcal{O}_X$-flat L there is a canonical isomorphism in $\mathcal{H}o\mathcal{BV}_u$*

$$C^{ch}(X, U(L)^\flat_\cdot) \xrightarrow{\sim} C(R\Gamma_{DR}(X, L))^\flat. \tag{4.8.3.1}$$

Sketch of a proof. One can either repeat the arguments in the non-twisted case or reduce the twisted situation to the non-twisted one using the fact that $U(L)^\flat$ is a specialization of $U(L^\flat)$ and applying 4.3.9. The details are left to the reader. $\square$

The picture of 4.8.2 admits the following twisted versions (the proofs are similar to the proof of the proposition in 4.8.2):

(i) Let T and U_T be as in 4.8.2, let L be an $\mathcal{O}$-flat Lie* algebra on U_T and $L^\flat$ its ω-extension. We have the Lie algebra $\Gamma(U_T, h(L_{\mathcal{P}}))$ and its central $\Gamma(U_T, h(\omega_{\mathcal{P}}))$-extension $\Gamma(U_T, h(L^\flat_{\mathcal{P}}))$.

Let $\{M_t\}$, $t \in T$, be chiral $U(L)^\flat$-modules supported at $t \in T$. Then $\otimes h(M_t)$ is naturally a $\Gamma(U_T, h(L_{\mathcal{P}}))$-module (indeed, it is a $\Gamma(U_T, h(L^\flat_{\mathcal{P}}))$-module, and the action of $\Gamma(U_T, h(\omega_{\mathcal{P}}))$ is trivial). We have an evident morphism

$$C(\Gamma(U_T, h(L_{\mathcal{P}})), \otimes h(M_t)) \to C^{ch}(X, U(L)^\flat, \{M_t\})_{\mathcal{P}} \tag{4.8.3.2}$$

which is a filtered quasi-isomorphism.

(ii) Let L be an $\mathcal{O}$-flat Lie* algebra on X and $L^\flat$ its ω-extension. For $T \in \mathcal{S}$ consider the corresponding Lie algebra $L^\natural_{\mathcal{P}X^T}$ in the tensor category of left $\mathcal{D}_{X^T}$-modules (see 3.7.6) and its central extension $L^{\flat\natural}_{\mathcal{P}X^T}$ by $\omega^\natural_{\mathcal{P}X^T}$.

Let $\{M_t\}$ be a T-family of chiral $U(L)^\flat$-modules. Then $j^{(T)*}\boxtimes M_t$ is an $L^\natural_{\mathcal{P}U^{(T)}}$-module (as in (i), $j^{(T)*}\boxtimes M_t$ carries a natural $L^{\flat\natural}_{\mathcal{P}U^{(T)}}$-action which factors through $L^\natural_{\mathcal{P}U^{(T)}}$). We have the Lie algebra homology complex $C(L^\natural_{\mathcal{P}U^{(T)}}, j^{(T)*}\boxtimes M_t)$ and an evident morphism of filtered complexes of $\mathcal{D}_{X^T}$-modules

$$j^{(T)}_* C(L^\natural_{\mathcal{P}U^{(T)}}, j^{(T)*}\boxtimes M_t) \to \mathcal{C}^{ch}(X, U(L)^\flat, \{M_t\})_{\mathcal{P}} \tag{4.8.3.3}$$

which is a filtered quasi-isomorphism. Notice also that $j^{(T)}_* C(L^\natural_{\mathcal{P}U^{(T)}}, j^{(T)*}\boxtimes M_t) = C(L^\natural_{\mathcal{P}X^T}, j^{(T)}_* j^{(T)*}\boxtimes M_t)$.

4.8.4. Chiral differential operators. Let R^ℓ be a commutative unital $\mathcal{D}_X$-algebra, $\mathcal{L}$ a Lie* R-algebroid, $\mathcal{L}^\flat$ its chiral R-extension (see 3.9.6). Below we fix $\mathcal{P}$, $\mathcal{Q}$ and write $C^{ch}(\cdot)$ for $C^{ch}(X,\cdot)_{\mathcal{PQ}}$.

Consider the chiral envelope $U(\mathcal{L})^\flat$ (see 3.9.11) equipped with the PBW filtration. It yields a filtered homotopy unital BV algebra $C^{ch}(U(R,\mathcal{L})^\flat)$ (see 4.3.2 and 4.3.4).

On the other hand, we have a homotopy unital commutative algebra $C^{ch}(R)$ and a homotopy unital $C^{ch}(R)$-module $C^{ch}(R,\mathcal{L}) := C^{ch}(X,R,\mathcal{L})_{\mathcal{PQ}}$ (see 4.3.2 and 4.3.4). The latter is naturally a Lie $C^{ch}(R)$-algebroid: the Lie bracket comes in the obvious manner from the Lie* bracket on $\mathcal{L}_{\mathcal{PQ}}$ and the $\mathcal{L}_{\mathcal{PQ}}$-action on $R_{\mathcal{PQ}}$, and the action on $C^{ch}(R)$ comes from the $\mathcal{L}_{\mathcal{PQ}}$-action on $R_{\mathcal{PQ}}$.

The complex $C^{ch}(R,\mathcal{L}^\flat)$ is an extension of $C^{ch}(R,\mathcal{L})$ by $C^{ch}(R,R)$. Let $C^{ch}(R,\mathcal{L})^\flat$ be its push-out by the obvious map $C^{ch}(R,R)\to C^{ch}(R)[-1]$. Our $C^{ch}(R,\mathcal{L})^\flat$ is naturally a homotopy unital BV extension of $C^{ch}(R,\mathcal{L})$ (see 4.1.9 and 4.1.15). Namely, the Lie bracket on $C^{ch}(R,\mathcal{L})^\flat$ comes from the Lie* bracket on $\mathcal{L}^\flat_{\mathcal{PQ}}$ and the $\mathcal{L}_{\mathcal{PQ}}$-action on $R_{\mathcal{PQ}}$, and the $C^{ch}(R)$-module structure is the obvious "exterior product" map $C^{ch}(R)\otimes C^{ch}(R,\mathcal{L})^\flat \to C^{ch}(R,\mathcal{L})^\flat$. The $C^{ch}(R)$-action does *not* commute with the differential due to the fact that $\mathcal{L}^\flat$ is *not* a central R-module; the discrepancy is given by axiom (ii) of BV extensions (see 4.1.9)[82] following from the definition of chiral extension (see 3.9.6).

There is an obvious map $C^{ch}(R,\mathcal{L})^\flat \to C^{ch}(U(R,\mathcal{L})^\flat)[-1]$ compatible with the embeddings of $C^{ch}(R)[-1]$. It is also compatible with the Lie bracket and the $C^{ch}(R)$-action, and its image lies in the first term of the filtration. By universality we get a morphism $C_{BV}(C^{ch}(R), C^{ch}(R,\mathcal{L}))^\flat \to C^{ch}(U(R,\mathcal{L})^\flat)$ of the homotopy unital filtered BV algebras, hence a morphism in $\mathcal{H}o\mathcal{BV}_u$ (see 4.1.9 and 4.1.15)

$$C^L_{BV}(C^{ch}(R), C^{ch}(R,\mathcal{L}))^\flat \to C^{ch}(U(R,\mathcal{L})^\flat). \tag{4.8.4.1}$$

THEOREM. *If R is homotopically $\mathcal{O}_X$-flat and $\mathcal{L}$ is a homotopically flat R^ℓ-module, then this is a filtered quasi-isomorphism.*

Proof. By PBW, $\mathrm{Sym}_R\mathcal{L} \xrightarrow{\sim} \mathrm{gr}\, U(R,\mathcal{L})^\flat$. Now use (4.2.18.3) and 4.6.4. □

EXERCISE. Suppose that R, $\mathcal{L}$ have degree 0. By (3.9.21.1), the category of chiral extensions of $\mathcal{L}$ is a torsor over the 2-term complex $\tau_{\leq 2}R\Gamma(X, h(F^1\mathcal{C}_R(\mathcal{L}))) = \tau_{\leq 2}R\Gamma_{DR}(X, F^1\mathcal{C}_R(\mathcal{L}))$ where F is the stupid filtration. The category of homotopy BV extensions of the homotopy Lie $C^{ch}(R)$-algebroid $C^{ch}(R,\mathcal{L})$ is a torsor over the 2-term complex $\tau_{\leq 2}\tau_{\geq 1}F^1\mathcal{C}_{C^{ch}(R)}(C^{ch}(R,\mathcal{L})) = \tau_{\leq 2}\tau_{\geq 1}C^{ch}(F^1\mathcal{C}_R(\mathcal{L}))$

[82] We sincerely apologize for a hideous incongruency of notation: R and $\mathcal{L}^\flat$ from 4.1.9 are now $C^{ch}(R)$ and $C^{ch}(R,\mathcal{L})^\flat$.

(see (4.6.10.1)). The canonical morphism $R\Gamma_{DR}(X, F^1\mathcal{C}_R(\mathcal{L})) \to C^{ch}(X, F^1\mathcal{C}_R(\mathcal{L}))$ maps the first truncated complex to the second one. Show that the functor $\mathcal{L}^\flat \mapsto C^{ch}(R,\mathcal{L})^\flat$ from the category of chiral extensions of $\mathcal{L}$ to the one of homotopy BV extensions of $C^{ch}(R,\mathcal{L})$ is affine with respect to the map $\mathfrak{e} : \tau_{\leq 2}R\Gamma_{DR}(X, F^1\mathcal{C}_R(\mathcal{L})) \to \tau_{\leq 2}\tau_{\geq 1}C^{ch}(F^1\mathcal{C}_R(\mathcal{L}))$.

Example. Let Y be a smooth affine variety, $X = \mathbb{P}^1$, $\mathfrak{J}Y_X = \operatorname{Spec} R^\ell$ the jet scheme for $Y \times X/X$, $\mathcal{L} = \Theta_R$. Then $\operatorname{Spec}\langle R\rangle = Y$ and $H^{ch}_{>0}(X,R) = 0$. So for a cdo A on Y the complex $C^{ch}(X,A)$ is the de Rham complex of Y with coefficients in $\mathcal{O}_Y$ with respect to certain right $\mathcal{D}_Y$-module structure on $\mathcal{O}_Y$, which is the same as a flat connection ∇^A on ω_Y^{-1}. The isomorphism classes of cdo form an $H^2_{DR}(X, DR(\mathfrak{J}Y_X/X)^{\geq 1}) = H^2(X, hDR(\mathfrak{J}Y_X/X)^{\geq 1})$-torsor. The “constant jet” embedding of $\mathcal{D}_X$-schemes $Y \times X \hookrightarrow \mathfrak{J}Y_X$ yields a morphism of $\mathcal{D}_X$-modules $DR(\mathfrak{J}Y_X/X) \to DR(Y) \otimes \omega_X$. Passing to de Rham cohomology and applying $\mathrm{tr} : R\Gamma_{DR}(X,\omega_X)[1] \xrightarrow{\sim} k$, we get a morphism $\mathfrak{e} : H^2_{DR}(X, DR(\mathfrak{J}Y_X/X)^{\geq 1}) \to H^0DR(Y)^{\geq 1}$ = the closed 1-forms on Y. By the exercise above, the map $A \mapsto \nabla^A$ is $\mathfrak{e}$-affine.

Notice that the canonical projection $\mathfrak{J}Y_X \to Y \times X$ yields a morphism of complexes $DR(Y) \otimes \mathcal{O}_X \to DR(\mathfrak{J}Y_X/X)^\ell$, hence the one $DR(Y)^{\geq 1} \otimes \omega_X \to DR(\mathfrak{J}Y_X/X)^{\geq 1} \to hDR(\mathfrak{J}Y_X/X)^{\geq 1}$. Applying the functor $H^2(X,\cdot)$, we get a right inverse to $\mathfrak{e}$. Thus $\mathfrak{e}$ is a surjective map, so ∇^A can have arbitrary irregular singularities at infinity.

Question. According to §12 of [**GMS2**], any right $\mathcal{D}_Y$-module structure on $\mathcal{O}_Y$, i.e., a flat connection ∇ on ω_Y^{-1}, yields a Virasoro vector in any cdo on the “universal” jet scheme of Y in the setting of graded vertex algebras. The latter produces an action of the the group ind-scheme $\operatorname{Aut}k[[t]]$ on the cdo, which can be planted then on the jet scheme of Y over any curve X. Take for A in the above example such a cdo. Is it true that $\nabla^A = \nabla$?

4.8.5. Corollary. *Suppose R^ℓ is a very smooth plain $\mathcal{D}_X$-algebra and A is a chiral R^ℓ-cdo. Then* $\dim H^{ch}_\cdot(X,A) < \infty$.

Proof. By the above theorem, 4.6.9, 4.6.8, and 4.6.6, $C^{ch}(X,A)$ is a perfect BV algebra (see 4.1.18), and we are done by the proposition in 4.1.18. $\square$

Question. Is it true that the chiral homology of a (formal) quantization of any symplectic coisson algebra is finite-dimensional?

Presumably, the finiteness can be seen from the first order of deformation, so the picture of 3.9.10 may provide the clue.

4.9. Chiral homology of lattice chiral algebras

The fact that conformal blocks of a lattice Heisenberg algebra with positive κ are appropriate θ-functions is standard in mathematical physics; for a mathematical proof see [**Ga**] 6.2.2 and also [**FS**]. The proof of the theorem in 4.9.3 presented below uses the Fourier-Mukai transform which helps to reduce it to the particular case of the simplest commutative lattice chiral algebra. The descent construction in 4.9.1 is a particular case of [**BD**] 4.3.12, 4.3.13.

4.9.1. We follow the notation of 4.10, so we have a lattice Γ and the corresponding torus $T = \mathbb{G}_m \otimes \Gamma$. Let $\mathcal{T}ors(X, T)$ be the algebraic Picard stack of T-torsors on X; this is an algebraic stack.

For a test affine scheme Z set $T(X)_{rat}(Z) := \varinjlim T(U)$ where the limit is over the set of all open $U \subset X \times Z$ such that the fiber of U over any point of Z is non-empty. In other words, an element of $T(X)_{rat}(Z)$ is a family of rational maps $X \to T$ parametrized by Z. One easily checks that $T(X)_{rat}$ is a sheaf for the fppf topology. We also have a sheaf of Γ-divisors $\mathcal{D}iv(X, \Gamma) := \mathcal{D}iv(X) \otimes \Gamma$ (see 3.10.7), a morphism of sheaves div: $T(X)_{rat} \to \mathcal{D}iv(X, \Gamma)$, and a canonical identification of the Picard stacks

$$\pi : \mathcal{C}one(\mathrm{div}) \xrightarrow{\sim} \mathcal{T}ors(X, T) \tag{4.9.1.1}$$

where the projection $\mathcal{D}iv(X, \Gamma) \to \mathcal{T}ors(X, T)$ is $D \otimes \gamma \mapsto \mathcal{O}(D)^{\otimes \gamma}$.

Proposition. *This projection yields an equivalence of the Picard groupoids of line bundles*

$$\mathcal{P}ic(\mathcal{T}ors(X, T)) \xrightarrow{\sim} \mathcal{P}ic(\mathcal{D}iv(X, \Gamma)). \tag{4.9.1.2}$$

Proof. It suffices to check that for any test scheme Z
- every regular function φ on $T(X)_{rat} \times Z$ comes from Z;
- every line bundle $\mathcal{M}$ on $T(X)_{rat} \times Z$ comes from Z.

It suffices to prove our statements for $T = \mathbb{G}_m$. Choose an ample line bundle $\mathcal{L}$ on X and set $V_n := H^0(X, \mathcal{L}^{\otimes n})$, $V'_n := V_n \smallsetminus \{0\}$. Define $p_n : V'_n \times V'_n \to \mathbb{G}_m(X)_{rat}$ by $(f, g) \mapsto f/g$.

Our φ defines a regular function $p_n^* \varphi$ on $V'_n \times V'_n \times Z$ which is invariant with respect to the obvious action of $\mathbb{G}_m$ on $V'_n \times V'_n$. Suppose that n is big enough, so $\dim V_n > 1$. Then $p_n^* \varphi$ extends to a $\mathbb{G}_m$-invariant regular function on $V_n \times V_n \times Z$, which necessarily comes from Z. Similarly, $p_n^* \mathcal{M}$ extends to a $\mathbb{G}_m$-equivariant line bundle on $V_n \times V_n \times Z$ whose restriction to the diagonally embedded $V_n \times Z$ comes from Z; such an object comes from a uniquely defined line bundle on Z. $\square$

4.9.2. By 3.10.7 and the above proposition we have canonical morphisms of the Picard groupoids

$$\mathcal{P}^\theta(X, \Gamma) \xrightarrow{\sim} \mathcal{P}ic^f(\mathcal{D}iv(X, \Gamma)) \to \mathcal{P}ic(\mathcal{T}ors(X, T)) \tag{4.9.2.1}$$

where $\mathcal{P}ic$ is the Picard groupoid of *super* line bundles. For $\lambda \in \mathcal{P}ic^f(\mathcal{D}iv(X, \Gamma))$ we denote the corresponding super line bundle on $\mathcal{T}ors(X, T)$ also by λ.

Example. Suppose $\Gamma = \mathbb{Z}$ and $\lambda \in \mathcal{P}ic^f(\mathcal{D}iv(X))$ is defined by formula (3.10.7.4). The corresponding super line bundle on $\mathcal{T}ors(X, \mathbb{G}_m) = \mathcal{P}ic(X)$ is $\lambda_{\mathcal{L}} = \det R\Gamma(X, \mathcal{L}) \otimes \det^{\otimes -1} R\Gamma(X, \mathcal{O}_X)$.

The group of connected components of $\mathcal{T}ors(X, T)$ equals Γ (the degree of the T-torsor), $\mathcal{T}ors(X, T) = \bigsqcup \mathcal{T}ors(X, T)_\gamma$. Each line bundle λ on $\mathcal{T}ors(X, T)$ yields a map $\delta_\lambda : \Gamma \to \Gamma^\vee$ so that for $\mathcal{F} \in \mathcal{T}ors(X, T)_\gamma$ the group $T = \mathrm{Aut}(\mathcal{F})$ acts on the fiber $\lambda_{\mathcal{F}}$ by the character $\delta_\lambda(\gamma) : T \to \mathbb{G}_m$. The map $\lambda \mapsto \delta_\lambda$ is $\mathbb{Z}$-linear: one has $\delta_{\lambda \otimes \lambda'} = \delta_\lambda + \delta_{\lambda'}$.

For $\theta \in \mathcal{P}^\theta(X, \Gamma)$ we write $\delta_\theta := \delta_\lambda$ where λ corresponds to θ by (4.9.2.1).

LEMMA. (i) *For* $\theta \in \mathcal{P}^\theta(X,\Gamma)^\kappa$ *(see 3.10.3) the map* $\delta_\theta : \Gamma \to \Gamma^\vee$ *is affine with the linear part equal to* κ, $\langle \kappa(\gamma), \gamma' \rangle = \kappa(\gamma, \gamma')$. *If* θ *is symmetric,*[83] *then* $\delta_\theta = \kappa$.

(ii) *For* $\theta \in \mathcal{P}^\theta(X,\Gamma)^0 = \mathcal{P}(X,\Gamma) = Tors(X, T^\vee)$ *(see (3.10.3.1) the map* δ_θ *is constant with image equal to the degree of the* $T^\vee$*-torsor.*

Proof. Let us check (ii) first. For $\theta \in \mathcal{P}(X,\Gamma)$ the corresponding λ is an extension of $\mathcal{T}ors(X,T)$ by $\mathbb{G}_m$ (see (3.10.7.3)). So δ_θ is constant and its value is clear from [SGA 4] Exp. XVIII 1.3. Let us prove (i). By linearity and the argument from the end of the proof of the proposition in 3.10.7, it suffices to check the first statement for $\Gamma = \mathbb{Z}$ and κ the product pairing, which follows from the example above. The second statement follows from the first one. □

REMARK. If $\delta_\lambda(\gamma) = 0$, then the restriction of λ to $\mathcal{T}ors(X,T)_\gamma$ is the pull-back of a uniquely defined line bundle on $\mathrm{Tors}\,(X,T)_\gamma$ (the coarse moduli space of classes of isomorphisms of T-torsors of degree γ) which we denote also by λ.

4.9.3. Let A be a lattice chiral algebra. Let $\theta \in \mathcal{P}^\theta(X,\Gamma)^\kappa$ be its θ-datum (see 3.10.4), λ the super line bundle on $\mathcal{T}ors(X,T)$ defined by θ, and $\lambda^* = \lambda^{\otimes -1}$ the dual bundle.

THEOREM. *There is a canonical quasi-isomorphism*

$$C^{ch}(X,A) \xrightarrow{\sim} R\Gamma(\mathcal{T}ors(X,T), \lambda^*)^* = \bigoplus_{\gamma \in \Gamma} R\Gamma(\mathcal{T}ors(X,T)_\gamma, \lambda^*)^* \tag{4.9.3.1}$$

compatible with the Γ*-gradings.*

Here the Γ-grading of $C^{ch}(X,A)$ comes from the Γ-grading of A.

REMARKS. (i) The γ-components of the right-hand side of (4.9.3.1) for which $\delta_\lambda(\gamma) \neq 0$ vanish. If $\delta_\lambda(\gamma) = 0$, then $R\Gamma(\mathcal{T}ors(X,T)_\gamma, \lambda^*) = R\Gamma(\mathrm{Tors}(X,T)_\gamma, \lambda^*)$. If κ is non-degenerate, then there is only one such γ; in particular, the chiral homology is finite-dimensional.

(ii) More precisely, the numerical class of λ^* on $Tors(X,T)_\gamma$ (which is a $J(X) \otimes \Gamma$-torsor) equals $\xi \otimes \kappa$ where ξ is the canonical polarization of the Jacobian $J(X)$. Therefore for κ non-degenerate, the only non-zero chiral homology group occurs in the degree equal to the product of g and the number of negative squares in κ; its dimension is equal to $|\det \kappa|^g$.

(iii) The description of the chiral homology of A with coefficients[84] is left to the inquisitive reader.

4.9.4. *Proof of the theorem.* First let us check that the chiral homology satisfies the same vanishing property as the right-hand side of (4.9.3.1):

LEMMA. *If* $\delta_\lambda(\gamma) \neq 0$, *then* $C^{ch}(X,A)^\gamma = 0$.

Proof. Consider the Lie* subalgebra $\alpha^\theta : \mathfrak{t}^\theta_{\mathcal{D}} \hookrightarrow A^0$ (see 3.10.9). Therefore $\mathfrak{t} = \Gamma(X, h(\mathfrak{t}_{\mathcal{D}}))$ acts on A by the adjoint action. By (ii) and (iv) in the proposition in 3.10.9, $\mathfrak{t}$ acts on each A^γ by the character $\kappa(\gamma) \in \Gamma^\vee$, so it acts on $C(X,A)^\gamma$ in the same manner. On the other hand, according to Remark in 4.5.3 and Remark (ii) in 3.10.9, $\mathfrak{t}$ acts on $C(X,A)$ by the character $-\deg \mathfrak{F}^\vee \in \Gamma^\vee$ where $\theta = \theta^{sym}\mathfrak{F}$, $\theta^{sym} \in \mathcal{P}^\theta(X,\Gamma)^\kappa$ is symmetric and $\mathfrak{F} \in \mathcal{T}ors(X,T^\vee) = \mathcal{P}(X,\Gamma)$. So if $C^{ch}(X,A)^\gamma \neq 0$, then $\kappa(\gamma) + \deg \mathfrak{F}^\vee = 0$, and we are done by the lemma from 4.9.2. □

[83] I.e., invariant with respect to the involution $\gamma \mapsto -\gamma$.

[84] The category of A-modules was described in 3.10.13 and 3.10.14.

4.9.5. We use the notation from 3.10.8.

For each $I \in \mathcal{S}$ we have an ind-X^I-scheme $\mathcal{D}iv(X,\Gamma)_{X^I} := \mathcal{D}iv(X,\Gamma)_{S,X^I}$ (where S is the standard divisor $\cup x_i$). These ind-schemes form naturally an $\mathcal{S}^\circ$-diagram. Interpreting Γ-divisors as T-torsors equipped with meromorphic trivializations, we get a canonical projection $\varphi = \varphi_I : \mathcal{D}iv(X,\Gamma)_{X^I} \to \mathcal{T}ors(X,T)$ which is formally smooth. So our diagram lives over $\mathcal{T}ors(X,T)$. Notice that the canonical connection ∇ from the remark in 3.10.8 acts along the fibers of φ. Our φ yields the projection $\phi = \phi_I : \mathcal{D}iv(X,\Gamma)_{X^I} \to \mathrm{Tors}(X,T)$.

Let us fix some $\gamma \in \Gamma$ such that $\delta_\lambda(\gamma) = 0$, and consider the γ-component of our picture. Set $\mathcal{P} := \mathcal{T}ors(X,T)_\gamma$, $P := \mathrm{Tors}(X,T)_\gamma$, and $\mathcal{G}_{X^I} := \mathcal{D}iv(X,\Gamma)^\gamma_{X^I}$ (the Γ-divisors of degree γ). The $\mathcal{G}_{X^I}$'s form an $\mathcal{S}^\circ$-diagram $\mathcal{G}_{X^\mathcal{S}}$ of ind-schemes. We have the projections $\pi : \mathcal{G}_{X^\mathcal{S}} \to X^\mathcal{S}$ and $\mathcal{G}_{X^\mathcal{S}} \xrightarrow{\varphi} \mathcal{P} \to P$; the composition of the latter arrows is ϕ. As was mentioned in the remark in 4.9.2, our λ is a super line bundle on P.

4.9.6. Let us recall some terminology from [**BD**] 7.11.4. Let Y be an ind-scheme of ind-finite type. An $\mathcal{O}^!$-*module* M on Y is a rule that assigns to any closed subscheme $Z \subset Y$ a quasi-coherent $\mathcal{O}_Z$-module M_Z, and to any $Z' \subset Z$ an embedding $M_{Z'} \subset M_Z$ which identifies $M_{Z'}$ with the submodule of sections of M_Z killed by the ideal $\mathcal{I}_{Z'}$ of Z'; the embeddings should be transitive. If L is a line bundle on Y, then $L \otimes M$, $(L\otimes M)_Z := L_Z \otimes M_Z$, is an $\mathcal{O}^!$-module on Y. For a morphism $p : Y \to B$ where B is a scheme, we set $p_! M := \bigcup p|_{Z*} M_Z$; for $B = \operatorname{Spec} k$ we write $p_! M =: \Gamma_c(Y, M)$. All $\mathcal{O}^!$-modules on Y form an abelian category.

For a scheme Q of finite type we denote by D_Q the dualizing complex of Q realized as the Cousin complex (see [**Ha**]). For Y as above the complexes D_Z form an $\mathcal{O}^!$-module D_Y on Y. If $p : Y \to B$ is ind-proper and B is of finite type, then we have the canonical trace map $\mathrm{tr}_p : p_! D_Y \to D_B$.

4.9.7. We will consider $\mathcal{O}^!$-modules on the ind-schemes $\mathcal{G}_{X^I}$. Recall (see 3.10.8) that $\mathcal{G}_{X^I} = \operatorname{Spf} R$ where $R = R_{X^I}$ is a topological $\mathcal{O}_{X^I}$-algebra which is the projective limit of its quotients $R_\alpha = R/I_\alpha$ which are finite and flat over $\mathcal{O}_{X^I}$. For any $\mathcal{O}^!$-module M on $\mathcal{G}_{X^I}$ its image $\pi_! M$ is naturally a discrete R-module; this establishes an identification of the category of $\mathcal{O}^!$-modules on $\mathcal{G}_{X^I}$ and that of discrete R-modules (which are quasi-coherent as $\mathcal{O}_{X^I}$-modules). The functor $\pi_!$ admits a right adjoint $\pi^!$ which assigns to an $\mathcal{O}_{X^I}$-module N the discrete R-module $\pi^! N = \mathcal{H}om_{\mathcal{O}_{X^I}}(R, N) := \bigcup \mathcal{H}om_{\mathcal{O}_{X^I}}(R_\alpha, N)$; both $\pi_!$ and $\pi^!$ are exact functors. A line bundle on $\mathcal{G}_{X^I}$ is the same as an invertible topological R-module.

One has $D_{\mathcal{G}_{X^I}} = \pi^! D_{X^I}$. Since D_{X^I} is a right $\mathcal{D}_{X^I}$-complex and Θ_{X^I} acts on R via ∇, our $\pi_! D_{\mathcal{G}_{X^I}}$ is a right $\mathcal{D}_{X^I}$-complex. Since ∇ acts along the fibers of ϕ, the line bundle $\phi^*\lambda$ on $\mathcal{G}_{X^I}$ is equipped with a left Θ_{X^I}-action. Therefore $\pi_!(\phi^*\lambda \otimes D_{\mathcal{G}_{X^I}}) = \phi^*\lambda \underset{R}{\otimes} \pi_! D_{\mathcal{G}_{X^I}}$ is a right $\mathcal{D}_{X^I}$-complex.

The above objects are compatible with the embeddings coming from arrows in $\mathcal{S}$, so the $D_{\mathcal{G}_{X^I}}$ form a !-complex on $\mathcal{G}_{X^\mathcal{S}}$, etc. The right $\mathcal{D}$-complex $\pi_!(\phi^*\lambda \otimes D_{\mathcal{G}_{X^\mathcal{S}}})$ on $X^\mathcal{S}$ is evidently admissible.

Proposition. *There is a canonical quasi-isomorphic embedding in $C\mathcal{M}(X^\mathcal{S})$*

$$C(A)^\gamma_{X^\mathcal{S}} \hookrightarrow \pi_!(\phi^*\lambda \otimes D_{\mathcal{G}_{X^\mathcal{S}}}). \tag{4.9.7.1}$$

Proof. According to (3.10.8.1), one has $A^{\ell\gamma}_{X^I} = \pi_!(\phi^*\lambda\otimes\pi^!\mathcal{O}_{X^I})$. Therefore $C(A)^\gamma := A^{\ell\gamma}_{X^I}\otimes C(\omega)_{X^I} = \pi_!(\phi^*\lambda\otimes\pi^! C(\omega)_{X^I})$. Since $C(\omega)_{X^I}$ is the Cousin resolution of $\omega_{X^I}[|I|]$ with respect to the diagonal stratification and D_{X^I} is its whole Cousin resolution, we have a canonical quasi-isomorphic embedding $C(\omega)\hookrightarrow D_{X^I}$. Our (4.9.7.1) is the embedding $\pi_!(\phi^*\lambda\otimes\pi^! C(\omega)_{X^I})\hookrightarrow\pi_!(\phi^*\lambda\otimes\pi^! D_{X^I})$. $\square$

4.9.8. Passing to the de Rham cohomology, the proposition yields an identification (see 4.2.6(iv) for the notation)

$$C^{ch}(X,A)^\gamma \xrightarrow{\sim} \Gamma_{\widetilde{DR}}(X^{\mathcal{S}},\pi_!(\phi^*\lambda\otimes D_{\mathcal{G}_{X^{\mathcal{S}}}})). \tag{4.9.8.1}$$

Let $DR_\nabla(D_{\mathcal{G}_{X^{\mathcal{S}}}})$ and $DR_\nabla(\phi^*\lambda\otimes D_{\mathcal{G}_{X^{\mathcal{S}}}})$ be the de Rham complexes along the ∇-foliation; denote by $\widetilde{DR}_\nabla$ the corresponding canonical nice resolutions (see 4.2.6(iv)). These are complexes of !-sheaves on $\mathcal{G}_{X^{\mathcal{S}}}$, and $\Gamma_{\widetilde{DR}}(X^{\mathcal{S}},\pi_!(\phi^*\lambda\otimes D_{\mathcal{G}_{X^{\mathcal{S}}}})) = \Gamma_c(\mathcal{G}_{X^{\mathcal{S}}},\widetilde{DR}_\nabla(\phi^*\lambda\otimes D_{\mathcal{G}_{X^{\mathcal{S}}}}))$.

The differential of the DR_∇ complexes is $\phi^{-1}\mathcal{O}_P$-linear. Therefore we have $\mathcal{S}^\circ$-systems of complexes $\phi_! DR_\nabla(D_{\mathcal{G}_{X^{\mathcal{S}}}})$ and $\phi_! DR_\nabla(\phi^*\lambda\otimes D_{\mathcal{G}_{X^{\mathcal{S}}}}) = \lambda\otimes\phi_! DR_\nabla(D_{\mathcal{G}_{X^{\mathcal{S}}}})$. The same is true for the $\widetilde{DR}_\nabla$-complexes. One has $\Gamma_c(\mathcal{G}_{X^{\mathcal{S}}},\widetilde{DR}_\nabla(D_{\mathcal{G}_{X^{\mathcal{S}}}}\otimes\phi^*\lambda)) = \Gamma(P,\lambda\otimes\varinjlim\phi_!\widetilde{DR}_\nabla(D_{\mathcal{G}_{X^{\mathcal{S}}}}))$ where $\varinjlim$ is the inductive $\mathcal{S}^\circ$-limit. Combining the identifications, we get

$$C^{ch}(X,A)^\gamma \xrightarrow{\sim} \Gamma(P,\lambda\otimes\varinjlim\phi_!\widetilde{DR}_\nabla(D_{\mathcal{G}_{X^{\mathcal{S}}}})). \tag{4.9.8.2}$$

4.9.9. We have the trace maps $\mathrm{tr}_\phi : \phi_! D_{\mathcal{G}_{X^I}}\to D_P$ which extend canonically to the morphisms $\phi_! DR_\nabla(D_{\mathcal{G}_{X^I}})\to D_P$ (killing all the other components of the DR_∇-complex). Passing to the inductive limit, we get $\varinjlim\phi_! DR_\nabla(D_{\mathcal{G}_{X^{\mathcal{S}}}})\to D_P$. Composing it with the projection $\widetilde{DR}_\nabla\to DR_\nabla$, we get a morphism of complexes of $\mathcal{O}_P$-modules

$$\mathrm{tr}_{\phi\nabla} : \varinjlim\phi_!\widetilde{DR}_\nabla(D_{\mathcal{G}_{X^{\mathcal{S}}}})\to D_P. \tag{4.9.9.1}$$

We define the γ-component of (4.9.3.1) as the composition of (4.9.8.2) and the morphisms $\Gamma(P,\lambda\otimes\varinjlim\phi_!\widetilde{DR}_\nabla(D_{\mathcal{G}_{X^{\mathcal{S}}}}))\to\Gamma(P,\lambda\otimes D_P)\xrightarrow{\sim}R\Gamma(P,\lambda^*)^*$ where the first arrow comes from $id_\lambda\otimes\mathrm{tr}_{\phi\nabla}$ and the second arrow is the Serre duality. To finish the proof of the theorem, we need to prove that the constructed arrow is a quasi-isomorphism. This follows from the next proposition:

PROPOSITION. $\mathrm{tr}_{\phi\nabla}$ *is a quasi-isomorphism of complexes of* $\mathcal{O}_P$*-modules.*

Proof. A morphism f of complexes of $\mathcal{O}_P$-modules is a quasi-isomorphism if (and only if) its Fourier-Mukai transform $\Phi(f)$ is. We will check that it happens with $\mathrm{tr}_{\phi\nabla}$.

The Fourier-Mukai transform[85] Φ takes values in the derived category of complexes of $\mathcal{O}$-modules on the (coarse) moduli space of line bundles on P which are algebraically equivalent to 0. The latter identifies in the usual way with the moduli space $Q := \mathrm{Tors}(X,T^\vee)^0$ of $T^\vee$-torsors of degree 0. Namely, for $q\in Q$ the corresponding line bundle λ_q on P is constructed as follows. Let $\mathfrak{F}_q$ be the $T^\vee$-torsor of degree 0. It defines a line bundle on $\mathcal{T}ors(X,T)$ according to, say, (3.10.7.3) and (4.9.1.2); since $\mathfrak{F}_q$ has degree 0, our line bundle comes from a (uniquely defined)

[85] See the recent book [**Po**] on the subject.

line bundle on $\mathrm{Tors}(X,T)$. Our $\mathcal{L}_q$ is its restriction to $P := \mathrm{Tors}(X,T)_\gamma$. Therefore λ_q coincides with the line bundle λ that corresponds to the commutative lattice chiral algebra A_q such that $\mathrm{Spec}\, A_q^\ell = \mathfrak{J}\mathfrak{F}_q$ (the jet scheme of $\mathfrak{F}$; see 3.10.2).

Recall that Φ is defined by means of a "kernel" which is a line bundle on $Q \times P$ whose restriction to the fiber over each q equals λ_q. Looking back at (4.9.8.2), we see that $\Phi(\mathrm{tr}_{\phi\nabla})$ is equal to the γ-component of the morphism (4.9.3.1) for the *family* of lattice chiral algebras A_Q parametrized by Q.[86] So, what we want to do is to prove our theorem for our *special* family of commutative lattice algebras.

Notice that $\phi(D_P) = \delta_0 :=$ the skyscraper $\mathcal{O}$-module at $0 \in Q$. Summing up with respect to all $\gamma \in \Gamma$, we get a morphism of $\mathcal{O}_Q$-complexes $\tau : C^{ch}(X, A_Q) \to k[\Gamma] \otimes \delta_0$. We want to check that this is a quasi-isomorphism.

First notice that $H^0\tau : \langle A_Q \rangle = H_0^{ch}(X, A_Q) \xrightarrow{\sim} k[\Gamma] \otimes \delta_0$. Indeed, the Q-scheme $\mathrm{Spec}\, \langle A_Q \rangle$ is the space of horizontal sections of $\mathrm{Spec}\, A_Q^\ell$, so it is a copy of $T^\vee$ which lives over $0 \in Q$. One checks in a moment that $H^0\tau$ does not vanish on each of the γ-components, so it is an isomorphism.

Since $H_\cdot^{ch}(X, A_Q)$ is a unital $\langle A_Q \rangle$-module, we see that it vanishes outside $0 \in Q$. It remains to check that the morphism $Li_0^*\tau$, where $i_0 : \{0\} \hookrightarrow Q$, is a quasi-isomorphism. We have $Li_0^* C^{ch}(X, A_Q) = C^{ch}(X, A_0)$ and $Li_0^* k[\Gamma] \otimes \delta_0 \simeq k[\Gamma] \otimes \mathrm{Tor}_\cdot(\delta_0, \delta_0) = k[\Gamma] \otimes \mathrm{Sym}(V[1])$ where V is the cotangent space to Q at 0. It follows from the construction that our map $H_\cdot^{ch}(X, A_0) \to k[\Gamma] \otimes \mathrm{Sym}(V[1])$ is a morphism of commutative algebras. It is isomorphism in degree 0 and surjective in degree -1 (since $H^0\tau$ is an isomorphism).

To finish the proof, it suffices to check that the commutative algebra $H_\cdot^{ch}(X, A_0)$ is generated by H_0^{ch} and H_1^{ch}, and $\dim H_1^{ch}$ is equal to $\dim Q = rk(\Gamma) g$ where g is the genus of X. Let us embed our torus $T^\vee$ into a vector space K of the same dimension. We have the open embedding of the jet schemes $\mathfrak{J}T^\vee = \mathrm{Spec}\, A_0^\ell \hookrightarrow \mathfrak{J}K =: \mathrm{Spec}\, R^\ell$. Therefore $H_\cdot^{ch}(X, A_0) = H_\cdot^{ch}(X, R) \underset{\langle R \rangle}{\otimes} \langle A_0 \rangle$ (see (4.3.12.2)), and $H_\cdot^{ch}(X, R) = \mathrm{Sym}(K^* \otimes R\Gamma(X, \omega)[1])$ by the proposition in 4.6.2. We are done. $\square$

4.9.10. Questions. What would be an analog of the above theorem for a chiral algebra coming from an arbitrary ind-finite chiral mononoid (see 3.10.16)?

Suppose A is the integrable quotient of the Kac-Moody algebra $U(\mathfrak{g}_{\mathcal{D}})^\kappa$ defined by a semi-simple algebra $\mathfrak{g}$ and a positive integral level κ. Is it true that all the higher chiral homologies of A vanish?

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[86] The above considerations generalize to the situation of families of lattice algebras in a straightforward way.