K-THEORY OF A WALDHAUSEN CATEGORY AS A SYMMETRIC SPECTRUM

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Abstract. If $\mathcal{C}$ is a Waldhausen category (i.e., a "category with cofibrations and weak equivalences"), it is known that one can define its K-theory $K(\mathcal{C})$ as a connective symmetric Ω-spectrum. The goal of these notes is to explain the construction in such a way that it can be not only understood, but also (hopefully) *remembered* by a non-expert. We do not claim that our exposition shows the process by which Waldhausen has originally discovered his definition of $K(\mathcal{C})$, nor do we assert that our approach leads to a deeper understanding of algebraic K-theory. We simply try to remove as much "mystery" as possible from the multi-step construction of the spectrum $K(\mathcal{C})$.

1. Introduction.

Let $\mathcal{C}$ be a Waldhausen category (the precise definition appears later in this text). The original construction of Waldhausen ([Wa83], § 1.3) produces from $\mathcal{C}$ a certain simplicial Waldhausen category, usually denoted by $\mathcal{S}_\bullet\mathcal{C}$, with the following property. If we look at the subcategory $w(\mathcal{S}_\bullet\mathcal{C})$ of $\mathcal{S}_\bullet\mathcal{C}$ whose objects are all the objects of $\mathcal{S}_\bullet\mathcal{C}$ and whose morphisms are the weak equivalences in $\mathcal{S}_\bullet\mathcal{C}$, take its nerve $N_\bullet(w(\mathcal{S}_\bullet\mathcal{C}))$, which is a bisimplicial set, and form its geometric realization $|N_\bullet(w(\mathcal{S}_\bullet\mathcal{C}))|$, the homotopy groups of the resulting topological space are the K-groups of $\mathcal{C}$ up to a shift; more precisely,

$$K_i(\mathcal{C}) = \pi_{i+1}|N_\bullet(w(\mathcal{S}_\bullet\mathcal{C}))| \qquad \forall i \geq 0.$$

[At least with the classical approach, this equality is an (easy) theorem for $i = 0$, and a definition for $i > 0$.]

In the rest of these notes, we ask the reader to take on faith the following three principles, which we are unable to motivate, but which (if understood appropriately) will lead to a modern definition of $K(\mathcal{C})$ as a connective symmetric Ω-spectrum.

- **(1):** The space whose homotopy groups compute the higher K-theory of $\mathcal{C}$ is obtained as the classifying space of a certain simplicial category $w(\mathcal{S}_\bullet\mathcal{C})$.
- **(2):** The n-th term of this simplicial category has something to do with n-step *filtrations* of objects in $\mathcal{C}$,

$$0 = A_0 \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail A_n = A.$$

- **(3):** If one wants to define $K(\mathcal{C})$ as a spectrum rather than a space, one needs to iterate the $\mathcal{S}_\bullet$-construction (this is possible because each term $\mathcal{S}_n\mathcal{C}$ will also be a Waldhausen category, as mentioned above). Moreover, the iterations can be performed in a symmetric way (i.e., one which is independent of the order in which the iterations are being made), and this will lead to a definition of $K(\mathcal{C})$ as a *symmetric* spectrum.

The reader is advised to look through at least §§ 1.1 – 1.3 of [Wa83], since a lot of what follows was motivated by those sections. In particular, a glance at the proof of Lemma 1.1.5 in op. cit. shows that Waldhausen was fully aware of the possibility of

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formulating his iterated $\mathcal{S}_\bullet$-construction in a symmetric way, only he did not have the adequate language of symmetric spectra to describe the result. The actual definition of $K(\mathcal{C})$ that we give at the end of these notes is contained in the *very densely* written § 6.1 of [GH97]. A lot of the work shown below was done in order to explain how one could deduce the various ingredients in the definition in loc. cit. from the three principles stated above.

2. Categories with cofibrations.

Almost all of our constructions will only deal with cofibrations, and weak equivalences will appear at the very end. Thus we begin with

Definition ([Wa83], § 1.1): A **category with cofibrations** is a pair $(\mathcal{C}, \mathrm{co}(\mathcal{C}))$ consisting of a category $\mathcal{C}$ that has a zero object 0 (i.e., an object which is both initial and terminal) and a subcategory $\mathrm{co}(\mathcal{C}) \subseteq \mathcal{C}$ whose morphisms are called **cofibrations** in $\mathcal{C}$ and are represented pictorially as

$$A \rightarrowtail B,$$

satisfying the following three axioms.

Cof 1.: All isomorphisms in $\mathcal{C}$ are contained in $\mathrm{co}(\mathcal{C})$ (in particular, as a subcategory, $\mathrm{co}(\mathcal{C})$ contains all objects of $\mathcal{C}$).

Cof 2.: For each object $A \in \mathcal{C}$, the unique arrow $0 \longrightarrow A$ is a cofibration.

Cof 3.: Cofibrations admit "cobase change". This means that if

$$\begin{tikzcd} A \arrow[r, tail, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

is a diagram in $\mathcal{C}$ (where f is a cofibration), then the pushout $C \underset{A}{\cup} B$ exists, and, moreover, the induced arrow $C \longrightarrow C \underset{A}{\cup} B$ is also a cofibration:

$$C \rightarrowtail C \underset{A}{\cup} B.$$

Informally speaking, one should think of cofibrations as being analogous to monomorphisms, and one should think of a category with cofibrations as a category where one has a good notion of a monomorphism which is stable under pushouts.

Example: Let $\mathcal{A}$ be an abelian category, and let $\mathcal{P} \subseteq \mathcal{A}$ be an exact subcategory, i.e., a strict full additive subcategory which is stable under extensions. We define the cofibrations in $\mathcal{P}$ to be the **admissible monomorphisms**, i.e., arrows $A \xrightarrow{f} B$ in $\mathcal{P}$ that are monomorphisms in $\mathcal{A}$ and are such that $\mathrm{Coker}(f) \in \mathcal{P}$. This makes $\mathcal{P}$ into a category with cofibrations; the only nontrivial axiom to verify is Cof 3. In order to check it, let $A \overset{f}{\rightarrowtail} B$ be an admissible monomorphism in $\mathcal{P}$, and let $A \xrightarrow{g} C$ be any arrow in $\mathcal{P}$. We have a diagram in $\mathcal{A}$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "{(f,-g)}"] \arrow[d, equal] & B \oplus C \arrow[r] \arrow[d, "\mathrm{proj}_1"] & C \underset{A}{\cup} B \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r] & \mathrm{Coker}\, f \arrow[r] & 0 \end{tikzcd}$$

with exact rows. It induces an isomorphism $C \cong \mathrm{Ker}(\mathrm{proj}_1) \cong \mathrm{Ker}\,\varphi$, which shows that the induced morphism $i : C \to C \cup_A B$ is a monomorphism in $\mathcal{A}$. In addition, φ is epimorphic, so since $C \in \mathcal{P}$ and $\mathrm{Coker}\, f \in \mathcal{P}$ by assumption, we see that $C \underset{A}{\cup} B \in \mathcal{P}$ as well. And, finally, we also see that $\mathrm{Coker}\, i \cong \mathrm{Coker}\, f \in \mathcal{P}$, which implies that i is an admissible monomorphism in $\mathcal{P}$ and completes the proof.

In what follows, the reader would not miss any part of the explanations by restricting attention to the special class of categories with cofibrations arising from exact categories $\mathcal{P}$ in the way described above. Indeed, one may even assume that $\mathcal{P} = \mathcal{A}$ is abelian; moreover, this case will constantly be used as a motivation for more general constructions involving cofibrations.

3. Filtrations and quotients.

It is clear that once we have a suitable analogue of a monomorphism, we can also define filtrations. It will be convenient for us to assume that all our filtrations start with the zero object. Thus, if $\mathcal{C}$ is a category with cofibrations, we define an **n-step filtration** of an object $A \in \mathcal{C}$ to be a sequence of cofibrations

$$0 = A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n = A.$$

The next observation is that the notion of "quotient by a subobject" makes sense in $\mathcal{C}$. Namely, if $A \overset{f}{\rightarrowtail} B$ is a cofibration in $\mathcal{C}$, then, by axiom Cof 3, the pushout of f and the unique arrow $A \longrightarrow 0$ exists; we will denote it by

$$B/A = 0 \cup_A B$$

and represent the natural arrow $B \longrightarrow B/A$ as follows: $B \twoheadrightarrow B/A$. Thus we have a pushout diagram

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & B/A \end{tikzcd}$$

A sequence of the form $A \rightarrowtail B \twoheadrightarrow B/A$ will be referred to as a **cofibration sequence**. Note that the choice of B/A is not quite unique: it is only unique up to canonical isomorphism.

Let us now return to principles (1) and (2) stated above. The n-th term $\mathcal{S}_n\mathcal{C}$ of the simplicial category $\mathcal{S}_\bullet\mathcal{C}$ will essentially be the category of all sequences of cofibrations of the form $0 = A_0 \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail A_n$. To understand why this is not good enough, recall that we need to have $n+1$ "face functors" $\mathcal{S}_n\mathcal{C} \to \mathcal{S}_{n-1}\mathcal{C}$ corresponding to the $n+1$ strictly increasing maps

$$\{0, 1, \ldots, n-1\} = [n-1] \to [n] = \{0, 1, \ldots, n\}.$$

It is easy to guess n of them: they are obtained by omitting A_j from the sequence (where $1 \le j \le n$) and replacing the two cofibrations $A_{j-1} \rightarrowtail A_j \rightarrowtail A_{j+1}$ with their composition (if $j \neq n$). However, we need one more face functor, and we can not define by simply omitting A_0 since we have agreed that all our filtrations start with the zero objects. What we do instead is omit A_0 and quotient out everything else by A_1. However, we now run into the problem of making functorial choices of the quotients A_j/A_1 for all $2 \le j \le n$. More generally, if we try to define other structure maps $\mathcal{S}_n\mathcal{C} \to \mathcal{S}_k\mathcal{C}$ of a simplicial category, we will see that we will need to make functorial choices of all the possible subquotients A_j/A_i $(1 \leq i < j \leq n)$. Waldhausen's solution is to keep track of these choices from the very beginning. Thus he defines a category which is equivalent to the category of n-step filtrations $0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n$, whose objects are such filtrations *together with compatible choices of all subquotients* A_j/A_i. More precisely, this means that we look at the category of commutative triangular diagrams of the form

$$(*)\qquad \begin{tikzcd} 0 = A_{00} \arrow[r, tail] & A_{01} \arrow[r, tail] \arrow[d, two heads] & A_{02} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & A_{0n} \arrow[d, two heads] \\ & 0 = A_{11} \arrow[r, tail] & A_{12} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & A_{1n} \arrow[d, two heads] \\ & & 0 = A_{22} \arrow[r, tail] & \cdots \arrow[r, tail] & A_{2n} \\ & & & \ddots & \vdots \\ & & & & 0 = A_{nn} \end{tikzcd}$$

with the property that each composition

$$A_{ij} \rightarrowtail A_{ik} \longrightarrow A_{jk}$$

is a cofibration sequence. One passes from such a diagram to an n-step filtration by ignoring everything except for the first row, and one goes the other way by starting with an n-step filtration $0 = A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n$, choosing subquotients $A_{ij} := A_j/A_i$, and filling in all the other arrows in the diagram above using the universal property of pushouts and the axiom Cof 3.

A nice way of restating the definition of the diagram $(*)$, which will be crucial for the more complicated constructions below, is as follows. Consider the linearly ordered set

$$[n] = \{0, 1, 2, \ldots, n\}$$

as a category in the usual way. Then a diagram of the form $(*)$ is the same thing as a functor

$$\begin{aligned} Ar[n] &\longrightarrow \mathcal{C}, \\ (i \to j) &\longmapsto A_{i\to j} = A_{ij}, \end{aligned}$$

where $Ar[n]$ is the *arrow category* of $[n]$, that has the following properties:

(i) : $A_{i\to j} = 0$ whenever $i = j$;

(ii) : for each composable pair of arrows $i \to j$ and $j \to k$, the morphism

$$A_{i\to j} \longrightarrow A_{i\to k}$$

is a cofibration;

(iii): for each composable pair of arrows $i \to j$ and $j \to k$, the sequence

$$A_{i\to j} \rightarrowtail A_{i\to k} \longrightarrow A_{j\to k}$$

is a cofibration sequence in $\mathcal{C}$.

We note that the property (iii) is an analogue of the Second Isomorphism Theorem in algebra:

$$(A_k/A_i)/(A_j/A_i) \cong A_k/A_j.$$

With this notation, we define $\mathcal{S}_n\mathcal{C}$ to be the category of functors $Ar[n] \to \mathcal{C}$ satisfying properties (i) – (iii) above, the morphisms in $\mathcal{S}_n\mathcal{C}$ being natural transformations. When formulated this way, the definition becomes manifestly functorial with respect to nondecreasing maps $[m] \to [n]$, and so we obtain a simplicial category $\mathcal{S}_\bullet\mathcal{C}$, as promised.

4. Iteration.

According to principle (3), we now need to iterate this construction. It already suffices to consider the case $n = 2$ because the general case is philosophically the same. Thus we consider the category $\mathcal{S}_2\mathcal{C}$, which can be naturally identified with the category of cofibration sequences in $\mathcal{C}$,

$$A \rightarrowtail B \twoheadrightarrow B/A\ , \tag{$**$}$$

the morphisms being the obvious commutative diagrams. If we want to be able to iterate the $\mathcal{S}_\bullet$-construction, we need to equip $\mathcal{S}_2\mathcal{C}$ with a class of cofibrations. A natural desire is to define cofibrations in $\mathcal{S}_2\mathcal{C}$ to be such that the three natural functors

$$\begin{tikzcd} \mathcal{S}_2\mathcal{C} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{C} \end{tikzcd}$$

obtained by sending $(**)$ to A, B, and B/A, respectively, are exact, and in particular preserve cofibrations. Thus the most obvious guess is that a cofibration in $\mathcal{S}_2\mathcal{C}$ is a commutative diagram of the form

$$\begin{tikzcd} A_1 \arrow[r, tail] \arrow[d, tail] & B_1 \arrow[r, two heads] \arrow[d, tail] & B_1/A_1 \arrow[d, tail] \\ A_2 \arrow[r, tail] & B_2 \arrow[r, two heads] & B_2/A_2 \end{tikzcd}$$

However, when we define cofibrations this way, we lose symmetry: the diagram above is not symmetric under switching A_2 and B_1. In order to keep the symmetry we discard the quotients and look at squares of the form

$$\begin{tikzcd} A_1 \arrow[r, tail] \arrow[d, tail] & B_1 \arrow[d, tail] \\ A_2 \arrow[r, tail] & B_2 \end{tikzcd} \tag{$***$}$$

In order to understand the condition we need to impose on this square, let us return to the motivating example where $\mathcal{C} = \mathcal{A}$ is an abelian category. Thus we are given an object B_2 of $\mathcal{A}$ and three subobjects $A_1, B_1, A_2 \subseteq B_2$ such that $A_1 \subseteq B_1 \cap A_2$. Then there is an induced morphism $B_1/A_1 \to B_2/A_2$, and it is a trivial exercise to check that it is a monomorphism if and only if $A_1 = B_1 \cap A_2$. Of course, this condition (taken literally) does not make sense in a general category with cofibrations; however, one of the first observations in [Wa83] is that in an abelian category, we have $A_1 = B_1 \cap A_2$ if and only if the induced morphism

$$B_1 \cup_{A_1} A_2 \longrightarrow B_2$$

is a monomorphism. Now this property certainly has an analogue in an arbitrary category with cofibrations, and we take it as our definition.

Proposition-Definition ([Wa83], Lemma 1.1.1): Define a cofibration in the category $F_1(\mathcal{C})$ of cofibrations $A \rightarrowtail B$ in $\mathcal{C}$ to be a commutative square of the form $(***)$ such that the induced arrow $B_1 \cup_{A_1} A_2 \longrightarrow B_2$ is also a cofibration. This makes $F_1(\mathcal{C})$ into a category with cofibrations.

The main point is that the condition that $B_1 \cup_{A_1} A_2 \rightarrowtail B_2$ is symmetric with respect to switching A_2 and B_1. Moreover, we can now iterate this construction, obtaining categories $F_1F_1(\mathcal{C})$, $F_1F_1F_1(\mathcal{C})$, and so on. These multiple iterations can be organized in an economical and symmetric way using the following notion.

5. Cofibration cubes.

Let Q be a finite set. Let $\mathcal{P}(Q) = 2^Q$ denote its power set, viewed as a partially ordered set with respect to inclusion, and hence as a category.

Definition: A **Q-cube** in a category $\mathcal{C}$ is a functor $\mathfrak{X} : \mathcal{P}(Q) \to \mathcal{C}$.

Examples: (0) If $Q = \varnothing$, then $\mathcal{P}(Q) = \{\varnothing\}$, and hence an $\varnothing$-cube in $\mathcal{C}$ is just an object of $\mathcal{C}$.

(1) If $Q = \{1\}$, then $\mathcal{P}(Q) = \{\varnothing, \{1\}\}$, and hence a 1-cube in $\mathcal{C}$ is an arrow in $\mathcal{C}$.

(2) If $Q = \{1,2\}$, then $\mathcal{P}(Q) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$, and the inclusion relations are

$$\begin{array}{ccc} \{2\} & \subset & \{1,2\} \\ \cup & & \cup \\ \varnothing & \subset & \{1\}, \end{array}$$

so a 2-cube in $\mathcal{C}$ is a commutative square

$$\begin{tikzcd} \mathfrak{X}_{01} \arrow[r] & \mathfrak{X}_{11} \\ \mathfrak{X}_{00} \arrow[u] \arrow[r] & \mathfrak{X}_{10} \arrow[u] \end{tikzcd}$$

(3) It should by now be obvious what is going on in higher dimensions. Here is a picture of a 3-cube:

$$\begin{tikzcd} & \mathfrak{X}_{011} \arrow[rr] & & \mathfrak{X}_{111} \\ \mathfrak{X}_{001} \arrow[ur] \arrow[rr] & & \mathfrak{X}_{101} \arrow[ur] & \\ & \mathfrak{X}_{010} \arrow[uu] \arrow[rr] & & \mathfrak{X}_{110} \arrow[uu] \\ \mathfrak{X}_{000} \arrow[uu] \arrow[ur] \arrow[rr] & & \mathfrak{X}_{100} \arrow[uu] \arrow[ur] & \end{tikzcd}$$

The point is that in order to specify the action of $\mathfrak{X}$ on the arrows of $\mathcal{P}(Q)$, it is enough to look at arrows of the form $S \to T$, where $S \subset T \subseteq Q$ are subsets such that S and T differ by exactly one element.

Definition (Important!): Let $\mathcal{C}$ be a category with cofibrations. A Q-cube $\mathfrak{X} : \mathcal{P}(Q) \to \mathcal{C}$ is said to be a **cofibration cube** if for each pair of subsets $S \subsetneq T \subseteq Q$, the induced morphism

$$\varinjlim_{S \subseteq \mathcal{U} \subsetneq T} \mathfrak{X}(\mathcal{U}) \longrightarrow \mathfrak{X}(T) \qquad (\dagger)$$

is a cofibration in $\mathcal{C}$.

As we explain below, the colimit in this definition can be formed by iterated pushouts along cofibrations in $\mathcal{C}$, and therefore exists.

Examples: (0) Every $\varnothing$-cube in $\mathcal{C}$ is a cofibration cube.

(1) A cofibration 1-cube in $\mathcal{C}$ is the same as an arrow $\mathfrak{X}_0 \to \mathfrak{X}_1$ in $\mathcal{C}$ which is a cofibration.

(2) A cofibration 2-cube in $\mathcal{C}$ is a commutative square of the form

$$\begin{tikzcd} \mathfrak{X}_{01} \arrow[r, tail] & \mathfrak{X}_{11} \\ \mathfrak{X}_{00} \arrow[r, tail] \arrow[u, tail] & \mathfrak{X}_{10} \arrow[u, tail] \end{tikzcd}$$

with the additional property that the induced arrow $\mathfrak{X}_{01} \cup_{\mathfrak{X}_{00}} \mathfrak{X}_{10} \to \mathfrak{X}_{11}$ is a cofibration. We see that cofibration 2-cubes in $\mathcal{C}$ are the same as cofibrations in the category $F_1(\mathcal{C})$. Thus, as promised, cofibration cubes provide a generalization of the latter notion.

(3) A cofibration 3-cube in $\mathcal{C}$ is a commutative 3-cube, drawn above, such that each of its faces is a cofibration 2-cube, and such that the induced arrow

$$\operatorname{colim}\left(\begin{tikzcd} & \mathfrak{X}_{100} \arrow[r] \arrow[dr] & \mathfrak{X}_{110} \\ \mathfrak{X}_{000} \arrow[ur] \arrow[r] \arrow[dr] & \mathfrak{X}_{010} \arrow[ur] \arrow[dr] & \mathfrak{X}_{101} \\ & \mathfrak{X}_{001} \arrow[r] \arrow[ur] & \mathfrak{X}_{011} \end{tikzcd}\right) \longrightarrow \mathfrak{X}_{111}$$

is a cofibration. It is a simple exercise to check that the colimit on the left can be constructed as

$$\left(\mathfrak{X}_{101} \bigcup_{\mathfrak{X}_{100}} \mathfrak{X}_{110}\right) \bigcup_{\mathfrak{X}_{001} \underset{\mathfrak{X}_{000}}{\cup} \mathfrak{X}_{010}} \mathfrak{X}_{011}$$

and therefore it exists. Moreover, in case $\mathcal{C}$ arises from an abelian category $\mathcal{A}$, we see that giving a cofibration cube in $\mathcal{A}$ amounts to giving an object $\mathfrak{X}_{111} = X$ of $\mathcal{A}$ together with three subobjects $A = \mathfrak{X}_{110}$, $B = \mathfrak{X}_{101}$, $C = \mathfrak{X}_{011}$, satisfying the properties

$$\begin{aligned} A \cap (B + C) &= (A \cap B) + (A \cap C), \\ B \cap (A + C) &= (B \cap A) + (B \cap C), \\ C \cap (A + B) &= (C \cap A) + (C \cap B). \end{aligned}$$

All the other vertices of the cube can be recovered from A, B, C as follows:

$$\begin{aligned} \mathfrak{X}_{100} &= A \cap B, & \mathfrak{X}_{010} &= A \cap C, \\ \mathfrak{X}_{001} &= B \cap C, & \mathfrak{X}_{000} &= A \cap B \cap C. \end{aligned}$$

Higher-dimensional cofibration cubes can be interpreted in a similar way, using induction on dimension. The main point to keep in mind is that a Q-cube is a cofibration cube if and only if its faces are cofibration cubes in lower dimensions, and it satisfies an extra condition corresponding to taking $S = \varnothing$ and $T = Q$ in the definition.

6. Multifiltrations.

We return to the main question of how to iterate the $\mathcal{S}_\bullet$-construction in general. Since one iteration corresponds to looking at filtrations, a natural guess is that in general one has to look at multifiltrations. As usual, we first illustrate the approach for an abelian category $\mathcal{A}$. Moreover, for now we will only work with bifiltrations, since passing to general Q-filtrations only requires an understanding of bifiltrations and of cofibration cubes, discussed above. Thus let $\vec{n} = (n_1, n_2)$ be a pair of nonnegative integers, let A be an object of an abelian category $\mathcal{A}$, and assume that A is equipped with two filtrations:

$$(0) = F_0^{(1)}A \subseteq F_1^{(1)}A \subseteq \ldots \subseteq F_{n_1}^{(1)}A = A$$

and

$$(0) = F_0^{(2)}A \subseteq F_1^{(2)}A \subseteq \ldots \subseteq F_{n_2}^{(2)}A = A.$$

In the case of ordinary filtrations, we saw that we needed to keep track of all possible subquotients. The correct analogue of this for bifiltrations is to keep track of all possible double subquotients, defined as follows. If (i_1, i_2) and (j_1, j_2) are pairs of integers satisfying $0 \leq i_1 \leq j_1 \leq n_1$ and $0 \leq i_2 \leq j_2 \leq n_2$, then we can form a subquotient

$$\mathrm{gr}^{(1)}_{i_1 \to j_1} A := F^{(1)}_{j_1} A / F^{(1)}_{i_1} A,$$

which inherits a filtration induced by the second filtration A, and we can form

$$\mathrm{gr}_{\vec{i} \to \vec{j}} A := \frac{F^{(2)}_{j_2}\left(\mathrm{gr}^{(1)}_{i_1 \to j_1} A\right)}{F^{(2)}_{j_1}\left(\mathrm{gr}^{(1)}_{i_1 \to j_1} A\right)}.$$

To see that this definition is independent of the order in which the subquotients are taken, note that we can rewrite

$$\mathrm{gr}_{\vec{i} \to \vec{j}} A = \frac{\left(F^{(1)}_{j_1} A\right) \cap \left(F^{(2)}_{j_2} A\right)}{\left(F^{(1)}_{j_1} A\right) \cap \left(F^{(2)}_{i_2} A\right) + \left(F^{(1)}_{i_1} A\right) \cap \left(F^{(2)}_{j_2} A\right)},$$

and this expression is manifestly symmetric with respect to $i_1 \leq j_1$ and $i_2 \leq j_2$.

In addition, this construction has the following properties:

(i): If $i_1 = j_1$ or $i_2 = j_2$, then

$$\mathrm{gr}_{\vec{i} \to \vec{j}} A = 0.$$

(ii): Let $(k_1, k_2) = \vec{k}$ be another pair satisfying $0 \leq i_q \leq j_q \leq k_q \leq n_q$ $(q = 1, 2)$. Then the square

$$\begin{array}{ccc} \mathrm{gr}_{(i_1,i_2)\to(j_1,k_2)} A & \subseteq & \mathrm{gr}_{(i_1,i_2)\to(k_1,k_2)} A \\ \cup\! \mid & & \cup\! \mid \\ \mathrm{gr}_{(i_1,i_2)\to(j_1,j_2)} A & \subseteq & \mathrm{gr}_{(i_1,i_2)\to(k_1,j_2)} A \end{array}$$

is a cofibration square in $\mathcal{A}$.

(iii): In the same situation, the sequence

$$\begin{tikzcd} 0 \arrow[r] & \left(\mathrm{gr}_{(i_1,i_2)\to(j_1,k_2)} A\right) \bigcup\limits_{\mathrm{gr}_{(i_1,i_2)\to(j_1,j_2)} A} \mathrm{gr}_{(i_1,i_2)\to(k_1,j_2)} A \arrow[d] & \\ & \mathrm{gr}_{(i_1,i_2)\to(k_1,k_2)} A \arrow[d] & \\ & \mathrm{gr}_{(j_1,j_2)\to(k_1,k_2)} A \arrow[r] & 0 \end{tikzcd}$$

is exact.

In fact, (ii) is a special case of (iii), but we separate them for psychological reasons. Property (iii) is a natural 2-dimensional analogue of the Second Isomorphism Theorem, stating that if $A \subseteq B \subseteq C$ are subobjects in an abelian category, then the sequence

$$0 \to B/A \to C/A \to C/B \to 0$$

is exact.

Finally, let Q be any finite set, and let Δ^Q be the product of copies of the category Δ indexed by Q. Thus the objects of Δ^Q are represented by $[\vec{n}]$, where

$$\vec{n} = (n_q)_{q \in Q}$$

is a Q-tuple of nonnegative integers. We recall that a Q-simplicial set in any category $\mathcal{D}$ is a functor

$$\left(\Delta^Q\right)^{\mathrm{op}} \to \mathcal{D}.$$

Now if $[\vec{n}] \in \Delta^Q$, then $[\vec{n}]$ itself can be viewed as a category since it is a partially ordered set:

$$\text{if } \vec{i} = (i_q)_{q \in Q} \text{ and } \vec{j} = (j_q)_{q \in Q},$$
$$\text{then } \vec{i} \le \vec{j} \iff i_q \le j_q \text{ for all } q.$$

If A is an object of an abelian category $\mathcal{A}$ equipped with a Q-filtration:

$$0 = F_0^{(q)} A \subseteq F_1^{(q)} A \subseteq \ldots \subseteq F_{n_q}^{(q)} A = A$$

for all q, then, generalizing the previous constructions, we define

$$\mathrm{gr}_{\vec{i} \to \vec{j}} A = \frac{\bigcap_{q \in Q} \left(F_{j_q} A\right)}{\sum_{q \in Q} \left(F_{i_q} A \cap \bigcap_{t \neq q} F_{j_t} A\right)}.$$

These subquotients satisfy natural analogues of the properties (i) – (iii) above, and with all this in mind we can finally give the

7. Construction of $K(\mathcal{C})$ as a symmetric spectrum.

First recall the following

Definition ([Wa83], § 1.2): If $\mathcal{C}$ is a category with cofibrations, a **category of weak equivalences in** $\mathcal{C}$ is a subcategory $w(\mathcal{C}) \subseteq \mathcal{C}$ whose morphisms are called **weak equivalences** in $\mathcal{C}$ and are represented graphically as $A \xrightarrow{\sim} B$, satisfying the following two axioms:

Weq 1: All isomorphisms in $\mathcal{C}$ are contained in $w(\mathcal{C})$ (in particular, all objects of $\mathcal{C}$ lie in $w(\mathcal{C})$).

Weq 2: *Gluing lemma for weak equivalences*: given a diagram

$$\begin{tikzcd} B \arrow[d, "\wr"] & A \arrow[l, tail] \arrow[r] \arrow[d, "\wr"] & C \arrow[d, "\wr"] \\ B' & A' \arrow[l, tail] \arrow[r] & C' \end{tikzcd}$$

the induced arrow

$$B \cup_A C \to B' \cup_{A'} C'$$

is also a weak equivalence.

Main construction: Let Q be a finite set and let $[\vec{n}] \in \triangle^Q$, viewed as a category in the way explained above.

Given any arrow $\vec{i} \to \vec{j}$ in $[\vec{n}]$ and a subset $\mathcal{U} \subseteq Q$, we define $(\vec{i} \to \vec{j})_{\mathcal{U}}$ to be the arrow in $[\vec{n}]$ whose q-th component is $i_q \to j_q$ for $q \in \mathcal{U}$, and $i_q \to i_q$ for $q \notin \mathcal{U}$. It is easy to see that the assignment

$$\mathcal{U} \longmapsto (\vec{i} \to \vec{j})_{\mathcal{U}}$$

defines a Q-cube in the category of arrows $\mathrm{Ar}[\vec{n}]$.

Let $\mathcal{S}^Q_{\vec{n}}\mathcal{C}$ be the full subcategory of the category of functors

$$\begin{aligned} A : \mathrm{Ar}[\vec{n}] &\longrightarrow \mathcal{C} \\ (\vec{i} \to \vec{j}) &\longmapsto A_{\vec{i}\to\vec{j}} \end{aligned}$$

(where $\mathcal{C}$ is a fixed category with cofibrations and weak equivalences) consisting of the functors satisfying the following three properties:

(i): If $i_q = j_q$ for some $q \in Q$, then

$$A_{\vec{i}\to\vec{j}} = 0.$$

(ii): For every pair of composable arrows $\vec{i} \to \vec{j} \to \vec{k}$, the cube

$$\mathcal{U} \longmapsto A_{(\vec{j}\to\vec{k})_{\mathcal{U}} \circ (\vec{i}\to\vec{j})}$$

is a cofibration cube in $\mathcal{C}$.

(iii): Under the same assumption, the sequence

$$\varinjlim_{\mathcal{U} \subsetneq Q} A_{(\vec{j}\to\vec{k})_{\mathcal{U}} \circ (\vec{i}\to\vec{j})} \rightarrowtail A_{\vec{i}\to\vec{k}} \longrightarrow A_{\vec{j}\to\vec{k}}$$

is a cofibration sequence in $\mathcal{C}$.

We define a **weak equivalence** in $\mathcal{S}^Q_{\vec{n}}\mathcal{C}$ to be a natural transformation

$$f : A \to A'$$

such that for each $(\vec{i} \to \vec{j}) \in \mathrm{Ar}[\vec{n}]$, the morphism $f_{\vec{i}\to\vec{j}} : A_{\vec{i}\to\vec{j}} \to A'_{\vec{i}\to\vec{j}}$ is a weak equivalence in $\mathcal{C}$.

Thus we obtain a Q-simplicial category $w\mathcal{S}^Q_\bullet\mathcal{C} = \left([\vec{n}] \longmapsto w\mathcal{S}^Q_{\vec{n}}\mathcal{C}\right)$.

Observe that $\mathcal{S}^\varnothing\mathcal{C} = \mathcal{C}$ in a natural way, and therefore $w\mathcal{S}^\varnothing\mathcal{C} = w\mathcal{C}$.

By taking the classifying space of each of the categories $w\mathcal{S}^Q_{\vec{n}}\mathcal{C}$, we obtain a pointed Q-simplicial topological space

$$\begin{aligned} \left(\triangle^Q\right)^{\mathrm{op}} &\longrightarrow \mathcal{T}\mathrm{op} = \left\{ \begin{matrix} \text{topological} \\ \text{spaces} \end{matrix} \right\}, \\ [\vec{n}] &\longmapsto B\left(w\mathcal{S}^Q_{\vec{n}}\mathcal{C}\right). \end{aligned}$$

The reason it is pointed is that whenever $n_q = 0$ for *some* q, the category $\mathcal{S}^Q_{\vec{n}}\mathcal{C}$ is the trivial one consisting just of the zero object of $\mathcal{C}$.

Moreover, if we think of $B\left(w\mathcal{S}^Q_\bullet\mathcal{C}\right)$ as a presheaf of topological spaces on the category $\triangle^Q$, this presheaf is equivariant with respect to the natural action of

$$\mathrm{Aut}(Q) \cong \begin{matrix} \text{symmetric group} \\ \text{on } \ |Q| \ \text{ letters} \end{matrix}$$

on $\triangle^Q$. This implies that when we pass to the diagonal, $\operatorname{diag}\left(B\left(w\mathcal{S}_\bullet^Q\mathcal{C}\right)\right)$ is a simplicial topological space with a natural action of $\operatorname{Aut}(Q)$.

Finally, for each $m \geq 0$, let us pick a finite set Q with m elements, and set

$$K(\mathcal{C})_m = \left|\operatorname{diag}\left(B\left(w\mathcal{S}_\bullet^Q\mathcal{C}\right)\right)\right|,$$

the geometric realization of the simplicial topological space $\operatorname{diag}\left(B\left(w\mathcal{S}_\bullet^Q\mathcal{C}\right)\right)$. This is a topological space (in fact a CW complex) with a natural action of the symmetric group $\sum_m$ on m letters.

To describe $K(\mathcal{C})$ as a symmetric spectrum, it remains to define the structure maps

$$S^1 \wedge K(\mathcal{C})_m \to K(\mathcal{C})_{m+1}$$

for all $m \geq 0$. However, observe that if $Q = \{1, 2, \ldots, m\}$ and $Q' = \{1, 2, \ldots, m+1\}$, and for each $\vec{n} = (n_1, \ldots, n_m)$ we define $\vec{n}' = (n_1, \ldots, n_m, 1)$, then the category $\mathcal{S}_{\vec{n}'}^{Q'}\mathcal{C}$ is naturally equivalent to $\mathcal{S}_{\vec{n}}^{Q}\mathcal{C}$. (Informally speaking, when we pass from $\mathcal{S}_{\vec{n}}^{Q}\mathcal{C}$ to $\mathcal{S}_{\vec{n}'}^{Q'}\mathcal{C}$, we add the extra data of a 1-step filtration in the $(m+1)$st direction, but the data of a 1-step filtration amounts to no data at all.)

The natural inclusion

$$\mathcal{S}_{\vec{n}}^{Q}\mathcal{C} \simeq \mathcal{S}_{\vec{n}'}^{Q'}\mathcal{C} \hookrightarrow \mathcal{S}_\bullet^{Q'}\mathcal{C}$$

yields, by adjunction, the maps

$$S^1 \wedge K(\mathcal{C})_m \to K(\mathcal{C})_{m+1},$$

which are the structure maps of the symmetric spectrum $K(\mathcal{C})$.

Theorem (Essentially proved by Waldhausen modulo the fact that he did not use the term "symmetric spectrum"): $K(\mathcal{C})$ is a connective symmetric spectrum, which is an Ω-spectrum beyond the term $K(\mathcal{C})_0$.

References

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