§1.0. The following three monographs (along with several papers) were used during the preparation of this series of lectures:

[BK] B. Bakalov and A. Kirillov, Jr., "Lectures on Tensor Categories and Modular Functors"

[Ka] C. Kassel, "Quantum Groups"

[Tu] V. G. Turaev, "Quantum Invariants of Knots and 3-Manifolds"

We begin by drawing a diagram summarizing the relationships between the various types of monoidal categories that will appear in these lectures (see below). In this diagram, an arrow from type A to type B means that the definition of type B is obtained from the definition of type A either by putting an additional structure on the category or by imposing an additional requirement.

We remark that despite the appearance of the word "category", the story we will try to explain will have the flavor not of abstract category theory, but rather of advanced linear algebra, due to the rather strong finiteness conditions we will soon impose on our categories.
§ 1.1. Monoidal categories. For the sake of time, we will not review the precise definition of a monoidal category. Let us recall, however, that a strict monoidal category is a triple 
\[ M = (M, \otimes, I) \]
consisting of a category \( M \), a bifunctor 
\[ \otimes : M \times M \to M \]
and an object \( I \in M \) such that the diagram
\[
\begin{array}{ccc}
M \times M \times M & \xrightarrow{\otimes \times \text{Id}} & M \times M \\
\downarrow{\text{Id} \times \otimes} & & \downarrow{\otimes} \\
M \times M & \xrightarrow{\otimes} & M
\end{array}
\]
of functors commutes on the nose, and the functors \( X \mapsto I \otimes X \) and \( X \mapsto X \otimes I \) are equal to the identity functor \( \text{Id} : M \to M \).

Whenever we are giving general definitions and stating general results, we will tacitly assume that all our monoidal categories are strict. This leads to no loss of generality due to MacLane's coherence theorem, which shows that every monoidal category is equivalent (via a monoidal functor) to a strict one.

On the other hand, there are at least two situations where it is much better to allow non-strict monoidal categories. On the one hand, almost all the examples of monoidal categories that arise "in nature" are non-strict.
On the other hand, whenever one is dealing with classification problems (e.g., trying to classify all fusion categories with the given based ring as their \(K_0\) group), it is convenient to make the categories one studies “as small as possible” at the expense of allowing the associativity constraints to be non-trivial.

**Example:** Suppose we try to classify all “pointed fusion categories over \(C\)”. For the time being, we define a pointed fusion category over \(C\) to be:

- a \(C\)-linear semisimple abelian category \(\mathcal{M}\) such that \(\mathcal{M}\) has only finitely many simple objects up to isomorphism, and every object of \(\mathcal{M}\) is a finite direct sum of simple ones, and \(\dim_C \text{End}(X) = 1\) for every simple \(X \in \mathcal{M}\) (which implies that the Hom-spaces in \(\mathcal{M}\) are finite dimensional)

- equipped with a \(C\)-bilinear monoidal structure such that every simple object of \(\mathcal{C}\) is invertible, i.e., if \(X \in \mathcal{C}\) is simple, then \(\exists Y \in \mathcal{C}\) with \(X \otimes Y \cong \text{Id}\).

Given such a category \(\mathcal{C}\), it is easy to check that \(\text{Id}\) is simple, and if \(X, Y \in \mathcal{C}\) are simple, then so is \(X \otimes Y\). Thus the set \(\mathcal{C}\) of isomorphism classes of simple objects of \(\mathcal{C}\) becomes a group.
For every \( g \in G \), let us choose a representative \( x_g \in C \) of the isomorphism class labeled by \( g \). Then \( C \) has a skeleton, \( C' \), formed by all the direct sums
\[
\bigoplus_{g \in G} x_g^{n(g)} \quad | \quad n : G \to \mathbb{Z}_{\geq 0} \text{ is a function}
\]
Thus \( C' \) inherits a pointed fusion category structure from \( C \) which is strictly unital \(^{(1)}\) but is usually not strictly associative \(^{(2)}\). In fact, using the \( C \)-bilinearity of \( \otimes \), one easily checks that the group \( G \) determines the bifunctor \( \otimes \) and the associativity constraint can be encoded in a function
\[
\omega : G \times G \times G \longrightarrow C^x
\]
so that, for \( (g, h, k) \in G^3 \), we have
\[
x_g \otimes x_h \otimes x_k = \omega(g, h, k) \cdot \text{id} x_{ghk}.
\]
The pentagon axiom for \( \otimes \) translates into the 3-cocycle condition for \( \omega \) (with respect to the trivial \( G \)-action on \( C^x \)), which leads to the classification of equivalence classes of pointed fusion categories over \( C \) with the given group \( G \) of isomorphism classes of simple objects by the elements of the cohomology group \( H^3(G, C^x) \) modulo the action of the group of (outer) automorphisms of \( G \).

\(^{(1)}\) Assuming \( C \) is strictly unital

\(^{(2)}\) Even if the original category \( C \) is strict
$k$-linear categories. For the most part we will work over an algebraically closed field $k$; we will often assume that $\text{char}(k) = 0$ as well. We remark that it is not always necessary to work with $k$-linear, or even additive, categories; for instance, the notion of a ribbon category is defined in the non-additive setting in [Tu1]. However, in these introductory lectures it will be psychologically easier to work in the linear setting.

Thus we make the following conventions:

- by a $k$-linear category we will mean a category enriched over $k$-vector spaces, which is also an abelian category
- all functors between $k$-linear categories are assumed to be $k$-linear as well
- in particular, whenever we consider $k$-linear monoidal categories, the monoidal bifunctor will be assumed to be $k$-bilinear

We say that a $k$-linear category $\mathcal{C}$ is finite if:

(i) $\forall X, Y \in \mathcal{C}$, $\dim_k \text{Hom}(X, Y) < \infty$;
(ii) $\mathcal{C}$ has finitely many simple objects up to isomorphism;
(iii) every object of $\mathcal{C}$ has finite length; and
(iv) every simple object of $\mathcal{C}$ admits a projective cover.

In fact, one can characterize finite $k$-linear categories in much more concrete terms.
Namely, a \( k \)-linear category is finite if and only if it is (\( k \)-linearly) equivalent to the category of finite dimensional modules over a finite dimensional \( k \)-algebra.

3.3 Rigid monoidal categories. We have a well-known and very useful duality theory for finite dimensional vector spaces over a field (or, more generally, for finitely generated projective modules over a commutative ring). There is a class of monoidal categories in which an abstract version of this duality can be defined; they are called "rigid" monoidal categories. We warn the reader that, in general, one can define the notion of a "left dual" and a "right dual" of an object, and they may not be the same.

Definition. Let \( \mathcal{M} \) be a monoidal category, and let \( X \in \mathcal{M} \) be an object.

(a) A left dual of \( X \) is a triple \((X^*, \text{ev}_X, \text{coev}_X)\), where \( X^* \) is an object of \( \mathcal{M} \) and

\[
\text{ev}_X : X^* \otimes X \longrightarrow I, \\
\text{coev}_X : I \longrightarrow X \otimes X^*
\]

are morphisms in \( \mathcal{M} \) satisfying

\[(\text{id}_X \otimes \text{ev}_X) \circ (\text{coev}_X \otimes \text{id}_X) = \text{id}_X \]

and

\[(\text{ev}_X \otimes \text{id}_{X^*}) \circ (\text{id}_{X^*} \otimes \text{coev}_X) = \text{id}_X .
\]

Pictorially, this means that the compositions
\( X = 1 \otimes X \xrightarrow{\text{coev}_x \otimes \text{id}_X} X \otimes X^* \otimes X \xrightarrow{\text{id}_X \otimes \text{ev}_X} X \otimes 1 = X \)

and
\( X^* = X^* \otimes 1 \xrightarrow{\text{id}_{X^*} \otimes \text{coev}_x} X^* \otimes X \otimes X^* \xrightarrow{\text{ev}_x \otimes \text{id}_{X^*}} 1 \otimes X^* = X^* \)

are equal to the identity. (Recall that we are implicitly assuming that our monoidal categories are strictly associative and strictly unital.)

(b) The notion of a right dual of \( X \) is defined similarly; as an object \( ^*X \) of \( M \) together with morphisms
\[ \text{ev}' : X \otimes ^*X \rightarrow 1 \quad \text{and} \quad \text{coev}' : 1 \rightarrow ^*X \otimes X \]
satisfying the obvious analogues of the identities in the previous definition.

Remark. The terminology we are using is taken from [Ka]. In [BK], the object we denote by \( X^* \) (resp., \(^*X\)) is called the right (resp., left) dual of \( X \). In [Tu], only \( X^* \) is considered, and is simply called the dual of \( X \).

Our choice is partially justified by the fact that if \( X \in M \) has a left dual, \( X^* \), then the functor \( X^* \otimes ? \) is left adjoint to \( X \otimes ? \), and also by the following

Example. If \( C \) is any (small) category and \( M \) is the category of all functors \( C \rightarrow C \), a left (resp., right) dual of an object \( F \in M \) is a left (resp., right) adjoint functor to \( F \).

Exercise. What can you say about \( M = (\text{A-mod}, \otimes) \), where \( A \) is a commutative ring?
Exercise. Let $\mathcal{C}$ be a monoidal category, and fix $X \in \mathcal{C}$. Show that a left dual of $X$ if it exists, is unique in the following sense.

Suppose $X_1^*$ and $X_2^*$ are two left duals of $X$, and let us denote the "structure morphisms" by

\[
\begin{align*}
&\text{ev}_1 : X_1^* \otimes X \to 1, \\
&\text{coev}_1 : 1 \to X \otimes X_1^*, \\
&\text{ev}_2 : X_2^* \otimes X \to 1, \\
&\text{coev}_2 : 1 \to X \otimes X_2^*.
\end{align*}
\]

For a morphism $f : X_1^* \to X_2^*$ consider the diagrams

\[
\begin{array}{ccc}
X_1^* \otimes X & \xrightarrow{\text{ev}_1} & 1 \\
\downarrow \otimes \text{id}_X & & \downarrow \text{id}_X \otimes f \\\nX_2^* \otimes X & \xrightarrow{\text{ev}_2} & X \otimes X_2^*
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
1 & \xrightarrow{\text{id} \otimes \text{coev}_2} & X_1^* \otimes X \otimes X_2^* \\
\downarrow & & \downarrow \text{id} \otimes \text{coev}_2 \otimes \text{id} \\
1 \otimes X_2^* & \xrightarrow{\text{coev}_1} & X \otimes X_2^*
\end{array}
\]

Show that if either of these diagrams commute, then $f$ must equal the composition

\[
X_1^* = X_1^* \otimes 1 \xrightarrow{id \otimes \text{coev}_2} X_1^* \otimes X \otimes X_2^* \xrightarrow{\text{ev}_1 \otimes id} 1 \otimes X_2^* = X_2^*.
\]

Conversely, show that taking $f$ to be this composition makes both of the diagrams above commute.

By symmetry, it follows that the morphism $f : X_1^* \to X_2^*$ defined above is an isomorphism. Thus the left dual of an object of a monoidal category, if it exists, is unique up to a unique isomorphism (in the appropriate sense).
Remark. In some situations (though not always) the fact that left and right duals are defined by “equations” rather than by a universal property makes working with duality rather convenient. For instance, every monoidal functor between two monoidal categories automatically preserves duals.

Definition. A monoidal category $M$ is said to be left (resp., right) rigid if every object of $M$ has a left (resp., right) dual. We say that $M$ is simply rigid if it is both left and right rigid. We note that in [Ka] the word “autonomous” is used in place of “rigid”.

Exercises. Let $M$ be a rigid monoidal category.
(a) For each $X \in M$, show that there are canonical identifications $(\ast X)^* \cong X \cong \ast (X^*)$.
(b) For $X, Y \in M$, construct natural isomorphisms $(X \otimes Y)^* \cong Y^* \otimes X^*$, $\ast (X \otimes Y) \cong \ast Y \otimes \ast X$.
(c) Given $X \in M$, show that the functor $X \otimes$ has left and right adjoints given by $X^* \otimes$ and $\ast X \otimes$, respectively, and the functor $\otimes X$ has left and right adjoints given by $\otimes ^* X$ and $\otimes \ast X$, respectively.
(d) Deduce that if $M$ is abelian and the bifunctor $\otimes$ is bi-additive, then it is bi-exact.
(e) If $M$ is as in (d), is the unit object $1$ of $M$ necessarily semisimple?
Remark. Suppose \( \mathcal{M} \) is a left rigid monoidal category. For every \( X \in \mathcal{M} \), choose a left dual \( (X^*, \Phi, \text{coev}_X) \) (we already know that it is unique up to a unique isomorphism). We can turn the assignment \( X \mapsto X^* \) into a contravariant functor as follows.

Given a morphism \( X \xrightarrow{f} Y \) in \( \mathcal{M} \), we define \( f^*: Y^* \rightarrow X^* \) to be the composition
\[
Y^* = Y^* \otimes \mathbb{1} \xrightarrow{id \otimes \text{coev}_X} Y^* \otimes X \otimes X^* \xrightarrow{id \otimes f \otimes id} Y^* \otimes Y \otimes X^* \xrightarrow{\text{ev}_Y \otimes id} \mathbb{1} \otimes X^* = X^*.
\]

Example. Let \( A \) be a finite dimensional bi-algebra over a field \( k \), and let \( A\text{-mod} \) denote the category of finite dimensional left \( A \)-modules. We make it a monoidal category in the standard way: if \( M, N \in A\text{-mod} \), then \( M \otimes N \) is defined to be \( M \otimes_k N \) with the \( A \)-action induced from the natural action of \( A \otimes A \) on \( M \otimes N \) via the comultiplication homomorphism \( \Delta: A \rightarrow A \otimes_k A \).

Let us investigate the question of when the category \( A\text{-mod} \) is left rigid.

First fix \( M \in A\text{-mod} \). Since the forgetful functor \( A\text{-mod} \rightarrow k\text{-mod} \) has an obvious monoidal structure, and since we know what duality on \( k\text{-mod} \) looks like, we see that if \( M \) has a left dual, it must be given by the unique \( A \)-module structure on the
dual vector space \( M^* = \text{Hom}_k(M, k) \) such that
the natural \( k \)-linear maps
\[
M^* \otimes_k M \to k \quad \text{and} \quad k \to M \otimes_k M^*
\]
are homomorphisms of \( A \)-modules.

Instead of trying to make this more explicit, let us observe that, again by
uniqueness, the \( A \)-action on \( M^* \) must commute
with the action of \( \psi^* \) for any \( \psi \in \text{End}_A(M) \).
Now let us apply this with \( M = A \), with the
\( A \)-action on itself given by left multiplication.
Then the \( A \)-action on \( A^* \) must commute
with \( \rho^* \) for every \( a \in A \), where \( \rho_a : A \to A \)
is the \( k \)-linear map given by \( x \mapsto xa \).
It follows easily that the \( A \)-action on \( A^* \)
must be of the form
\[
(a \cdot f)(b) = f(S(a) \cdot b) \quad a, b \in A, f \in A^*
\]
where \( S : A \to A \) is an anti-homomorphism
of \( k \)-algebras, i.e., \( S \) is a \( k \)-linear map
satisfying \( S(1) = 1 \) and \( S(ab) = S(b)S(a) \).

The condition that the natural \( k \)-linear
maps \( A^* \otimes_k A \to k \) and \( k \to A \otimes_k A^* \) are
\( A \)-module homomorphisms, become the
requirement that the compositions
\[
A \xrightarrow{\Delta} A \otimes_k A \xrightarrow{S \otimes \text{id}_A} A \otimes_k A \xrightarrow{\text{mult}} A
\]
and
\[
A \xrightarrow{\Delta} A \otimes_k A \xrightarrow{\text{id}_A \otimes S} A \otimes_k A \xrightarrow{\text{mult}} A
\]
are equal to \( \eta \circ \varepsilon : A \to k \to A \).
Definition. If $A$ is a bi-algebra over a field $k$, an antipode for $A$ is a $k$-linear map $S : A \rightarrow A$ such that

$$\mu \circ (S \otimes \text{id}_A) \circ \Delta = \eta \circ \varepsilon = \mu \circ (\text{id}_A \otimes S) \circ \Delta,$$

where

$$\mu : A \otimes A \rightarrow A, \quad \Delta : A \rightarrow A \otimes A,$$

$$\eta : k \rightarrow A \quad \text{and} \quad \varepsilon : A \rightarrow k$$

are the multiplication, comultiplication, unit and counit, respectively.

Remark. It is not hard to check that if $A$ is a bialgebra as above, then the space $\text{End}_k(A)$ of $k$-linear maps $A \rightarrow A$ can be made into a $k$-algebra with respect to the “convolution” operation, given by

$$f \ast g = \mu \circ (f \otimes g) \circ \Delta.$$

The identity with respect to $\ast$ is $\eta \circ \varepsilon$. Thus an antipode for $A$ is the same thing as an element $S \in \text{End}_k(A)$ which is a two-sided inverse of $\text{id}_A$ with respect to $\ast$. So we see that if an antipode exists, it is necessarily unique.

For more details, we refer the reader to Chapter III of [Ka1], where it is proved, in particular, that if $S : A \rightarrow A$ is an antipode for a bi-algebra $A$, then $S$ is an anti-homomorphism both of $k$-algebras and of $k$-coalgebras. Also, if $A$ is either commutative or cocommutative, and $S'$ is an antipode for $A$, then $S'^2 = \text{id}_A$. 
So far, we have shown that if $A$ is a finite dimensional $k$-bialgebra such that the monoidal category $A$-mod is left rigid, then $A$ has an antipode. The converse is easy to check: if $A$ has an antipode, then for every finite dimensional $A$-module $M$, the dual space $M^* = \text{Hom}_k (M, k)$ becomes an $A$-module with the action
\[(a \cdot f)(m) = f(S(a) \cdot m),\]
and the natural $k$-linear maps
\[M^* \otimes M \longrightarrow k \quad \text{and} \quad k \longrightarrow M \otimes M^* \]
are $A$-module homomorphisms.

**Example.** Let $G$ be a monoid. The "monoid algebra" $A = k[G]$ has an obvious (cocommutative) bialgebra structure, given by $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ for all $g \in G$. A $k$-linear map $S: A \longrightarrow A$ is an antipode if and only if $S(g)g = 1 = gS(g)$ for all $g \in G$. Hence an antipode for $A$ exists if and only if $G$ is a group.

In particular, if $G$ is a finite monoid which is not a group, the monoidal category $k[G]$-mod is not left rigid.

**Exercise.** (a) Consider a finite dimensional $k$-bialgebra $A$. Show that the monoidal category $A$-mod is right rigid if and only
If the bialgebra $A^\text{cop}$ obtained from $A$ by replacing the comultiplication with the opposite one (while keeping the same multiplication, unit and counit) has an antipode.

(b) If $A$ is an arbitrary $k$-bialgebra with an antipode $S$, show that $A^\text{cop}$ also has an antipode if and only if $S$ is invertible, and in this case, $S^{-1}$ is the antipode for $A^\text{cop}$.

**Definition.** A Hopf algebra over a field $k$ is a $k$-bialgebra which has an invertible antipode.

**Remark.** Some authors, such as Kassel, do not require the antipode to be invertible in the definition of a Hopf algebra.

**Corollary.** Let $A$ be a $k$-bialgebra, and $M = A\text{-mod}$ the monoidal category of left $A$-modules that are finite dimensional over $k$.

(a) If $A$ is a Hopf algebra, then $M$ is rigid.
(b) If $M$ is rigid and $A$ is finite dimensional, then $A$ is a Hopf algebra.
Tensor and fusion categories. We briefly list some special classes of $k$-linear monoidal categories that appear in practice. Throughout this section $k$ will be an algebraically closed field of arbitrary characteristic.

**Definitions.**
1. A tensor category over $k$ is a rigid monoidal $k$-linear category $M$ such that $\dim_k \text{Hom}(X,Y) < \infty$ for all $X, Y \in M$, and such that $\text{End}_M(I) = k$.
2. A finite tensor category over $k$ is a tensor category over $k$ which is finite as a $k$-linear category (see above).
3. A fusion category over $k$ is a semisimple finite tensor category over $k$.

**Remark.** Dropping the requirement $\text{End}_M(I) = k$ in the definitions above, one arrives at the notions of a multitensor category and a multifusion category. However, there is little loss of generality in restricting attention to multitensor categories, due to the fact that if $M$ is any multitensor category over $k$, then one can write $I = \bigoplus_{i \in I} I_i$, a finite direct sum of pairwise nonisomorphic simples with $I_i \otimes I_j = \delta_{ij} I_i$ and $I_i^* \cong I_i$ for all $i, j \in I$.

(At least, this is true if we assume that $I$ has finite length. I am not sure if this follows automatically from the other requirements.)
Pivotal and spherical categories. In linear algebra, the notion of the trace of an endomorphism of a finite dimensional vector space plays an important role. However, in an arbitrary tensor (or even fusion) category, this notion cannot be defined unless we equip the category with some extra structure. Roughly speaking, the reason is as follows. If $V$ is a finite dimensional vector space over a field $k$ and $f : V \to V$ is a linear map, the "abstract" definition of $\text{tr}(f)$ is as the composition

$$k \to V \otimes V^* \xrightarrow{f \otimes \text{id}} V \otimes V^* \to k,$$

where $V^* = \text{Hom}_k(V, k)$ and the first and last maps are the canonical ones. However, this definition cannot be repeated in an arbitrary tensor category, because we would have to replace $V^*$ with either the left or the right dual of an object $X$; but the first choice does not work because there is no natural morphism $X \otimes X^* \to \mathbb{1}$, and the second choice does not work because there is no natural morphism $\mathbb{1} \to X \otimes X^*$.

The most straightforward way to deal with this problem is to choose an isomorphism between $X^*$ and $^*X$; or, equivalently, between $X$ and $X^{**}$. We will do so shortly. First let us explain that what can be canonically defined is the trace of a morphism $f : X \to X^{**}$ in a tensor category $M$. 
Namely, consider the composition
\[ 1 \xrightarrow{\text{coev}_X} \mathcal{X} \otimes \mathcal{X}^* \xrightarrow{f \otimes \text{id}} \mathcal{X}^{**} \otimes \mathcal{X}^* \xrightarrow{\text{ev}_{\mathcal{X}^*}} 1. \]

It gives an element of \( \mathcal{K} = \text{End} (\mathcal{I}) \) which is called the trace of \( f \) and denoted \( \text{tr} (f) \).

(or, sometimes, "quantum trace")

Exercise. Show that the functor \( \mathcal{X} \mapsto \mathcal{X}^{**} \) has a natural structure of a tensor functor \( \mathcal{M} \to \mathcal{M} \).

(By definition, a tensor functor between tensor categories over a field \( k \) is a \( k \)-linear exact monoidal functor.) Then check that if \( X \xrightarrow{f} X^{**} \) and \( Y \xrightarrow{g} Y^{**} \) are morphisms in \( \mathcal{M} \), then \( \text{tr} (f \otimes g) = \text{tr} (f) \cdot \text{tr} (g) \).

Definition. A pivotal structure on a tensor category \( \mathcal{M} \) over \( k \) is an isomorphism of tensor functors \( a : \text{Id}_\mathcal{M} \xrightarrow{\sim} (\mathcal{X} \mapsto \mathcal{X}^{**}) \). A pivotal category is a pair \((\mathcal{M}, a)\), where \( \mathcal{M} \) is a tensor category and \( a \) is a pivotal structure on \( \mathcal{M} \).

If \((\mathcal{M}, a)\) is a pivotal category, \( X \in \mathcal{M} \), and \( f \in \text{End} (X) \), we define \( \text{tr} (f) = \text{tr} (axof) \in k \), where \( \text{tr} (X \xrightarrow{axof} X^{**}) \) is defined as above.

We put \( \dim (X) = \text{tr} (\text{id}_X) \in k \). If \( X \in \mathcal{M} \) and \( g \in \text{End} (Y) \), it follows that \( \text{tr} (f \otimes g) = \text{tr} (f) \cdot \text{tr} (g) \), and, in particular, \( \dim (X \otimes Y) = \dim (X) \dim (Y) \).

A pivotal structure \( a \) on a tensor category \( \mathcal{M} \) is said to be spherical if \( \dim (X) = \dim (X^*) \) for all \( X \in \mathcal{M} \).
Caution. Note that we defined the dimension of every object of a pivotal category \((M, a)\) as an element of \(k\), not as an integer! Hence this notion must be used with some care. For instance, if \(M\) is the tensor category of finite dimensional vector spaces over \(k\) with the obvious pivotal structure, and \(\text{char}(k) = p > 0\), then \(\dim(V) = 0\) whenever \(V \in M\) has its usual dimension divisible by \(p\).

Exercise. If \(M\) is a semisimple tensor category over \(k\) and \(X \in M\) has finite length, then \(X \cong X^*\) and hence \(X^{**} \cong X\). If \(X\) is simple, then for any isomorphism \(f : X \cong X^{**}\), we have \(\text{tr}(f) \neq 0\).

Open question. Does every fusion category admit a pivotal structure?

Exercise. If \(A\) is a Hopf algebra over \(k\) whose antipode is involutive \((S^2 = 1d_A)\), it follows from a previous discussion that for any \(M \in A\text{-mod}\), there is a natural identification \(*M \cong M^*\), which leads to a natural pivotal structure on \(A\text{-mod}\). The set of all pivotal structures on \(A\text{-mod}\) can then be identified with the group of tensor automorphisms of the identity functor on \(A\text{-mod}\). Use this to show that if \(G\) is a finite group, there is a bijection between the set of pivotal (resp., spherical) structures on \(k[G]\text{-mod}\) and \(\mathbb{Z}(G)\) (resp., \(\{g \in \mathbb{Z}(G) | g^2 = 1\}\)).