§2.0. General plan for an ideal second lecture

- pivotal and spherical structures
- braided monoidal categories and twists
- Drinfeld center of a monoidal category
- ribbon categories
- modular categories and Verlinde's formula
- elementary examples

§2.1. A few words about examples: Before continuing with the general theory, we will list a few examples of monoidal categories that one should keep in mind while reading these notes. Some of the terms appearing on this list will be defined later in this lecture. The examples we will mention will also play a role in Nikshych's talks.

We remark that the monoidal categories that will appear in this lecture, and probably also in Nikshych's talks, originated in (geometric) representation theory, rather than low-dimensional topology. Thus we will present the theory as a purely algebraic one in this lecture. In particular, for the sake of time, we will not discuss the graphical calculus for morphisms in ribbon categories, and the connection between BMCs and braid groups.
Now we list some of the examples of monoidal categories that are useful to keep in mind. In what follows, $G$ is a fixed finite group.

(1) Given a 3-cocycle $\omega: G \times G \times G \to \mathbb{C}^\times$ with respect to the trivial $G$-action on $\mathbb{C}^\times$, which is normalized in the sense that

\[ \omega(1, g_2, g_3) = \omega(g_1, 1, g_3) = \omega(g_1, g_2, 1) = 1 \]

for all $g_1, g_2, g_3 \in G$, we described in the previous lecture a monoidal groupoid $\mathcal{M}(G, \omega)$, defined as follows:

- objects = elements of $G$
- $\text{Aut}(g) = \text{End}(g) = \mathbb{C}^\times \quad \forall g \in G,$
  and $\text{Hom}(g, h) = \emptyset$ if $g \neq h$
- the bifunctor

\[ \otimes: \mathcal{M}(G, \omega) \times \mathcal{M}(G, \omega) \to \mathcal{M}(G, \omega) \]

is the product in $G$ at the level of objects, and the product in $\mathbb{C}^\times$ at the level of morphisms

- the left and right unit constraints are the trivial ones (given by $1 \in \mathbb{C}^\times$)
- the associativity constraint

\[ \otimes_{g, h, k} : (g \otimes h) \otimes k \xrightarrow{\sim} g \otimes (h \otimes k) \]

is the element

\[ \omega(g, h, k) \in \mathbb{C}^\times = \text{Aut}(ghk) \]

for all $g, h, k \in G$.

Note that the category $\mathcal{M}(G, \omega)$ is "skeletal" (i.e., is its own skeleton; equivalently, if two objects in it are isomorphic, then they are equal).
(1') The category described in (1) has a C-linear analogue (which appeared in the notes for the first lecture). Namely, for each \( g \in G \), let \( X_g \) be a symbol, and consider the category \( \mathcal{V}(G, \omega) \) whose objects are formal direct sums
\[
\bigoplus_{g \in G} X_g^{n_g}, \quad \text{where } n_g \in \mathbb{Z}_{\geq 0},
\]
for every \( g \in G \), with morphisms determined uniquely by
\[
\text{Hom}(X_g, X_h) = \begin{cases} C & \text{if } g = h \\ 0 & \text{if } g \neq h \end{cases}
\]
(using bi-additivity). Note that \( \mathcal{M}(G, \omega) \) is naturally a (non-full) subcategory of \( \mathcal{V}(G, \omega) \). It is clear that the monoidal category structure on \( \mathcal{M}(G, \omega) \) can be extended uniquely to a C-linear monoidal structure on \( \mathcal{V}(G, \omega) \), making the latter a fusion category over \( C \) (§1.4), which is moreover pointed (every simple object is invertible).

In fact, as we explained in the first lecture, this example is universal in the sense that every pointed fusion category over \( C \) is equivalent to one of the form \( \mathcal{V}(G, \omega) \) for some finite group \( G \) and some (normalized) 3-cocycle \( \omega : G \times G \times G \to C^* \).

(1") The category \( \mathcal{V}(G, \omega) \) has a different "model" namely, the category \( \text{Vec}(G, \omega) \) (Nikshych and co-authors denote it by \( \text{Vec}^\omega \)). Its objects are finite dimensional \( G \)-graded \( C \)-vector spaces:
\[
\mathcal{V} = \bigoplus_{g \in G} V_g, \quad \dim_C(V) < \infty.
\]
The morphisms in this category are linear maps preserving the grading. The bifunctor
\[ \text{Vec}(G, \omega) \times \text{Vec}(G, \omega) \longrightarrow \text{Vec}(G, \omega) \]
should be thought of as the convolution:
\[ (V \ast W)_q = \bigoplus_{h, k \in G, \ h \cdot k = q} V_h \otimes W_k. \]
The unit object is defined by
\[ I_1 = C, \quad I_q = 0 \quad \text{for} \quad q \in G \setminus \{1\}. \]
The left and right unit constraints are
induced by the natural ones in the category of
\[ C \text{-vector spaces under tensor product.} \]
Finally, the associativity constraints are induced by
\[ \alpha : \left( U_q \otimes V_h \right) \otimes W_k \longrightarrow U_q \otimes \left( V_h \otimes W_k \right), \]
\[ (u \otimes v) \otimes w \longrightarrow \omega(g, h, k) \cdot u \otimes (v \otimes w). \]

(2) An example which is perhaps even better known
is the category \( \text{Rep}(G) \) of finite-dimensional
representations of \( G \) over \( C \). It is in fact a
symmetric monoidal category.

(3) One of the most basic examples of a modular
category is the category \( \mathcal{D}_G(G) \) of \( G \)-equivariant
sheaves of finite dimensional \( C \)-vector spaces
on \( G \), where \( G \) acts on itself by conjugation. The
precise definition of this category will be given
later in these notes. It has a natural braiding
and a natural pivotal structure.
The category $\mathcal{D}_G(G)$ is sometimes also called the "Drinfeld double" or "quantum double" of $G$. The reason for this terminology is that $\mathcal{D}_G(G)$ can be naturally identified with the Drinfeld center of either of the two monoidal categories $\text{Rep}(G)$ or $\text{Vec}(G, 1)$, the latter being the special case of example (1') corresponding to the trivial 3-cocycle $G \times G \times G \rightarrow \mathbb{C}^\times$, $(g, h, k) \mapsto 1$.

(4) It is natural to generalize example (3) in the following manner. Choose an arbitrary (normalized) 3-cocycle $\omega: G \times G \times G \rightarrow \mathbb{C}^\times$ and define $\mathcal{D}_G^\omega(G)$ to be the Drinfeld center of the category $\text{Vec}(G, \omega)$ (or, if you prefer, of the category $\text{Vec}(G, \omega)$). This category is sometimes called a "twisted Drinfeld double" or a "twisted quantum double" of $G$. (The "twisting" is by the 3-cocycle $\omega$.)

(5) Finally, we will discuss another class of examples of ribbon and modular categories. In the situation of example (1'), assume that $G$ is commutative and $\omega \equiv 1$. For any bimultiplicative map $B: G \times G \rightarrow \mathbb{C}^\times$, the category $\mathcal{D}(G, 1)$ has a natural braiding determined by $B(g, h)B(h, k) = B(g, h)$, and $B(g, h) = 1$. Moreover, there is a natural compatible pivotal structure which makes $\mathcal{D}(G, 1)$ a ribbon category. This ribbon category is modular if and only if the symmetrization $(g, h) \mapsto B(g, h)B(h, g)$ is nondegenerate.
Monoidal categories and monoidal functors

The material appearing in this subsection will not be discussed in detail during the lecture itself, due to the lack of time. However, this material is rather important for anyone who wishes to learn the general theory properly.

The basic picture is as follows. In abstract algebra, we have the notion of a monoid, and we have the notion of a homomorphism of monoids. In the theory of monoidal categories, the first notion has two counterparts ("monoidal category" and "strict monoidal category"), while the second notion has three different analogues ("weak monoidal functor", "strong monoidal functor", and "strict monoidal functor"). In addition, we have the notion of a morphism of monoidal functors, which has no counterpart in the world of monoids. Note also that a weak or strong monoidal functor is a functor equipped with an additional structure, whereas a morphism of monoidal functors is a natural transformation between the underlying functors, satisfying a certain property.

We now define all these notions precisely. For more details, we recommend (in addition to the references that were already mentioned) MacLane's book "Categories for the Working Mathematician".
We start with the notion of a monoidal category, which was explained during the first lecture, but did not appear in these notes yet.

**Definition.** A monoidal category is a 6-tuple $M = (M, \otimes, I, \alpha, \lambda, \rho)$ consisting of:

- a category $M$
- a bifunctor $\otimes : M \times M \to M$, called the "monoidal bifunctor", or the "tensor product in $M"$
- an object $I \in M$, called the "unit object"
- a collection of trifunctorial isomorphisms
  $$\lambda_{X,Y,Z} : (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z),$$
  one for each (ordered) triple $X, Y, Z \in M$, called the "associativity constraint" for $\otimes$
- collections of functorial isomorphisms
  $$\lambda_X : I \otimes X \cong X$$ and $p_X : X \otimes I \cong X$, for each $X \in M$, called the "left unit constraint" and the "right unit constraint" respectively.

These data are required to satisfy a certain coherence condition. Before stating it, let us explain why a condition is needed at all. For concreteness, let us choose five objects of $M$, say $X_1, X_2, X_3, X_4, X_5$, and consider the following expression:

$$\left( I \otimes \left((X_1 \otimes I) \otimes (I \otimes X_2)\right) \otimes \left(X_3 \otimes (X_4 \otimes X_5)\right) \right) \otimes I$$
This expression determines a well defined object of $M$. Furthermore, thanks to the mere existence of the constraints $\alpha$, $\beta$ and $\rho$, this object is isomorphic to

$$(((X_1 \otimes X_2) \otimes X_3) \otimes X_4) \otimes X_5.$$ 

We would like to be able to replace "isomorphic" with "canonically isomorphic" in the previous sentence. However, using the isomorphisms $\alpha$, $\beta$ and $\rho$, as well as their inverses, we can construct many (in fact, infinitely many: PTAI isomorphisms between the two expressions we wrote down, and a priori there is no reason to expect all these isomorphisms to be the same.

This is precisely what the coherence axiom asks for. At this point the reader can probably guess the general statement of this axiom (of course, instead of five objects, we must allow any finite (possibly empty) collection of objects $X_1, \ldots, X_n \in M$, and we must also allow the insertion of any finite number of copies of the unit object). For a precise formulation of the coherence axiom along these lines, we refer the reader to [BK], Definition 1.1.7. (There is no point in copying their formulation.)

In practice, whenever one works with nontrivial associativity constraints, it is useful to know that the verification of the coherence axiom can
be reduced to checking a finite number of compatibility conditions. This follows from Mac Lane’s Coherence Theorem. Consider a 6-tuple $(M, \otimes, I, \otimes, \lambda, \rho)$ as in the definition of a monoidal category, but without assuming the coherence axiom.

Then the coherence axiom is satisfied if and only if the following two conditions hold.

(a) The pentagon axiom: for all $X, Y, Z \in M$, the following diagram commutes:

\[
\begin{array}{c}
\downarrow \otimes (Z \otimes W) \\
\downarrow \otimes (X \otimes Y) \otimes W \\
\downarrow \otimes \otimes \otimes W \\
\end{array}
\]

(b) The triangle axiom: for all $X, Y \in M$, the following diagram commutes:

\[
\begin{array}{c}
\downarrow \otimes (Y \otimes Z) \otimes W \\
\downarrow \otimes \otimes \otimes W \\
\end{array}
\]

Remarks. (1) Some authors use the conditions (a) and (b) as the definition of a monoidal category. (2) Of course, the “only if” direction of Mac Lane’s coherence theorem is obvious from the definition we gave. The “if” part is useful in practice. (3) It is sometimes useful to remember that, in addition to the “axioms” (a) and (b), the
Definition. A monoidal category \((\mathbf{M}, \otimes, 1, \lambda, \rho)\) is said to be:

(i) **strictly associative** if the bifunctor \(\otimes\) is strictly associative (in the sense explained at the beginning of §1.1), and
\[
\lambda_{X,Y,Z} := \text{id}_X \otimes \lambda_{Y,Z} \quad \forall X, Y, Z \in \mathbf{M};
\]

(ii) **strictly unital** if the functors \(X \mapsto 1 \otimes X\) and \(X \mapsto X \otimes 1\) are equal to the identity functor \(\text{Id}_\mathbf{M} : \mathbf{M} \rightarrow \mathbf{M}\), and
\[
\lambda_X = \text{id}_X = \rho_X \quad \forall X \in \mathbf{M};
\]

(iii) **strict** if it is both strictly associative and strictly unital.

Caution. If \(\mathbf{M}\) is a category equipped with a strictly associative bifunctor \(\otimes : \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{M}\),
for which a strict unit object exists, then there is a unique structure of a strict monoidal category on \( \mathbf{M} \). On the other hand, a monoidal category such that the underlying bifunctor \( \otimes \) is strictly associative need not itself be strictly associative! An example of this phenomenon is provided by the category \( \mathbf{M}(G, c) \) constructed in §2.1(4), for a nontrivial (normalized) 3-cocycle \( c \).

Now we move on to the three different notions of a monoidal functor between monoidal categories. Our recommendation for the reader is to try to avoid using the term “monoidal functor” without an additional adjective (“weak”, “strong” or “strict”). In the literature, the term “monoidal functor” or “tensor functor” without any extra adjectives usually refers to a strong monoidal functor.

**Definition.** Consider monoidal categories \( \mathbf{M} = (\mathbf{M}, \otimes, I, \alpha, \lambda, \rho) \) and \( \mathbf{M}' = (\mathbf{M}', \otimes', I', \alpha', \lambda', \rho') \), and a triple \( \mathbf{F} = (F_0, F_1, F_2) \) consisting of
- a functor \( F_1 : \mathbf{M} \to \mathbf{M}' \) between the underlying categories;
- a collection of bifunctorial morphisms
  \[ F_2(X, Y) : F_1(X) \otimes' F_1(Y) \to F_1(X \otimes Y) \]
  for all \( X, Y \in \mathbf{M} \); and
- a morphism \( F_0 : I' \to F_1(I) \).
(a) The triple $F$ is said to define a weak monoidal functor from $M$ to $M'$ if the following diagrams commute: $\forall X, Y, Z \in M$:

\[
\begin{align*}
&F_1((X \otimes Y) \otimes Z) \xrightarrow{F_2(\otimes X, X, Z)} F_1(X \otimes (Y \otimes Z)) \\
&F_1(X \otimes Y) \otimes F_1(Z) \xrightarrow{F_2(X, Y, \otimes Z)} F_1(X) \otimes F_1(Y) \otimes F_1(Z)
\end{align*}
\]

\[
\begin{align*}
&F_1(I) \otimes F_1(X) \xleftarrow{F_0 \otimes \text{id}} I' \otimes F_1(X) \\
&F_2(I, X) \downarrow \quad F_1(\otimes X) \xrightarrow{F_1(\otimes X)} F_1(X)
\end{align*}
\]

\[
\begin{align*}
&F_1(X) \otimes F_1(I) \xleftarrow{\text{id} \otimes F_0} F_1(X) \otimes I' \\
&F_2(X, I) \downarrow \quad F_1(p_X) \xrightarrow{F_1(p_X)} F_1(X)
\end{align*}
\]

(b) The triple $F$ is said to define a strong monoidal functor from $M$ to $M'$ if it defines a weak monoidal functor, and $F_0$ and all the morphisms $F_2(X, Y)$ are isomorphisms.

c) The triple $F$ is said to define a strict monoidal functor from $M$ to $M'$ if it defines a weak monoidal functor, and $F_0$ and all the morphisms $F_2(X, Y)$ are identity morphisms in $M'$. 
Remark. Thus, giving a strict monoidal functor between \( \mathcal{M} \) and \( \mathcal{M}' \) is the same as giving a functor \( F_1 : \mathcal{M} \to \mathcal{M}' \) between the underlying categories such that \( F_1(1) = 1' \), 
\( \otimes' \circ (F_1 \times F_1) = F_1 \circ \otimes \) as bifunctors \( \mathcal{M} \times \mathcal{M} \to \mathcal{M}' \), and 
\[ F_1 (x, y, z) = F_1(x) \cdot F_1(y) \cdot F_1(z) \quad \forall x, y, z \in \mathcal{M}. \]

Finally, we arrive at the last important definition.

**Definition.** Let \( \mathcal{M} \) and \( \mathcal{M}' \) be monoidal categories as in the previous definition, and let \( F = (F_0, F_1, F_2) \) and \( G = (G_0, G_1, G_2) \) be weak monoidal functors from \( \mathcal{M} \) to \( \mathcal{M}' \). A morphism of monoidal functors between \( F \) and \( G \) is a natural transformation \( \eta : F_1 \to G_1 \), \( \eta = \{ \eta_x : F_1(x) \to G_1(x) \mid x \in \mathcal{M} \} \), such that the following diagrams commute:

\[
\begin{align*}
F_0 & \xrightarrow{F_1(1)} F_1(1) \\
G_0 & \xrightarrow{G_1(1)} G_1(1)
\end{align*}
\]
and

\[
\begin{align*}
F_1(x) \otimes' F_1(y) & \xrightarrow{\eta_x \otimes \eta_y} G_1(x) \otimes G_1(y) \\
F_1(x \otimes y) & \xrightarrow{\eta_{x \otimes y}} G_1(x \otimes y)
\end{align*}
\]

for all \( x, y \in \mathcal{M} \).

Exercise. Suppose that \( F : \mathcal{M} \to \mathcal{M}' \) is a strong monoidal functor such that the underlying functor \( F_1 : \mathcal{M} \to \mathcal{M}' \) is an equivalence. Show that if \( G_1 : \mathcal{M}' \to \mathcal{M} \) is quasi-inverse to \( F_1 \), then \( G_1 \) can be “upgraded” to a strong monoidal functor from \( \mathcal{M}' \) to \( \mathcal{M} \) such that \( F_0 G \cong \text{Id}_{\mathcal{M}'} \) and \( G_0 F \cong \text{Id}_{\mathcal{M}} \) as monoidal functors.
Braided monoidal categories. Now we will introduce one of the most important definitions of this series of lectures. Our approach will be purely algebraic (in a sense which will soon become clear). It is possible to give the definition of a BMC in the same spirit as the definition of a monoidal category we explained in §2.2, and there is an analogue of MacLane’s coherence theorem for BMCs. For this, we refer the reader to §1.2 of [BK]. The advantage of this definition is that it brings braid groups into the picture from the very beginning, and, in particular, explains the term “braided monoidal.”

However, we decided to take the statement of the coherence theorem for BMCs as the definition of a BMC.

**Definition.** A braided monoidal category (BMC) is a 7-tuple \( (\mathcal{M}, \otimes, 1, \alpha, \beta, \lambda, \rho) \), where 
\( (\mathcal{M}, \otimes, 1, \alpha, \lambda, \rho) \) is a monoidal category and \( \beta \) is a collection of bifunctorial isomorphisms 
\[ \beta_{X,Y} : X \otimes Y \xrightarrow{\alpha} Y \otimes X \quad \forall X, Y \in \mathcal{M} \]
satisfying the following “hexagon axioms,” stating that for all \( X, Y, Z \in \mathcal{M} \), the next two diagrams are commutative:

\[
\begin{array}{ccc}
(X \otimes Y) \otimes Z & \xrightarrow{\beta_{X \otimes Y, Z}} & Z \otimes (X \otimes Y) & \xrightarrow{\Delta_{X, Y}} & (Z \otimes X) \otimes Y \\
\downarrow & & & & \\
X \otimes (Y \otimes Z) & & \xrightarrow{id \otimes \beta_{Y, Z}} & X \otimes (Z \otimes Y) & \xrightarrow{\Delta_{X, Z, Y}} & (X \otimes Z) \otimes Y
\end{array}
\]
and
\[ X \otimes (Y \otimes Z) \xrightarrow{\beta_{X,Y \otimes Z}} (Y \otimes Z) \otimes X \xrightarrow{\alpha_{Y \otimes Z, X}} Y \otimes (Z \otimes X) \]
\[ \downarrow_{\lambda_{X,Y \otimes Z}} \]
\[ (X \otimes Y) \otimes Z \xrightarrow{\beta_{X,Y \otimes Z} \otimes 1} (Y \otimes X) \otimes Z \xrightarrow{\alpha_{Y \otimes X, Z}} Y \otimes (X \otimes Z) \]

**Exercise.** Show that in any BMC, the diagram
\[ 1 \otimes X \xrightarrow{1 \otimes \beta_{1,X}} X \otimes 1 \xrightarrow{\beta_{X,1}} 1 \otimes X \]
\[ \downarrow_{\lambda_{X}} \quad \leftarrow \quad \downarrow_{\lambda_{X}} \]
commutes for every object \( X \). Deduce that
\[ \beta_{X,1} \circ \beta_{1,X} = 1 \otimes X \quad \text{and} \quad \beta_{1,X} \circ \beta_{X,1} = 1 \otimes X \]
and that \( \beta_{1,1} = 1 \otimes 1 \).

**Definition.** In the situation of the definition above, the braiding \( \beta \) is said to be symmetric if \( \beta_{Y,X} \circ \beta_{X,Y} = 1 \otimes X \otimes Y \) for all \( X, Y \in M \).

A symmetric monoidal category is a BMC in which the braiding is symmetric.

**Remark.** Most of the BMCs that we will consider will NOT be symmetric. In particular, a modular category over C is not symmetric unless it is equivalent to the category of finite dimensional C-vector spaces with its standard braided monoidal and pivotal structures.
Remark. The definition of a BMC is easier to remember if the underlying monoidal category is strict. In this case the hexagon axioms reduce to the following identities \( \forall X, Y, Z \in M: \)

\[
\beta_{X,Y,Z} = (\beta_{X,Z} \otimes \text{id}_Y) \circ (\text{id}_X \otimes \beta_{Y,Z})
\]

\[
\beta_{X,Y,Z} = (\text{id} \otimes \beta_{X,Z}) \circ (\beta_{X,Y} \otimes \text{id}_Z)
\]

The intuitive meaning of these identities should be fairly clear. For instance, the first one says that if we want to “commute \( X \otimes Y \) past \( Z \),” it does not matter whether we do it “at once,” or we first commute \( Y \) past \( Z \), and then commute \( X \) past \( Z \). The second identity has a similar interpretation. Thus, even without knowing the connection between BMCs and braid groups, the axioms of a BMC will hopefully seem natural.

Example. Let \( G \) be a finite group, and consider the strict monoidal category \( M(G,1) \) defined in Example 2.1 (1). What are all the possible braidings on this category? First of all, a monoidal category \( M \) can only have a braiding provided \( X \otimes Y \cong Y \otimes X \) for all \( X, Y \in M \). Thus \( M(G,1) \) cannot have a braiding unless \( G \) is commutative.

Let us assume that \( G \) is commutative. In this case, a collection of bifunctorial isomorphism

\[
\beta_{X,Y}: X \otimes Y \cong Y \otimes X \quad \text{for all} \ X, Y \in M(G,1)
\]

to a map \( B: G \times G \to \mathbb{C}^x \). Furthermore, the
hexagon axioms reduce to the identities
\[ B(gh, k) = B(g, k)B(h, k) \quad \forall g, h, k \in C^x, \]
\[ B(g, hk) = B(g, h)B(g, k) \]
which mean that the map \( B \) is bimultiplicative.

By way of notation, given any bimultiplicative map \( B : G \times G \to C^x \), we will denote by \( B(G, B) \) the monoidal category \( M(G, 1) \) equipped with the braiding given by \( \beta_{g, h} = B(g, h) \). Thus \( B(G, B) \) is our first example of a BMC. Observe that \( B(G, B) \) is symmetric if and only if the map \( B \) is skew symmetric in the sense that
\[ B(g, h) = B(h, g)^{-1} \quad \forall g, h \in G. \]

We will return to this example soon. First, however, we should discuss the relationship between rigidity and braidings in monoidal categories. Some parts of the notes that follow will overlap with my notes for the first lecture, since in my first talk I did not have enough time to cover tensor/fusion and pivotal/spherical categories (i.e., \$\S\$1.4-1.5 in my notes).

**§2.4. Duality in BMCs.** We begin with a simple-minded remark. Let \( V \) be a finite dimensional vector space over a field \( k \), let \( V^* = \text{Hom}_k(V, k) \), and let \( \text{ev}_V : V^* \otimes V \to k \) denote the natural linear map \( (f \otimes v) \mapsto f(v) \). We also have a natural map \( k \to V^* \otimes V \) determined by \( 1 \mapsto \sum v_i^* \otimes v_i \), where
\{v_i^*\} is any basis of $V$ and \{v_i\} is the dual basis of $V^\ast$ (it is easy to check that the element $\sum v_i^* \otimes v_i \in V^\ast \otimes_k V$ does not depend on the choice of the basis \{v_i\} of $V$). Now if $f : V \rightarrow V$ is any linear map, then the trace of $f$, call it $\text{tr}(f)$, equals the scalar by which the following composition acts on $k$:

$$
\begin{align*}
    k & \rightarrow V^\ast \otimes_k V \\
    & \xrightarrow{id \otimes f} V^\ast \otimes_k V \\
    & \xrightarrow{\text{ev}_V} k.
\end{align*}
$$

Now we could try to repeat the same definition in an arbitrary monoidal category $\mathcal{M}$. Of course, the first thing to note is that $k$ has to be replaced by the unit object $I$ of $\mathcal{M}$, so we could try to define the trace of an endomorphism $X \xrightarrow{\ell} X$ as an element of the monoid $\text{End}(I)$. However, even if we assume that $\mathcal{M}$ is rigid, we cannot imitate the formula recalled above, because there is no natural morphism $I \rightarrow X^\ast \otimes X$, and there is no natural morphism $^*X \otimes X \rightarrow I$.

However, if we assume that $\mathcal{M}$ is braided, there is an obvious way out: namely, consider the composition

$$
\begin{align*}
    1 & \xrightarrow{\text{coev}_X} \ X \otimes X^\ast \\
    & \xrightarrow{\beta_X, X^*} \ X^\ast \otimes X \\
    & \xrightarrow{id \otimes f} X^\ast \otimes X \\
    & \xrightarrow{\text{ev}_X} 1.
\end{align*}
$$

Unfortunately, if $\mathcal{M}$ is not symmetric, this presents two problems (related to each other):

(a) The choice made above is not quite canonical, because we could replace $\beta_X, X^*$ with $\beta_{X^*}^{-1}, X$. 

(b) ...
(b) If we define $t(f)$ to be the composition above, the map $f \mapsto t(f)$ may fail to be multiplicative, in the sense that we do not necessarily have $t(f \circ g) = t(f)t(g)$ for all $X, Y \in \mathcal{M}$, $f \in \text{End}(X)$, $g \in \text{End}(Y)$.

The latter problem can be rephrased in a different way. Let us recall from §1.5 that for any morphism $F : X \to X^{**}$ in a rigid monoidal category $\mathcal{M}$, the (quantum) trace, $\text{tr}(F)$, can be canonically defined as the composition

$$1 \xrightarrow{\text{coev}_X} X \otimes X^* \xrightarrow{F \otimes \text{id}} X^{**} \otimes X^* \xrightarrow{\text{ev}_X^*} 1.$$ 

We can relate this to the discussion above as follows.

**Exercise.** If $\mathcal{M}$ is a rigid BMC, show that for any $X \in M$, the following diagram commutes:

```
\[\begin{array}{ccc}
X & \xrightarrow{\text{coev}_X \otimes \text{id}} & X^* \otimes X^{**} \otimes X \\
\downarrow{\text{id} \otimes \text{coev}_X^*} & & \downarrow{\beta_{X^*, X^{**}} \otimes \text{id}} \\
X & \xrightarrow{\beta_{X^*, X^{**}} \otimes \text{id}} & X^{**} \otimes X \otimes X^* \\
\end{array}\]
```

If $\psi_X : X \to X^{**}$ denotes the morphism defined by this diagram, show that $\psi_X$ is an isomorphism. Finally, if, for a morphism $f : X \to X$, we define $t(f) \in \text{End}(1)$ as above, then $t(f) = \text{tr}(\psi_X \circ f)$.

Thus the problem is that, in general, $\psi$ may not be an isomorphism of monoidal functors.
This problem can be fixed as follows.

**Definition.** Let \( \mathcal{M} \) be a rigid BMC. A balancing on \( \mathcal{M} \) is an automorphism \( \Theta \) of the identity functor on \( \mathcal{M} \), \( \Theta = \{ \Theta_X : X \rightarrow X \mid X \in \mathcal{M} \} \), satisfying the following two conditions:

1. \( \Theta_{X \otimes Y} \circ (\Theta_X \otimes \Theta_Y)^{-1} = \Phi_{Y,X} \circ \Phi_{X,Y} \quad \forall X,Y \in \mathcal{M} \),
2. \( \Theta_X^* = \Theta_X \) for all \( X \in \mathcal{M} \).

**Exercise.** Show that if \( \Theta \) is a balancing on a rigid BMC \( \mathcal{M} \), then \( a_X := \Psi_X \circ \Theta_X \) defines a pivotal structure on \( \mathcal{M} \) (i.e., \( a_X \otimes a_Y = a_X \otimes a_Y \)), which is moreover automatically spherical (i.e., \( tr(a_X) = tr(a_X^*) \) for all \( X \in \mathcal{M} \)).

**Remark.** If a rigid BMC \( \mathcal{M} \) has a balancing, the set of all balancings on \( \mathcal{M} \) is a torsor under the group of invertible automorphisms \( \Psi \) of the identity functor on \( \mathcal{M} \) satisfying \( \Psi_X^* = \Psi_X \).

**Definition.** A ribbon category is a rigid BMC \( \mathcal{M} \) equipped with a balancing \( \Theta \). Sometimes \( \Theta \) is also called the "twist" of "the collection of twists" in \( \mathcal{M} \) and the individual automorphisms \( \Theta_X : X \rightarrow X \) could also be called the "twists".
Let us now return to the example $M = B(G, B)$ considered in the previous subsection. Recall that $G$ is a finite abelian group, $B : G \times G \to C^\times$ is a bi-multiplicative map, and $B(G, B)$ is the strict monoidal category $M(G, 1)$ equipped with the braiding defined by $B_{g, h} = B(g, h)$.

First we note that $B(G, B)$ is rigid. Indeed, for any $g \in G$, the identity morphisms $g^{-1} g \to 1$ and $1 \to g g^{-1}$ realize $g^{-1}$ as the left dual of $g$. For any morphism $g \to g'$ represented by $\lambda \in C^\times$, the dual morphism $g^{-1} \to g'^{-1}$ is also represented by $\lambda$.

In fact, $B(G, B)$ has a natural balancing given by $\Theta_g := B(g, g)$. Indeed, the first condition in the definition of a balancing follows from the "polarization identity"

$$B(g h, g h) B(g, g)^{-1} B(h, h)^{-1} = B(h, g) B(g, h),$$

while the second condition follows from

$$B(g^{-1}, g^{-1}) = B(g, g) \quad \forall g \in G.$$

According to an earlier remark, in order to describe all possible balancing structures on $B(G, B)$, we need to find all monoidal automorphisms $\psi$ of the identity functor on $B(G, B)$ that satisfy $\psi x^* = \psi x^*$ for all $x \in B(G, B)$. However, this amounts to saying that $\psi : G \to C^\times$ is a homomorphism with $\psi(g^{-1}) = \psi(g)^{-1}$ for all $g \in G$. 
Conclusion. The BMC $B(G,B)$ has a ribbon structure, and the set of all such structures is in bijection with the set of homomorphisms $\psi : G \to \{1, -1\}$. Given such a $\psi$, the corresponding balancing on $B(G,B)$ is given by $g \mapsto \psi(g) B(g, g)$.

§2.5. Modular categories. We are now ready to give the key definition of this series of lectures. Roughly speaking, a modular category is a ribbon category satisfying a certain nondegeneracy property. However, in order to formulate this property, we need additional structures and assumptions. For instance, we need $\text{End}(1)$ to be at least a ring in which multiplication is given by composition of endomorphisms. We also need some sort of semisimplicity and finiteness conditions on the underlying category.

The definition of a modular category is formulated under the minimal collection of extra assumptions in §II.1.4 of [Tu]. We will work in a more restrictive setting, which will suffice for the purpose of this lecture (and of Nikshych’s talks).

Let $k$ be an algebraically closed field, and let $M$ be a ribbon fusion (§1.4) category over $k$.

* Recently, several definitions of a modular category not relying on the semisimplicity assumption were proposed.
Recall that for any endomorphism \( f : X \to X \) of an object \( X \in M \), we can speak of the (quantum) trace, \( \text{tr}(f) \in k \), of \( f \), defined by
\[
\text{tr}(f) = \text{tr}(\psi_X \circ \Theta_X \circ f : X \to X^{**}),
\]
where \( \psi_X : X \cong X^{**} \) is the isomorphism that arises from the braiding, and \( \Theta \) is the balancing.

Now choose a set of representatives \( \{X_i\}_{i=1}^n \) of the isomorphism classes of simple objects of \( M \), and form the \( n \)-by-\( n \) matrix
\[
S = (S_{ij})_{1 \leq i, j \leq n}, \quad S_{ij} = \text{tr}(\beta_{X_i,j} \circ X_i \circ \beta_{X_i,j} \circ X_j).
\]
It is called the \( S \)-matrix of the category \( M \).

**Definition (Turaev).** A ribbon fusion category \( M \) is said to be modular if its \( S \)-matrix is invertible.

**Example.** The category \( \mathcal{B}(G, B) \) discussed earlier is not fusion, so we cannot ask whether it is modular or not. However, let us replace the category \( \mathcal{U}(G, 1) \) by its \( C \)-linear analogue \( \mathcal{V}(G, 1) \), defined in \( \S 2.1(1') \). It is clear that \( \mathcal{B}(G, B) \) also determines a braiding and a balancing on \( \mathcal{V}(G, 1) \) such that \( \beta_{X_j, X_k} = \mathcal{B}(g, h) \) and \( \Theta_{X_g} = \mathcal{B}(g, g) \) for all \( g \in G \). Thus \( \mathcal{V}(G, 1) \) becomes a ribbon fusion category which we will denote by \( \mathcal{R}(G, B) \). Of course, its \( S \)-matrix is the matrix
\[
(B(h, g) B(g, h))_{g, h \in G} \in C^x.
\]
Claim. If $G$ is a finite abelian group and $B : G \times G \to C^\times$ is a bimultiplicative map, the ribbon fusion category $R(G, B)$ is modular if and only if the symmetrized form $C : G \times G \to C^\times$, $(g, h) \to B(h, g) B(g, h)^{-1}$, is nondegenerate, i.e., induces an isomorphism $G \cong G^\ast = \text{Hom}(G, C^\times)$.

Proof. One direction is clear: if $C$ is degenerate, there exists $g \in G$, $g \neq 1$, such that $C(g, h) = 1$ for all $h \in G$, which means that the $S$-matrix $(C(g, h))_{g, h \in G}$ has two equal rows, and thus cannot be invertible.

Conversely, if $C$ is nondegenerate, let us use $C$ to identify $G$ with $G^\ast$ in the obvious way, so that the $S$-matrix becomes the matrix $S' = (C(g))_{g \in G, f \in G^\ast}$. However, this matrix is invertible because it is the matrix of the Fourier transform from functions on $G$ to functions on $G^\ast$, in a suitable basis. More concretely, we have $S' S = |G| I$, where $S^\ast$ is the conjugate transpose of $S$ and $I$ is the identity matrix of size $|G|$.

This example is the first concrete illustration of the principle that "symmetric and modular categories live on the opposite ends of the world of braided monoidal categories."
Another characterization of modularity.

In view of the previous example, the next result should not be completely unexpected (although it is rather strong). To state it, let us first introduce a

Definition (Müger). An object $X$ of a BMC $\mathcal{M}$ is said to be central if

$$\beta_{Y,X} \circ \beta_{X,Y} = \text{id}_{X \otimes Y}$$

for all $Y \in \mathcal{M}$.

(According to Müger, central objects have also previously been called “degenerate,” “transparent” and “pseudotrivial”; however, “central” is the most natural terminology.) The center$^2$ of a BMC $\mathcal{M}$ is the full subcategory $Z_2(\mathcal{M}) \subseteq \mathcal{M}$ formed by the central objects.

It is clear that $Z_2(\mathcal{M})$ is a symmetric monoidal subcategory of $\mathcal{M}$.

Theorem (A. Bruguières, M. Müger...). Let $\mathcal{M}$ be a ribbon fusion category over an algebraically closed field $k$ of characteristic 0 (this restriction is essential). Then $\mathcal{M}$ is modular if and only if $I$ is the only central simple object of $\mathcal{M}$.

$^2$Perhaps the term “Müger center” should be used?