§3.0. In this lecture we study some properties of modular fusion categories from a purely algebraic viewpoint (i.e., we do not explore the connection with low-dimensional topology).

We do provide one justification for the term "modular." Namely, we show that every modular fusion category gives rise to certain natural projective (finite dimensional) representations of the group $SL_2(\mathbb{Z})$.

§3.1. Conventions. If $\mathcal{C}$ is a category, we denote by $Z_0(\mathcal{C})$ the Bernstein center of $\mathcal{C}$, i.e., the monoid of all endomorphisms of the identity functor $\text{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. It is a commutative monoid. The group of invertible elements of $Z_0(\mathcal{C})$, i.e., the group of automorphisms of $\text{Id}_{\mathcal{C}}$, will be denoted by $Z_0(\mathcal{C})^\times$.

Let us recall, once again, that in all of our general definitions, constructions and results, it is tacitly assumed that all the monoidal categories we deal with are strictly associative and strictly unital.

We fix, once and for all, a field $k$. For simplicity, by a $k$-category we will mean a $k$-linear category that is equivalent to a finite direct sum of copies of the category $k$-vect of finite dimensional $k$-vector spaces.
§3.2. We now explain an important construction. Namely, given a pivotal BMC \( \mathcal{M} \), we will define a homomorphism of monoids
\[
\mu : (1_{\mathcal{M}}, \otimes) \to \mathbb{Z}_0(\mathcal{M}),
\]
where \((1_{\mathcal{M}}, \otimes)\) is the monoid of isomorphism classes of objects of \( \mathcal{M} \) with the multiplication induced by the bifunctor \( \otimes \).

The end goal is to show that if \( \mathcal{M} \) is a modular category over an (algebraically closed) field \( k \), then the map \( \mu \) induces an isomorphism of \( k \)-algebras
\[
K_0(\mathcal{M}) \otimes_k k \xrightarrow{\cong} \mathbb{Z}_0(\mathcal{M}),
\]
which moreover is closely related to the \( S \)-matrix of \( \mathcal{M} \) introduced in the last lecture.

Let us first recall that to say that \( \mathcal{M} \) is a pivotal BMC means the following:
- \( \mathcal{M} \) is a braided monoidal category;
- the underlying monoidal category of \( \mathcal{M} \) is rigid (see Lecture 4);
- \( \mathcal{M} \) is equipped with a pivotal structure, i.e., an isomorphism \( \alpha \) of monoidal functors between the identity functor \( \text{Id}_\mathcal{M} : \mathcal{M} \to \mathcal{M} \) and the "double dual" functor \( \mathcal{M} \to \mathcal{M}^*, \quad X \mapsto X^{**} \).

Now we fix an object \( X \in \mathcal{M} \). We wish to associate to \( X \) a functorial collection of endomorphisms \( \mu(X)_Z : Z \to Z \) for all \( Z \in \mathcal{M} \).
Given $Z \in \mathcal{M}$, we define $\mu(X)_Z$ as the composition

$$
Z = 1 \otimes Z \xrightarrow{\text{coev}_X \otimes \text{id}_Z} X \otimes X^* \otimes Z \\
\xrightarrow{\text{id} \otimes \text{ev}_{x, Z}} X \otimes Z \otimes X^* \xrightarrow{\text{id} \otimes \text{ev}_{x, Z}} X \otimes X^* \otimes Z \\
\xrightarrow{\alpha_X \otimes \text{id} \otimes \text{id}} X^* \otimes X^* \otimes Z \xrightarrow{\text{ev}_{X^*} \otimes \text{id}} 1 \otimes Z = Z.
$$

Remark. This definition of $\mu(X)_Z$ is taken from the statement of Theorem 3.1.12 in [BK]. The reader who is familiar with the graphical calculus for morphisms as described in §2.3 of op. cit. will verify without difficulty that the morphism $\mu(X)_Z$ can be represented by the following diagram:

It is completely obvious that the collection of morphisms $\mu(X)_Z$ is functorial in $Z$.

§3.3. Lemma. If $X, Y \in \mathcal{M}$, then

$$
\mu(X \otimes Y)_Z = \mu(X)_Z \circ \mu(Y)_Z \quad \forall Z \in \mathcal{M}
$$

Proof. The argument relies on the assumption that $\mathcal{A} = \{ \alpha_X : X \rightarrow X^{**} \mid X \in \mathcal{M} \}$ is an isomorphism of monoidal functors.
To apply this assumption, we need to describe the monoidal structure on the functor \( X \mapsto X^{**} \) more explicitly.

Let \( X, Y \in \mathcal{M} \). One can easily check that the morphisms
\[
Y^* \otimes X^* \otimes X \otimes Y \xrightarrow{\text{id} \otimes \text{ev} \otimes \text{id}} \text{Id} \\
\xrightarrow{\text{ev} \otimes \text{id}} Y^* \otimes X = Y^* \otimes Y
\]
and
\[
\text{Id} \xrightarrow{\text{coev} \otimes \text{id}} X \otimes Y \otimes Y^* \otimes X^* \\
\xrightarrow{\text{id} \otimes \text{coev} \otimes \text{id}} X \otimes Y \otimes Y^* \otimes X^* = X \otimes \text{Id} \otimes X^*
\]
satisfy the axioms needed to identify \( Y^* \otimes X^* \) with \((X \otimes Y)^*\). This identification yields the structure of a monoidal functor on the duality functor
\[
\mathcal{M} \xrightarrow{\text{duality}} \mathcal{M}^{\text{op}, \text{rev}} \xrightarrow{X \mapsto X^*} 
\]
where \( \mathcal{M}^{\text{op}, \text{rev}} \) denotes the monoidal category obtained from \( \mathcal{M} \) by reversing all the arrows and also replacing the monoidal bifunctor \( \otimes \) with the reverse one: \( \overset{\text{rev}}{\otimes} \mathcal{W} = W \otimes V \).

Thus we obtain a natural monoidal structure on the composition
\[
\mathcal{M} \xrightarrow{\text{duality}} \mathcal{M}^{\text{op}, \text{rev}} \xrightarrow{\text{duality}} \mathcal{M}, \ X \mapsto X^{**}.
\]
Now we proceed with the proof of the lemma. By the hexagon axioms in the definition of a BMC, we have a commutative diagram

\[
\begin{array}{ccc}
Y^* \otimes X^* \otimes Z & \xrightarrow{\beta^2_{Y^* \otimes X^*, Z}} & Z \otimes Y^* \otimes X^* \\
\downarrow & & \downarrow \beta_{Y^* \otimes X^*, Y^* \otimes X^*} \\
Y^* \otimes Z \otimes X^* & \xrightarrow{\beta^2_{Y^* \otimes Z, Z}} & Y^* \otimes Z \otimes X^*
\end{array}
\]

We can rewrite it more succinctly as follows:

\[
\begin{array}{ccc}
Y^* \otimes X^* \otimes Z & \xrightarrow{\beta^2_{Y^* \otimes X^*, Z}} & Y^* \otimes X^* \otimes Z \\
\downarrow & & \downarrow \beta_{Y^* \otimes X^*, Y^* \otimes X^*} \\
Y^* \otimes Z \otimes X^* & \xrightarrow{\beta^2_{Y^* \otimes Z, Z}} & Y^* \otimes Z \otimes X^*
\end{array}
\]

Hereafter, by a slight abuse of notation, if \( V, W \) are objects of any BMC, we write

\[
\beta^2_{V, W} := \beta_{W, V} \circ \beta_{V, W} : V \otimes W \cong W \otimes V.
\]

Now, recall that \( \mu(X \otimes Y) \) is defined as the composition

\[
\begin{array}{ccc}
1 \otimes Z & \xrightarrow{\text{ev}_{X \otimes Y}} & (X \otimes Y) \otimes (X \otimes Y)^* \otimes Z \\
\downarrow & & \downarrow \text{id}_{X \otimes Y} \otimes \beta^2_{X \otimes Y, Z} \\
(X \otimes Y) \otimes (X \otimes Y)^* \otimes Z & \xrightarrow{\alpha_{X \otimes Y} \otimes \text{id} \otimes \text{id}} & (X \otimes Y)^* \otimes (X \otimes Y)^* \otimes Z \\
\downarrow & & \downarrow \text{ev}_{(X \otimes Y)^*} \otimes \text{id} \\
1 \otimes Z & = & 1 \otimes Z.
\end{array}
\]

In view of the previous discussion, this composition equals the following one:
\[ Z = \mathbb{I} \otimes Z \xrightarrow{\text{cov}_X \otimes \text{id}_Z} X \otimes X^* \otimes Z = X \otimes \mathbb{I} \otimes X^* \otimes Z \]
\[
\text{id} \otimes \text{cov}_Y \otimes \text{id} \otimes \text{id} \xrightarrow{} X \otimes Y \otimes Y^* \otimes X^* \otimes Z
\]
\[
\text{id} \otimes \text{id} \otimes \text{id} \otimes \text{proj}_Y^* \otimes \text{id} \xrightarrow{} X \otimes Y \otimes Y^* \otimes Z \otimes X^*
\]
\[
\text{id} \otimes \text{id} \otimes \text{id} \otimes \text{proj}^*_X \otimes \text{id} \xrightarrow{} X \otimes Y \otimes Y^* \otimes Z \otimes X^*
\]
\[
\text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \xrightarrow{} X^* \otimes Y \otimes Y^* \otimes X^* \otimes Z
\]
\[
\text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \xrightarrow{} X^* \otimes \mathbb{I} \otimes X^* \otimes Z = X^* \otimes Z \otimes X^* \otimes Z
\]
\[
\text{env}_X^* \otimes \text{id} \xrightarrow{} \mathbb{I} \otimes Z = Z.
\]

We can clearly rewrite it as follows:
\[
Z = \mathbb{I} \otimes Z \xrightarrow{\text{cov}_X \otimes \text{id}_Z} X \otimes X^* \otimes Z \xrightarrow{\text{id} \otimes \text{proj}^*_X \otimes \text{id}} X \otimes Z \otimes X^* = X \otimes \mathbb{I} \otimes Z \otimes X^*
\]
\[
\text{id} \otimes \text{cov}_Y \otimes \text{id} \otimes \text{id} \xrightarrow{} X \otimes Y \otimes Y^* \otimes Z \otimes X^*
\]
\[
\text{id} \otimes \text{id} \otimes \text{id} \otimes \text{proj}^*_Y \otimes \text{id} \xrightarrow{} X \otimes Y \otimes Y^* \otimes Z \otimes X^*
\]
\[
\text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \xrightarrow{} X \otimes Y^* \otimes Y^* \otimes Z \otimes X^*
\]
\[
\text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \xrightarrow{} X \otimes \mathbb{I} \otimes Z \otimes X^* = X \otimes Z \otimes X^*
\]
\[
\text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \xrightarrow{\text{env}_Y^* \otimes \text{id} \otimes \text{id}} X \otimes X^* \otimes Z \xrightarrow{\text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id}} X^* \otimes X^* \otimes Z
\]
\[
\text{env}_X^* \otimes \text{id} \xrightarrow{} \mathbb{I} \otimes Z = Z.
\]

Using the definition of \( \mu(Y)_Z \), we reduce the previous composition to the following one:
\[ z = 1 \otimes z \xrightarrow{\text{corev}_x \otimes \text{id}_z} X \otimes X^* \otimes z \xrightarrow{\text{id}_x \otimes \beta_x^* \otimes z} \]

\[ \xrightarrow{\text{id} \otimes \mu(Y)_z \otimes \text{id}} X \otimes z \otimes X^* \]

\[ \xrightarrow{\text{id} \otimes j_z^* \otimes x} X \otimes X^* \otimes z \xrightarrow{\alpha_x \otimes \text{id} \otimes \text{id}} X^* \otimes X \otimes z \]

\[ \xrightarrow{\text{ev}_x \otimes \text{id}} 1 \otimes z \rightarrow z. \]

However, since the braiding is functional, we have

\[(\mu(Y)_z \otimes \text{id}_x^*) \circ \beta_x^* \circ z = \beta_x^* \circ z \circ (\text{id}_x^* \otimes \mu(Y)_z),\]

which immediately implies that the last composition equals \( \mu(X)_z \circ \mu(Y)_z \), as claimed.

\[ \text{§3.4.} \]

The homomorphism

\[ \mu : (\mathcal{M}, \otimes) \rightarrow \mathbb{Z}_0(\mathcal{M}), \]

which we constructed for any pivotal BMC \( \mathcal{M} \), is a convenient tool in the study of modular categories. Let us first introduce the following notion.

**Definition.** Let \( k \) be a field. A premodular category over \( k \) is a \( k \)-category \( \mathcal{M} \) (§3.1) equipped with a \( k \)-linear braided monoidal structure such that the unit object \( \mathbf{1} \in \mathcal{M} \) is simple, which is also rigid and equipped with a spherical structure.

In particular, once again, \( \mathcal{M} \) is rigid. We recall that a spherical structure on a rigid monoidal category is a pivotal structure such that for any object \( X \) of the category, the
Dimensions of $X$ and $X^*$ (which are defined in terms of the pivotal structure) are equal.

Remark. If $k$ is algebraically closed, a premodular category over $k$ is the same thing as a ribbon fusion category over $k$ in the sense of Lecture 2 and of Nikshych's earlier lectures in the seminar.

In particular, a premodular category $\mathcal{M}$ over $k$ automatically has a balancing structure $\Theta \in \mathcal{Z}_0(\mathcal{M})^\times$, i.e., a functorial collection of "twists" $\Theta_x : X \rightarrow X$ for all $X \in \mathcal{M}$, satisfying

$$\Theta_{X \otimes Y} = \beta_{X,Y}^{-\frac{1}{2}} \circ (\Theta_X \otimes \Theta_Y) \quad \forall X,Y \in \mathcal{M}$$

and $\Theta_{x^*} = \Theta_x^* \quad \forall X \in \mathcal{M}$.

The balancing structure is compatible with the spherical structure and the braiding in a suitable sense. For the time being, it is unnecessary to spell out this condition.

§3.5. Next we will recall the definition of a modular category given at the end of the previous lecture. We will formulate it only a little differently, in order to make our approach consistent with the approach of §3.1 of [BK]. We will see that the two definitions of modularity are obviously equivalent to each other.
§3.6. Notation. The following notation and conventions will be in effect for the rest of this lecture.

We fix a field $k$. The extra assumptions we need to make about $k$ (such as having characteristic zero or being algebraically closed) will be stated explicitly whenever they become necessary.

We fix a premodular category $\mathcal{M}$ over $k$. The braiding, the spherical structure, and the balancing on $\mathcal{M}$ will be denoted (as usual) by $\beta$, $\alpha$ and $\Theta$, respectively.

We fix a complete collection $\{X_i\}_{i \in I}$ of representatives of the isomorphism classes of simple objects of $\mathcal{M}$, so that the unit object $I \in \mathcal{M}$ appears in this collection. We denote by $0 \in I$ the unique index such that $X_0 = I$.

For every $i \in I$, we define $i^* \in I$ to be the unique index for which $X_{i^*} \cong X_i^\ast$. This gives an involution $i \mapsto i^*$ of the finite set $I$.

Note that $0^* = 0$.

Since $k \cong \text{End}_k(X_i)$ (by the definition of a $k$-category), we see that $\Theta_{X_i} = \Theta_i \cdot \text{id}_{X_i}$ for each $i \in I$, where $\Theta_i \in k^\times$ is uniquely determined.

Note that we have $\Theta_0 = 1$, for instance, because the definition of $\Theta$ implies that $\Theta_1 = \Theta \otimes 1 = \Theta \otimes \Theta_1 = \Theta$, so that $\Theta^2 = \Theta_0$.

Finally, we will write $d_i := \dim(X_i) = \text{tr}(\alpha_{X_i} : X_i \to X_{i^\ast}) \in k$. 

Lemma. Let $C$ be any semi-simple $k$-linear rigid monoidal category, and let $X \in C$ be such that $\dim_k \text{End}_k(X) = 1$ (in particular, $X$ is necessarily simple). For any isomorphism $f : X \to X^{**}$, we have $\text{tr}(f) \neq 0$ in $\text{End}_k(1)$. Thus the elements $d_i \in k$, defined at the end of §3.6, are all nonzero.

Proof. Recall that $\text{tr}(f) \in \text{End}_k(1)$ is defined as the composition (cf. §1.5)

$$
1 \xrightarrow{\text{coev}_X} X \otimes X^* \xrightarrow{f \otimes \text{id}_{X^*}} X^{**} \otimes X^* \xrightarrow{\text{ev}_{X^*}} 1.
$$

Suppose that this composition equals 0. Note that $\text{coev}_X$ and $\text{ev}_{X^*}$ are both nonzero (because $X \neq 0$), and since $f \otimes \text{id}_{X^*}$ is an isomorphism, there exists a nonzero morphism $\text{coker}(\text{coev}_X) \to 1$. Since $C$ is semi-simple, this implies that there exist two nonzero morphisms $1 \to X \otimes X^*$ whose images have zero intersection. Hence $\dim_k \text{Hom}_k(1, X \otimes X^*) \geq 2$. On the other hand, we have

$$
\text{Hom}_k(1, X \otimes X^*) \cong \text{Hom}_k(X, X),
$$

because the functor $? \otimes X^*$ is right adjoint to the functor $? \otimes X$. This contradicts the assumption that $\dim_k \text{End}_k(X) = 1$. //
§3.8. We return to the more specialized setup of §3.6. Remark. The definition of modularity that we will now give is independent of Lemma 3.7 and of much of the notation introduced in §3.6. Thus, in a revised version of these notes, it might be better to give the definition of a modular category right after the definition of a premodular category.

Note that if $\mathcal{M}$ is a premodular category over $k$, then the Grothendieck group $K_0(\mathcal{M})$ is a commutative ring, which is free of finite rank as an abelian group. Moreover, it is clear that $\mu(x \oplus y) = \mu(x) + \mu(y)$ for all $x, y, z \in \mathcal{M}$, where the map $\mu$ was defined in §3.2, which means that $\mu$ induces a ring homomorphism (cf. Lemma 3.3)

$$\mu : K_0(\mathcal{M}) \to Z_0(\mathcal{M})$$

On the other hand, $Z_0(\mathcal{M})$ is isomorphic, as a $k$-algebra, to a finite product of copies of $k$. With the notation of §3.6, we can canonically identify $Z_0(\mathcal{M})$ with the $k$-algebra of all functions $I \to k$ under pointwise operations. Note also that $K_0(\mathcal{M})$ has rank $|I|$. Thus the next definition at least has a chance of being useful.
3.9. Definition: A premodular category $\mathcal{M}$ over a field $k$ is said to be modular if the map $\mu : k_0(\mathcal{M}) \rightarrow z_0(\mathcal{M})$ induces an isomorphism of $k$-algebras $k \otimes k_0(\mathcal{M}) \cong z_0(\mathcal{M})$.

3.10. Let us explore some immediate properties of modular categories and connect Definition 3.9 with the definition given in the previous lecture and the one appearing in §3.1 of [BK].

First, both $k \otimes k_0(\mathcal{M})$ and $z_0(\mathcal{M})$ have natural bases labeled by the set $I$. Namely, for the first space the basis is given by the classes $[X_i] \in k_0(\mathcal{M})$ of the simple objects $\{X_i\}_{i \in I}$, and for $z_0(\mathcal{M}) \cong Fun(I)$ the basis is given by the delta-functions at the elements of $I$.

Let us compute the matrix of the map $k \otimes k_0(\mathcal{M}) \rightarrow z_0(\mathcal{M})$ induced by $\mu$ with respect to these bases. This means that we need to compute the scalar by which the endomorphism $\mu(X_i)X_j$ acts on $X_j$, for all pairs $(i,j) \in I \times I$.

To this end, we first prove some auxiliary results, the logical place for which is much earlier in this series of lecture notes.
Lemma. Let \( \mathcal{M} \) be a rigid monoidal category.

(a) If \( \mathcal{F} = \{ \phi_X : X \to X | X \in \mathcal{M} \} \) is any automorphism of the identity functor \( \text{Id}_\mathcal{M} : \mathcal{M} \to \mathcal{M} \) as a monoidal functor (i.e., \( \phi_X \otimes \phi_Y = \phi_{X \otimes Y} \)), then

\[
\phi_X = \text{id}_X \quad \text{and} \quad \phi_X^* = (\phi_X)^{-1} \quad \forall X \in \mathcal{M}
\]

(b) If \( \alpha \) is any pivotal structure on \( \mathcal{M} \), then with the natural identification \( \mathbb{1} \simeq \mathbb{1} \), we have \( \alpha_{\mathbb{1}} = \text{id}_{\mathbb{1}} \), and also:

\[
\alpha_{X^*} = (\alpha_X)^{-1} \quad \forall X \in \mathcal{M}.
\]

Note that the last equality is meaningful, since \( \alpha_X : X^* \to X^{***} \) and \( \alpha_X^* : X^{***} \to X^* \).

Proof. (a) The identity \( \phi_{\mathbb{1}} = \text{id}_{\mathbb{1}} \) follows immediately from \( \phi_{\mathbb{1}} = \phi_{\mathbb{1} \otimes \mathbb{1}} = \phi_{\mathbb{1}} \otimes \phi_{\mathbb{1}} = \phi_{\mathbb{1}} \).

For the second identity, we will verify that

\[
\phi_X^* \circ \phi_X^* = \text{id}_X^*. \quad \text{To this end, recall that}
\]

\[
\phi_X^* : X^* \longrightarrow X^{***} \quad \text{is defined as the composition}
\]

\[
X^* = X^* \otimes \mathbb{1} \xrightarrow{\text{id} \otimes \text{coev}_X} X^* \otimes X \otimes X^* \xrightarrow{\text{id} \otimes \phi_X \otimes \text{id}}
\]

\[
\longrightarrow X^* \otimes X \otimes X^* \xrightarrow{\text{ev}_X \otimes \text{id}} \mathbb{1} \otimes X^* = X^*.
\]

Hence \( \phi_X^* \circ \phi_X^* \) is equal to the composition

\[
X^* = X^* \otimes \mathbb{1} \xrightarrow{\text{id} \otimes \text{coev}_X} X^* \otimes X \otimes X^* \xrightarrow{\text{id} \otimes \phi_X \otimes \phi_X^*}
\]

\[
\longrightarrow X^* \otimes X \otimes X^* \xrightarrow{\text{ev}_X \otimes \text{id}} \mathbb{1} \otimes X^* = X^*.
\]

Now we also have \( \phi_X \otimes \phi_X^* = \phi_X \otimes \phi_X^* \), and
since \( y \) is functorial, we have
\[
\eta_{\mathcal{x} \otimes \mathcal{x}} \circ \operatorname{coev}_\mathcal{x} = \operatorname{coev}_\mathcal{x} \circ \eta_{\mathcal{x}} = \operatorname{coev}_\mathcal{x}.
\]
Thus the composition above is equal to
\[
\mathcal{x}^* = \mathcal{x}^* \otimes 1 \xrightarrow{\operatorname{id} \otimes \operatorname{coev}_\mathcal{x}} \mathcal{x}^* \otimes \mathcal{x} \otimes \mathcal{x}^* \xrightarrow{\operatorname{ev}_\mathcal{x} \otimes \operatorname{id}} \mathcal{1} \otimes \mathcal{x}^* = \mathcal{x}^*
\]
which is \( \operatorname{id}_{\mathcal{x}^*} \) by the definition of duality.

(b) The argument here is similar, but with a small additional complication. Namely, the identity \( a_x = \operatorname{id}_1 \) is clear, and to prove that \( a_{\mathcal{x}^*} = (a^*_x)^{-1} \), we proceed as before, and are quickly reduced to showing that the following composition equals \( \operatorname{id}_{\mathcal{x}^{***}} \): 
\[
\mathcal{x}^{***} = \mathcal{x}^{***} \otimes \mathcal{1} \xrightarrow{\operatorname{id} \otimes \operatorname{coev}_\mathcal{x}} \mathcal{x}^{***} \otimes \mathcal{x} \otimes \mathcal{x}^*
\]
Moreover, the functoriality of \( a \) implies that
\[
\eta_{\mathcal{x} \otimes \mathcal{x}^*} \circ \operatorname{coev}_\mathcal{x} = \operatorname{coev}_\mathcal{x} \circ \eta_{\mathcal{x}^*} = \operatorname{coev}_\mathcal{x}.
\]
If we could show that \( \operatorname{coev}^{**} = \operatorname{coev}^* \) up to the usual identification \( (\mathcal{x} \otimes \mathcal{x}^*)^{**} = \mathcal{x}^{**} \otimes \mathcal{x}^{**} \), then the proof would be complete.

Hence we are reduced to two applications of

**Sublemma.** If \( \mathcal{M} \) is any rigid monoidal category and \( \mathcal{x} \in \mathcal{M} \), then \( \operatorname{ev}_{\mathcal{x}^*} = \operatorname{coev}_\mathcal{x}^* \) and \( \operatorname{coev}_{\mathcal{x}^*} = \operatorname{ev}_{\mathcal{x}^*} \) modulo the usual identifications \( (\mathcal{x} \otimes \mathcal{x}^*)^{**} = \mathcal{x}^{**} \otimes \mathcal{x}^{**} \) and \( (\mathcal{x}^* \otimes \mathcal{x})^{**} = \mathcal{x}^* \otimes \mathcal{x}^{**} \).
The sublemma is proved by a direct computation. Let us recall once again that the strong monoidal structure on the duality functor

\[ M \rightarrow M^{op}, x \mapsto x^* \]

is defined in such a way that the diagrams

\[ z^* \otimes y^* \otimes y \otimes z \xrightarrow{\sim} (y \otimes z)^* \otimes (y \otimes z) \xrightarrow{\text{ev}_{y \otimes z}} 1 \]

\[ \text{id} \otimes \text{ev}_{y \otimes z} \otimes \text{id} \]

\[ z^* \otimes 1 \otimes z = z^* \otimes z \]

and

\[ 1 \xrightarrow{\text{coev}_{y \otimes z}} (y \otimes z) \otimes (y \otimes z)^* \xrightarrow{\sim} y \otimes z \otimes 2 \otimes y^* \]

\[ \text{coev} \]

\[ \text{id} \otimes \text{coev}_{y \otimes z} \otimes \text{id} \]

\[ y \otimes y^* = y \otimes 1 \otimes y^* \]

commute for all \( y, z \in M \).

In particular, \( \text{ev}_x^* \) is identified with the following composition:

\[ 1 \xrightarrow{\text{coev}_x^*} x^* \otimes x^{**} = x^* \otimes 1 \otimes x^{**} \]

\[ \text{id} \otimes \text{coev}_x \otimes \text{id} \]

\[ x^* \otimes x \otimes x^* \otimes x^{**} \xrightarrow{\text{ev}_x \otimes \text{id} \otimes \text{id}} \]

\[ 1 \otimes x^* \otimes x^{**} = x^* \otimes x^{**} \]

But by definition, the composition

\[ x^* = x^* \otimes 1 \xrightarrow{\text{id} \otimes \text{coev}_x} x^* \otimes x^{**} \xrightarrow{\text{ev}_x \otimes \text{id}} 1 \otimes x^* = x^* \]

equals \( \text{id}_{x^*} \), so we deduce that \( \text{ev}_x^* = \text{coev}_x^* \).

The proof of the identity \( \text{coev}_x^* = \text{ev}_x^* \) is completely analogous.
§3.12. **Remark.** Lemma 3.11 shows that the definition of a ribbon category given in [BK, Definition 2.2.1] is incorrect (or, at least, weaker than the one appearing in [Ka] and [Tu]). Namely, we see that their definition amounts to the following one: a ribbon structure on a rigid BMC is a pivotal structure on the underlying monoidal category (indeed, in view of Lemma 3.11, the identities (2.2.3) and (2.2.4) in [BK] follow from (2.2.2)). This implies that with their definition, if a rigid BMC $C$ has a ribbon structure, then the set of all ribbon structures on $C$ is a torsor under the group $\text{Aut}_\otimes (\text{Id}_C)$ of monoidal automorphisms of $\text{Id}_C : C \to C$.

On the other hand, with the definition of a ribbon category used in [Ka] and [Tu] (which we reproduced in §2.4), we saw that if a rigid BMC has a ribbon structure, then the set of all such structures is a torsor under the group of monoidal automorphisms

$$\varphi = \{ \varphi_X : X \xrightarrow{\simeq} X \mid X \in C \}$$

of $\text{Id}_C$ satisfying $\varphi_{X^\ast} = \varphi_X^\ast$ for all $X \in C$. In view of Lemma 3.11(a), the last requirement is equivalent to $\varphi_X^2 = \text{id}_X$ for all $X \in C$. Thus the set of all ribbon structures on $C$ is a torsor under the subgroup of $\text{Aut}_\otimes (\text{Id}_C)$ formed by the elements of order 1 or 2.
Lemma. With the notation of §3.6, we have
\[
\text{tr} \left( \mu(x_i)_j : X_j \to X_j \right) =
\]
\[
\text{tr} \left( \beta x_i^* x_j : X_i^* \otimes X_j \to X_i^* \otimes X_j \right)
\]
for all \( i, j \in I \).

Proof. It will be convenient to let \( F \) denote the following composition:
\[
X_i^* \otimes X_j \otimes X_j^* \xrightarrow{\beta x_i^* x_j \otimes \text{id} x_j^*} X_i^* \otimes X_j \otimes X_j^* \xrightarrow{F} X_i^* \otimes X_j^* \otimes X_j^* \xrightarrow{\text{id} x_i^* \otimes a x_j \otimes \text{id} x_j^*} X_i^* \otimes X_j^* \otimes X_j^*
\]
The composition
\[
X_i^* = X_i^* \otimes I \xrightarrow{\text{id} \otimes \text{coev}_x^* \otimes \text{coev}_x^*} X_i^* \otimes X_j \otimes X_j^* \xrightarrow{F} X_i^*
\]
is a scalar operator, since \( X_i^* \) is simple.
Let us denote this scalar by \( \lambda \in \mathbb{F} \).

Now the trace of \( \mu(x_i)_j \) is the scalar determined by the following composition:
\[
1 \xrightarrow{\text{coev}_x^* \otimes \text{id} \otimes \text{id}} X_j \otimes X_j^* \xrightarrow{\text{id} \otimes X_j \otimes X_j^*} X_i \otimes X_i^* \otimes X_j \otimes X_j^* \xrightarrow{\text{id} \otimes \beta x_i^* x_j \otimes \text{id}} X_i \otimes X_i^* \otimes X_j \otimes X_j^* \xrightarrow{\text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id}} X_i^* \otimes X_i^* \otimes X_j \otimes X_j^* \xrightarrow{\lambda} 1 \otimes X_j \otimes X_j^* \xrightarrow{a x_j \otimes \text{id}} X_j \otimes X_j \xrightarrow{\text{id} \otimes X_j \otimes X_j} 1.
\]
It is clear that this composition can be rewritten as follows:

\[ 1 \xrightarrow{\text{coev}_{x_i}} x_i \otimes x_i^* = x_i \otimes x_i^* \otimes 1 \xrightarrow{\text{id} \otimes \text{id} \otimes \text{coev}_{x_j}} \]

\[ \rightarrow x_i \otimes x_i^* \otimes x_j \otimes x_j^* \xrightarrow{\text{id} \otimes x_i^* \otimes x_j \otimes \text{id}} x_i \otimes x_i^* \otimes x_j \otimes x_j^* \]

\[ \xrightarrow{\text{id} \otimes \text{id} \otimes x_j \otimes \text{id}} x_i \otimes x_i^* \otimes x_j \otimes x_j^* \xrightarrow{\text{id} \otimes \text{id} \otimes \text{ev}_{x_j}} x_i \otimes x_i^* \otimes 1 \xrightarrow{\text{ax}_i \otimes \text{id}} x_i^* \otimes x_i^* \otimes x_i \xrightarrow{\text{ev}_{x_i}^*} 1 \]

Using the definition of \( F \) and \( A \), we see that this composition equals

\[ 1 \xrightarrow{\text{coev}_{x_i}} x_i \otimes x_i^* \xrightarrow{F \otimes x_i^*} x_i \otimes x_i^* \xrightarrow{\text{ax}_i \otimes \text{id}} x_i^* \otimes x_i^* \xrightarrow{\text{ev}_{x_i}^*} 1 \]

which is the scalar \( \lambda \cdot \dim(x_i) \). Thus

\[ \text{tr}(\mu(x_i) x_j) = \lambda \cdot \dim(x_i) \quad (3.13.1) \]

On the other hand, the trace of \( B_{x_i, x_j} \) is the scalar determined by the composition

\[ 1 \xrightarrow{\text{coev}_{x_i}^*} x_i^* \otimes x_i^* = x_i^* \otimes 1 \otimes x_i^* \xrightarrow{\text{id} \otimes \text{coev}_{x_j} \otimes \text{id}} \]

\[ \rightarrow x_i^* \otimes x_j \otimes x_i^* \otimes x_i^* \xrightarrow{\text{ax}_i^* \otimes x_j \otimes \text{id} \otimes \text{id}} x_i^* \otimes x_j \otimes x_i^* \otimes x_i^* \]

\[ \xrightarrow{\text{ax}_i^* \otimes x_j \otimes \text{id} \otimes \text{id}} x_i^* \otimes x_j \otimes x_i^* \otimes x_i^* \xrightarrow{\text{id} \otimes \text{ev}_{x_i} \otimes \text{id}} x_i^* \otimes 1 \otimes x_i^* \xrightarrow{\text{ev}_{x_i}^*} 1 \]

Using the definition of \( F \) and \( A \), we can rewrite this composition as follows:

\[ 1 \xrightarrow{\text{coev}_{x_i}^*} x_i^* \otimes x_i^* \xrightarrow{\lambda \cdot \text{id}} x_i^* \otimes x_i^* \xrightarrow{\text{ax}_i^* \otimes \text{id}} x_i^* \otimes x_i^* \xrightarrow{\text{ev}_{x_i}^*} 1 \]
By definition, this composition is given by the scalar \( A \cdot \dim(X^*_i) \). Thus

\[
\text{tr}(\beta^2_{X^*_i, X^*_j}) = A \cdot d_i^* \quad (3.13.2)
\]

Comparing (3.13.1) with (3.13.2) and using the fact that the pivotal structure \( \mathcal{M} \) is spherical, so that (by definition) \( d_i^* = d_i \), we obtain the equality claimed in Lemma 3.13.

\[\text{§3.14.}\]

We are ready to relate our definition of a modular category to the ones given in [BK] and in our previous lecture. We retain the notation of §3.6.

In Section 3.1 of [BK], the following square matrix of size \(|I|\) is introduced:

\[
\mathbf{S} = (S_{ij})_{i,j \in I}, \quad S_{ij} = \text{tr}(\beta^2_{X^*_i, X^*_j}).
\]

Let us also recall that \( d_i \neq 0 \) by Lemma 3.7. Now Lemma 3.13 implies at once the following observation.

**Observation.** For all \( i,j \in I \), the endomorphism \( \mu(X_i) : X_j \rightarrow X_j \) is given by the scalar \( \frac{S_{ij}}{d_j} \).

**Corollary.** The premodular category \( \mathcal{M} \) is modular in the sense of our Definition 3.9 if and only if the matrix \( \mathbf{S} \) is invertible.

Thus our definition of a modular category is equivalent to the more standard one given in [BK].
§3.15. Let us also recall that in §2.5 we introduced a different square matrix, which we called the $S$-matrix of the premodular category $M$: 

$$S = (S_{ij})_{i,j \in I}, \quad S_{ij} = \text{tr} \left( \beta_{\overline{i} \overline{j}}^2 X_{\overline{i}} X_{\overline{j}} \right).$$

Using the notation of §3.6, we see that the matrices $S$ and $\tilde{S}$ are related to each other by an involution of the rows, namely,

$$\tilde{S}_{ij} = S_{i* j} \quad \text{and} \quad S_{ij} = \tilde{S}_{i* j} \quad \forall i,j \in I.$$

Thus $\tilde{S}$ is invertible if and only if $S$ is invertible, which shows that the definitions of modularity given in [BK] and in our previous lecture are also equivalent.

§3.16. Let us introduce some more notation, taken from [BK, Theorem 3.1.7].

We let $t$ be the diagonal matrix with diagonal entries $\Theta_{\overline{i}}$:

$$(t_{\overline{i} \overline{j}}) = t = \text{diag}(\Theta_{\overline{i}})_{\overline{i} \in I}, \quad i.e., \quad t_{\overline{i} \overline{j}} = \delta_{\overline{i} \overline{j}} \Theta_{\overline{i}}.$$

We also let $c$ be the permutation matrix defined by the involution $i \mapsto i^*$ of $I$:

$$c = (c_{\overline{i} \overline{j}})_{\overline{i} \overline{j} \in I}, \quad c_{\overline{i} \overline{j}} = \delta_{\overline{i} \overline{j}^*}.$$

It follows that

$$S^t = c \cdot \tilde{S}.$$
The next result is good to keep in mind.

**Lemma.** The matrix $S$ is symmetric, i.e., $S_{ij} = S_{ji}$ for all $i, j \in I$. In particular, it commutes with the matrix $c$, whence $S = c \cdot S = S \cdot c$ is also symmetric.

**Proof.** If $\Theta$ is the balancing on $\mathcal{M}$, the identity $\Theta_x \otimes \Theta_y = (\Theta_x \otimes \Theta_y) \circ f^2_{x,y}$ for all $x, y \in \mathcal{M}$ implies that

$$S_{ij} = \Theta_i \otimes \Theta^{-1}_j \cdot \text{tr}(\Theta_x \otimes x_j)$$

for all $i, j \in I$. On the other hand, by the functoriality of $\Theta$, we have

$$\Theta_{x_j \otimes x_i} = \theta_{x_i, x_j} \circ \Theta_{x_i \otimes x_j} \circ \theta^{-1}_{x_i, x_j},$$

which immediately implies that

$$\text{tr}(\Theta_{x_j \otimes x_i}) = \text{tr}(\Theta_{x_i \otimes x_j}),$$

proving the lemma. \(/\)

Diagonalizing the fusion rules. Let us fix a modular category \( \mathcal{M} \) over \( k \) and keep the notation of \( \S 3.6 \). In this subsection we deduce some interesting consequences of the fact that the map \( \mu : \mathbb{K}_0(\mathcal{M}) \rightarrow \mathbb{Z}_0(\mathcal{M}) \) induces an isomorphism of \( k \)-algebras \( \mathbb{K} \cong \mathbb{K}_0(\mathcal{M}) \rightarrow \mathbb{Z}_0(\mathcal{M}) \).

The main point is that with respect to the natural basis \( \{ [x_i] \}_{i \in I} \) of \( \mathbb{K} \otimes \mathbb{K}_0(\mathcal{M}) \), the multiplication in \( \mathbb{K} \otimes \mathbb{K}_0(\mathcal{M}) \) looks rather complicated. The formulas describing this multiplication are called the “fusion rules” for \( \mathcal{M} \). Explicitly, let us introduce a collection of nonnegative integers \( \{ N_{ij} \} \) determined by

\[
x_i \otimes x_j = \bigoplus_{k \in I} x_k^{\oplus N_{ij}^k}.
\]

On the other hand, multiplication in the \( k \)-algebra \( \mathbb{Z}_0(\mathcal{M}) \cong \text{Fun}(I) \) is very simple. Using the matrix \( S \) we can relate the two multiplications.

Namely, let \( \{ \xi_i \}_{i \in I} \) denote the natural basis of \( \mathbb{Z}_0(\mathcal{M}) \) (formed by the minimal idempotents in \( \mathbb{Z}_0(\mathcal{M}) \), if you will). Define \( y_i = \mu^{-1}(\xi_i) \in \mathbb{K} \otimes \mathbb{K}_0(\mathcal{M}) \). Then \( \{ y_i \}_{i \in I} \) gives a new basis of \( \mathbb{K} \otimes \mathbb{K}_0(\mathcal{M}) \).
Let us compute the matrix of the operator of multiplication by $[X_i]$ with respect to the basis $\{y_j\}_{j \in I}$. We have

$$[X_i] \cdot y_j = [X_i] \cdot m^{-1}(y_j)$$

$$= m^{-1}(m([X_i]) \cdot y_j)$$

$$= m^{-1}\left( \frac{1}{d_j} \tilde{s}_{ij} \cdot y_j \right)$$

$$= \left( \frac{\tilde{s}_{ij}}{d_j} \right) \cdot y_j.$$ 

On the other hand, if $N_i$ denotes the matrix $(N_i)_{j,k}$ of the operator of multiplication by $[X_i]$ with respect to the basis $\{[X_j]\}_{j \in I}$, then, by definition,

$$(N_i)_{j,k} = N_{i,k}.$$ 

To find the matrix which conjugates $N_i$ to the diagonal matrix $D_i := \text{diag}(\tilde{s}_{ij}/d_j)_{j \in I}$, we need to relate the bases $\{y_j\}_{j \in I}$ and $\{[X_j]\}_{j \in I}$ of $k \otimes K_0(M)$.

But by Observation 3.14, we have

$$M([X_j]) = \sum_{k \in I} \tilde{s}_{jk} \cdot d_k \cdot y_k,$$

which is equivalent to

$$[X_j] = \sum_{k \in I} \frac{\tilde{s}_{jk}}{d_k} \cdot m^{-1}(y_k).$$

This leads at once to the formula

$$\tilde{s} \cdot N_i \cdot \tilde{s}^{-1} = D_i \quad \forall i \in I.$$
Another formula for $\mathcal{S}$. We proved in §3.17 that the fusion rules in a modular category can be expressed easily in terms of the matrix $\mathcal{S}$. Namely, if $N_i = ((N_{ij})_{j,k \in I}$ are the matrices encoding the fusion rules, then

$$N_i = \mathcal{S}^{-1} D_i \mathcal{S},$$

where $D_i$ is the diagonal matrix

$$D_i = \text{diag} \left( \frac{\mathcal{S}_{ij}}{\mathcal{S}_{0j}} \right)_{j \in I}.$$

(Indeed, since $\beta_{X_i \otimes X_j}^2 = id_{X_i \otimes X_j}$, we have $\mathcal{S}_{0j} = \text{dim}(X_j) = d_j$.)

In this subsection we record the (much more elementary) observation that, conversely, for any premodular category, the matrix $\mathcal{S}$ can be easily recovered from the fusion rules and the scalars $\{\Theta_i \in k^2\}_{i \in I}$ (defined by $\Theta_i = \Theta_i \cdot id_{X_i}$).

Indeed, recall that

$$\Theta_i \otimes \Theta_j = \beta_{X_i \otimes X_j}^2 \circ (\Theta_i \otimes \Theta_j) = \Theta_i \cdot \Theta_j \cdot \beta_{X_i \otimes X_j}^2,$$

so that, in particular,

$$S_{ij} = tr (\beta_{X_i \otimes X_j}^2) = \Theta_i^{-1} \Theta_j^{-1} tr (\Theta_i \otimes \Theta_j).$$

On the other hand, from the functoriality of $\Theta$ and the fact that

$$X_i \otimes X_j \cong \bigoplus_{k \in I} X_k \otimes N_{ij}^k,$$
we obtain
\[ \text{tr}(\Theta_{x_i \otimes x_j}) = \sum_{k \in I} N_{ij}^k \Theta_k \dim(x_k), \]
and therefore
\[ S_{ij} = \Theta_{i}^{-1} \Theta_{j}^{-1} \sum_{k \in I} N_{ij}^k \Theta_k d_k. \]
Using the fact that \( \Theta_i^* = \Theta_i \), we deduce
\[ S_{ij} = \Theta_{i}^{-1} \Theta_{j}^{-1} \sum_{k \in I} N_{ij}^k \Theta_k d_k \]
(cf. formula (3.1.2) in [BKJ]).

§3.19. The rank and the central charge of a modular category
In the remainder of this lecture there will be almost no proofs. We will record some properties of modular categories and identities that hold on these categories, culminating with the construction of a projective representation of the group \( SL_2(\mathbb{Z}) \) that can be naturally associated to any modular category.

We retain the notation of §3.6, and let \( \mathcal{M} \) be any premodular category over \( k \). The \textbf{categorical dimension} of \( \mathcal{M} \) is defined as
\[ \dim(\mathcal{M}) = \sum_{c \in I} d_c^2. \]
We say that \( \mathcal{M} \) is \underline{nondegenerate} if \( \dim(\mathcal{M}) \neq 0 \). Later on we will see that every modular category is nondegenerate.
Definition. A rank of a premodular category \( \mathcal{M} \) is an element \( D \in k \) such that \( D^2 = \text{dim}(\mathcal{M}) \). The Gauss sums for \( \mathcal{M} \) are the elements \( p^+, p^- \in k \) defined by

\[
p^\pm = \sum_{i \in I} \Theta_i^{-1} d_i^2
\]

Fact. We have

\[
p^+, p^- = \text{dim}(\mathcal{M}).
\]

Thus, if \( \mathcal{M} \) is nondegenerate (e.g., if \( \mathcal{M} \) is modular), then \( p^\pm \neq 0 \).

From now on we will assume that \( \text{dim}(\mathcal{M}) \) is a square in \( k \), and we fix \( D \in k \) with \( D^2 = \text{dim}(\mathcal{M}) \). In practice, one often works in a situation where \( k \) is algebraically closed, so that a rank always exists. If we also have \( \text{char}(k) = 0 \), then there is a preferred choice of \( \sqrt{\text{dim}(\mathcal{M})} \), but I don’t want to discuss this story.

Let us assume now that \( \mathcal{M} \) is modular, and let the matrices \( \Sigma, t, \) and \( c \) be defined as in \( \S 3.16 \).

Theorem ([BK], Theorem 3.1.7). We have:

\[
(\Sigma t)^5 = p^+ \Sigma^2,
\]

\[
(\Sigma t^{-1})^3 = p^- \Sigma^2 c^2,
\]

\[ct = t c, \quad c \Sigma = \Sigma c, \quad c^2 = 1,\]

\[\Sigma^2 = p^+ p^- c = \text{dim}(\mathcal{M}) \cdot c.\]
The last identity implies that $\mathcal{M}$ is nondegenerate, i.e., $D \neq 0$. The multiplicative central charge of $\mathcal{M}$ is defined by

$$\mathcal{F}(\mathcal{M}) = \frac{p^+}{p^-} = \frac{D}{p^-}.$$ 

We note that $\mathcal{F}(\mathcal{M})$ is a square root of $p^+/p^-$ and that it depends on the choice of $D$.

Remarks. (1) In Chapter 5 of [BK], it is $p^+/p^-$ that is called the multiplicative central charge of $\mathcal{M}$. We will return to this issue next time.

(2) For $k = \mathbb{C}$, according to Remark 3.1.20 in [BK], one sometimes writes

$$\Theta_i = e^{2\pi i \frac{\Delta_i}{\mathcal{F}(\mathcal{M})}} = e^{\pi \sqrt{\mathcal{F}(\mathcal{M})} \cdot c/4},$$

where the numbers $\Delta_i$ are called the conformal dimensions and $c$ is called the Virasoro central charge of $\mathcal{M}$.

§3.20. Modular categories and representations of $\text{SL}_2(\mathbb{Z})$.

We keep the same setup as in §3.19. In particular, we have a modular category $\mathcal{M}$ over $k$ with a chosen rank $D \in k$. Let us define

$$s = \frac{1}{\mathcal{F}(\mathcal{M})^2}.$$ 

sometimes $s$ is called the $s$-matrix of $\mathcal{M}$.

The relations listed in the last result can then be rewritten as follows:

$$(st)^3 = \mathcal{F}(\mathcal{M}) \cdot s^2, \quad s^2 = c, \quad ct = tc, \quad c^2 = 1.$$ 

Using these relations, we will now define a projective representation of $\text{SL}_2(\mathbb{Z})$. 
To this end, let us recall that $SL_2(\mathbb{Z})$ is generated by the two elements

\[ \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \]

subject to the relations

\[(\sigma \tau)^3 = \sigma^2 \quad \text{and} \quad \sigma^2 = 1.\]

It follows immediately that sending $\sigma \mapsto s$ and $\tau \mapsto t$ defines a projective representation of $SL_2(\mathbb{Z})$ on the vector space $k^I$.

**Remark.** If $\zeta(M) = 1$ (i.e., $\sigma^+ = \sigma^-$), this projective representation is a true one. If $\zeta(M) \neq 1$, suppose that there exists $\zeta \in k$ such that $\zeta^3 = \zeta(M)$ (e.g., this holds if $k$ is algebraically closed). Then the projective representation of $SL_2(\mathbb{Z})$ on $k^I$ constructed above comes from a true one, given by $\sigma \mapsto s$ and $\tau \mapsto t/\zeta$.

**\S 3.21.** An example. We conclude this section by exhibiting a modular category over $k = \mathbb{C}$ such that the associated representation of $SL_2(\mathbb{Z})$ comes from the Weil representation of the group $SL_2(\mathbb{F}_p)$.

Let us make this statement more precise.

**Exercise.** For any $n \in \mathbb{N}$, the "reduction mod $p"$ homomorphism $SL_n(\mathbb{Z}) \rightarrow SL_n(\mathbb{F}_p)$ is surjective.
Hint. We sketch the proof in the case \( n = 2 \); the general case is very similar. It is enough to show that the image of the homomorphism \( \text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{F}_p) \) contains all of the following matrices:

\[
\begin{pmatrix}
  a & 0 \\
  0 & c
\end{pmatrix} \quad \forall a, c \in \mathbb{F}_p, \\
\begin{pmatrix}
  1 & b \\
  0 & 1
\end{pmatrix} \quad \forall b \in \mathbb{F}_p, \\
\begin{pmatrix}
  0 & -1 \\
  1 & 0
\end{pmatrix}.
\]

For the last two, this is obvious. If \( a, c \in \mathbb{F}_p \) are such that \( a \cdot c = 1 \), one can find \( \bar{a}, \bar{c} \in \mathbb{Z} \) such that

\[
a = \bar{a} \mod p, \quad c = \bar{c} \mod p, \quad \bar{a} \bar{c} \equiv 1 \pmod{p^2}.
\]

Writing \( \bar{a} \bar{c} = 1 + p^m \) for some \( m \in \mathbb{Z} \), one sees that

\[
\begin{pmatrix}
  \bar{a} & -p \\
  pm & \bar{c}
\end{pmatrix} \in \text{SL}_2(\mathbb{Z}), \\
\begin{pmatrix}
  \bar{a} & -p \\
  pm & \bar{c}
\end{pmatrix} \mod p = \begin{pmatrix}
  a & 0 \\
  0 & c
\end{pmatrix},
\]

as desired.

Now let us fix a prime \( p > 2 \) and a nontrivial homomorphism \( \Psi : (\mathbb{F}_p^+, \cdot) \to \mathbb{C}^\times \). One knows that \( \Psi \) can be used to define a \( p \)-dimensional complex representation of the group \( \text{SL}_2(\mathbb{F}_p) \), called the Weil representation. Due to the lack of space and time, we cannot recall what this means. Our goal is to produce a modular category over \( \mathbb{C} \) such that the associated representation of \( \text{SL}_2(\mathbb{Z}) \) factors through the Weil representation of \( \text{SL}_2(\mathbb{F}_p) \).
To this end, we consider the category $\text{Vec}_{\mathbb{Z}/p\mathbb{Z}}$ of finite dimensional complex $(\mathbb{Z}/p\mathbb{Z})$-graded vector spaces, with the monoidal structure given by convolution, and with the obvious associativity and unit constraints. As the representatives of isomorphism classes of simple objects we choose the objects $\{C_x \mid x \in \mathbb{Z}/p\mathbb{Z}\}$ determined by

$$(C_x)_y = \begin{cases} C, & x = y; \\ 0, & x \neq y. \end{cases}$$

We have a (symmetric) biadditive map

$$B_{\psi} : \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}^*$$

given by $B_{\psi}(x, y) = \psi(\frac{x \cdot y}{2})$. As explained in the last lecture (see the examples in §2.4 and §2.5), $B_{\psi}$ induces a braiding on $\text{Vec}_{\mathbb{Z}/p\mathbb{Z}}$ which is determined by

$$\beta_{(C_x, C_y)} = B_{\psi}(x, y).$$

Moreover, the braided fusion category $(\text{Vec}_{\mathbb{Z}/p\mathbb{Z}}, B_{\psi})$ has a balancing (which is actually unique in our case, because $p$ is odd, so there are no non-trivial homomorphisms $\mathbb{Z}/p\mathbb{Z} \to \{\pm 1\}$), determined by

$$\theta_{C_x} = B_{\psi}(x, x) = \psi(\frac{x^2}{2}).$$
Finally, the symmetrization of the form $B_\psi$ is the form $(x, y) \mapsto \psi(xy)$, which is nondegenerate, and thus we see that the ribbon (or, if you prefer, premodular) category $(\text{Vec}_{\mathbb{Z}/p\mathbb{Z}}, B_\psi)$ is in fact modular.

\[ \text{§3.22} \]

Let us now compute the representation of $\text{SL}_2(\mathbb{Z})$ associated to this modular category. First, we have $\dim(C_x) = 1$ for all $x \in \mathbb{Z}/p\mathbb{Z}$. Indeed, $\dim(C_x)$ is the scalar determined by the composition

\[ 1 = C_0 \xrightarrow{id} C_x \ast C_{-x} \xrightarrow{\Theta_{C_x} \ast id} C_x \ast C_{-x} \xrightarrow{\beta_{C_x, C_{-x}}} C_{-x} \ast C_x \xrightarrow{id} C_0 = 1. \]

But $\beta_{C_x, C_{-x}}$ is the scalar $B_\psi(x, -x) = B_\psi(x, x)^{-1}$, and $\Theta_{C_x}$ is the scalar $B_\psi(x, x)$, so $\dim(C_x) = 1$.

It follows that the categorical dimension of $(\text{Vec}_{\mathbb{Z}/p\mathbb{Z}}, B_\psi)$ equals $p$. For concreteness, let us choose the rank, $D$, to be the positive square root of $p$, which we write as $\sqrt{p}$.

Next, since $\dim(C_x) = 1$ for all $x \in \mathbb{Z}/p\mathbb{Z}$, we see that $\text{tr}(\Theta_{C_x} \ast C_y) = B_\psi(y-x, y-x)$, and therefore the matrix $\tilde{S}$ is given by

\[ \tilde{S}_{x, y} = B_\psi(x, x)^{-1} B_\psi(y, y)^{-1} B_\psi(y-x, y-x) \]

\[ = \psi(-\frac{x^2}{2}) \psi(-\frac{y^2}{2}) \psi\left(\frac{(y-x)^2}{2}\right) = \psi(xy)^{-1}. \]
Hence the matrix $S$ is given by
\[ S_{x,y} = \frac{1}{\sqrt{p}} \psi(xy^{-1}) \quad \forall x,y \in \mathbb{Z}/p\mathbb{Z}. \]

The matrix $t$ is given by the formula
\[ t_{x,y} = \begin{cases} \psi(x^2/2), & x = y \\ 0, & x \neq y \end{cases}. \]

Next, the Gauss sums for the modular category $(\text{Vec}_{\mathbb{Z}/p\mathbb{Z}}, B_\psi)$ reduce to the standard Gauss sums:
\[ p^\pm = \sum_{x \in \mathbb{F}_p} \psi(\pm \frac{x^2}{2}) \]

One has an alternate formula for these sums. Namely, let us denote by $\chi$ the Legendre multiplicative character
\[ \chi : \mathbb{F}_p^\times \to \mathbb{C}^\times, \]
\[ \chi(x) = \begin{cases} 1 & \text{if } x \in (\mathbb{F}_p^\times)^2 \\ -1 & \text{if } x \not\in (\mathbb{F}_p^\times)^2. \end{cases} \]

Then we have
\[ p^\pm = \sum_{x \in \mathbb{F}_p^\times} \psi(\pm x) \chi(x) \]

In particular, $p^- = \chi(-1) p^+$, so we see that the multiplicative central charge of our category equals 1 if $p \equiv 1 \pmod{4}$, and is $\sqrt{-1}$ if $p \equiv 3 \pmod{4}$.
Now we can compare our formulas for the matrices $s$ and $t$ with the explicit formulas for the Weil representation of $SL_2(\mathbb{F}_p)$, which we take from the dissertation of S. Gurevich, available on the arXiv (see §3.3.2 in that paper). For convenience, we change the variables, so to speak, by conjugating Gurevich's picture by \( (01) \) and replacing the character \( \psi \) with \( \psi^{-1} \).

Recall our generators of $SL_2(\mathbb{Z})$, 
\[
\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]

and let $\sigma_p$, $\tau_p$ denote their images in $SL_2(\mathbb{F}_p)$.

By Exercise 3.21, $\sigma_p$ and $\tau_p$ generate $SL_2(\mathbb{F}_p)$.

With our notation, the formulas written out in Gurevich's dissertation imply that the elements $\sigma_p$ and $\tau_p$ act on the Weil representation of $SL_2(\mathbb{F}_p)$ on the space 
\[
\text{Fun}(\mathbb{F}_p) = \left\{ \text{functions } \mathbb{F}_p \rightarrow \mathbb{C} \right\}
\]
by the matrices 
\[
(\sigma_p)_{x,y} = \frac{1}{p} \psi(xy)^{-1}, \quad (\tau_p)_{x,y} = \psi(x^2 y^2/2)
\]

Thus, the matrix of $\tau_p$ coincides with $t$, and the matrix of $\sigma_p$ is proportional to $s$.

so the lift of the Weil representation of $SL_2(\mathbb{F}_p)$ to $SL_2(\mathbb{Z})$ gives rise to the same projective representation as the one associated to the modular fusion category $(\text{Vec}_{\mathbb{Z}/p\mathbb{Z}}, B_4)$. 