In these notes I will use the language developed in my earlier series of three lectures “Introduction to modular categories.” (The notes for these lectures are available on the seminar webpage.) The goal of this lecture is to explain the relationship between the notions of:

- a modular category,
- a 2-dimensional topological modular functor

Due to time constraints, we will have to restrict our attention to modular categories whose multiplicative central charge equals 1.

[The multiplicative central charge of a modular category was defined in Lecture 3. At this point it is not necessary to remember its precise definition. However, it is useful to keep in mind that for every modular category over a field $k$ we defined a projective representation of $SL_2(\mathbb{Z})$ over $k$ by explicit formulas: if the multiplicative central charge of the category equals 1, then the very same formulas define a linear (i.e., a “true”) representation of $SL_2(\mathbb{Z})$.]
We will focus our attention on the following

"THEOREM." Let \( k \) be a field, and let \( \mathcal{M} \) be a \( k \)-category. There is a natural bijection between the next two collections of data:

(i) structures of a modular category on \( \mathcal{M} \) (compatible with the given \( k \)-linear structure) such that the associated multiplicative central charge equals 1, and

(ii) nondegenerate \( \mathcal{M} \)-extended 2-dimensional topological modular functors.

The terms appearing in (ii) will be explained during today's lecture. The reason the word "Theorem" appears in quotation marks is that the statement above is not quite correct, and the correct reformulation is somewhat more involved. On the other hand, this statement is much easier to remember than the precise theorem that we state later in the lecture.

We recall that a \( k \)-category is defined as a \( k \)-linear category which is \( (k \text{-linearly}) \) equivalent to a finite direct sum of copies of the category of finite dimensional vector spaces over \( k \). (What we call a "\( k \)-category" is what some authors may call a "2-vector space over \( k \).")
The material that will be explained in this lecture can be roughly divided into three components, as follows.

(1) The first part has to do with the topology of surfaces with boundary. In this part one can forget about modular categories altogether. The goal is essentially to organize all compact orientable surfaces with boundary into one mathematical object which "encodes" the mapping class groups of the individual surfaces as well as the gluing operation (given two orientable surfaces and an orientation-reversing homeomorphism between parts of their respective boundaries, we obtain a third orientable surface by gluing).

A $2$-dimensional topological modular functor is something like a "representation" of this mathematical object.

(2) The second part has to do with an important ingredient in the proof of the main result stated above, namely, the following "Sub-Theorem". If $k$ is a field and $\mathcal{U}$ is a $k$-category, then to give a structure of a premodular category on $\mathcal{U}$ is the same as to give an $\mathcal{U}$-extended $2$-dimensional topological modular functor on genus $0$. 
Once again, this statement is not quite correct, and the correct reformulation will be explained in this lecture.

(3) Finally, one starts with a premodular category \( \mathcal{M} \) over \( k \), considers the corresponding 2D topological MF in genus 0, and shows that the latter can be extended to all genera if and only if \( \mathcal{M} \) is modular. This will complete the proof of the main result.

In this lecture no proofs will be given. Our goal is to explain carefully all the necessary definitions (and some of the background), and to give precise formulations of the main results. The main reference for the lecture is Chapter 5 of the book [BK] (see the first page of these notes for Lecture 1). We remark that in §5.7 of their book, Balakov and Kirillov explain what happens when one considers modular categories of arbitrary multiplicative central charge.

\underline{Also:} This lecture can by no means serve as a replacement of Chapter 5 of [BK]. At best, it can be used as an elementary introduction which could prepare the less experienced readers for reading §§5.1–5.6 of op. cit.
The Lego - Teichmüller game

The name of this section of the lecture is due to Grothendieck (it appeared in his 1984 text "Esquisse d'un programme"). It should be noted that I am by no means an expert on the theory of mapping class groups and the "Teichmüller tower" (defined below), so my approach to this subject will be very naive, and the presentation of the material will by no means be optimal or comprehensive.

The basic idea, which also goes back to (at least) Grothendieck, is as follows. If $\Sigma$ is a compact oriented surface, possibly with boundary, one can speak about the mapping class group $\Gamma'(\Sigma)$ of $\Sigma$. Actually, there are a few different versions of this group (distinguished by what happens on the boundary of $\Sigma$); to fix ideas, let us define

$$\Gamma'(\Sigma) = \left\{ \begin{array}{l} \text{orientation-preserving homeomorphisms} \\ \Sigma \to \Sigma \text{ fixing } \partial \Sigma \text{ pointwise} \\ \text{those self-homeomorphisms that} \\ \text{are homotopic to } \text{id}_{\Sigma} \text{ through} \\ \text{homeomorphisms fixing } \partial \Sigma \text{ pointwise} \end{array} \right\}$$

For instance, if $\Sigma = S^1 \times S^1$ is the usual 2-dimensional torus, then $\Gamma'(\Sigma) \cong SL_2(\mathbb{Z})$.

(1) In fact, $\Gamma'(\Sigma)$ is what later will be called the "pure mapping class group of $\Sigma". \)
It is a universally accepted fact that mapping class groups are very important objects, and their study is useful for several different branches of mathematics.

However, if $\Sigma$ is fixed, the study of $\Gamma'(\Sigma)$ can be rather complicated: for instance, it is often difficult to write down an explicit presentation of $\Gamma'(\Sigma)$ by generators and relations (especially if $\Sigma$ has genus $\geq 2$).

Grothendieck's idea was that one can obtain a more understandable object by keeping track of the mapping class groups of all compact oriented surfaces at once together with the various "gluing operations".

A typical example of how gluing is related to mapping class groups is as follows. Let $\Sigma$ be a compact oriented surface, and suppose we pick two different boundary components, $C_1$ and $C_2$, of $\Sigma$. (Of course, $C_1$ and $C_2$ are circles.) If we pick an orientation-reversing homeomorphism between $C_1$ and $C_2$, we can use it to glue $C_1$ with $C_2$ and obtain a new compact oriented surface, call it $\Sigma'$.  

(2) In full generality, such presentations have only been obtained relatively recently (in the mid 1990's).
Furthermore, every orientation-preserving self-homeomorphism of \( \Sigma \) which fixes \( \Sigma \) and \( \Sigma' \) pointwise descends to an orientation-preserving self-homeomorphism of \( \Sigma' \) which leads to a natural homomorphism \( \Gamma'(\Sigma) \to \Gamma'(\Sigma') \).

Our next goal is to introduce a convenient language for formalizing Grothendieck’s idea.

**Definition.** An extended surface is a tuple

\[
(\Sigma, \{p_a\}_{a \in \pi_0(\partial \Sigma)}
\]

where \( \Sigma \) is a (possibly disconnected) compact oriented surface with boundary \( \partial \Sigma \), and for every \( a \in \pi_0(\partial \Sigma) \), if we let \( (\partial \Sigma)_a \) denote the corresponding boundary component of \( \Sigma \), then \( p_a \) is a chosen point of \( (\partial \Sigma)_a \).

By a usual abuse of notation, we will often denote an extended surface by a single symbol, such as \( \Sigma \).

We turn the collection of all extended surfaces into a category (in fact, a groupoid) as follows.

**Definition.** A morphism of extended surfaces between \( (\Sigma, \{p_a\}) \) and \( (\Sigma', \{p'_a\}) \) is an isotopy class of orientation-preserving homeomorphisms \( \Sigma \to \Sigma' \) which take the set \( \{p_a\} \) onto \( \{p'_a\} \).
Remark. Of course, in the definition above it is tacitly assumed that the word "isotopy" means "isotopy through homeomorphisms, all of which take \( \{p_a\} \) onto \( \{p_b\} \).

**Definition.** The (extended) Teichmüller groupoid is the category \( \text{Teich} \) whose objects are extended surfaces and whose morphisms are described in the previous definition.

Remark. The groupoid \( \text{Teich} \) has two other equivalent realizations, which are also useful. They are described in Definitions 5.1.6 and 5.1.10 of the book [BK].

If \( \Sigma = (\Sigma, \{p_a\}) \in \text{Teich} \), the **mapping class group** of \( \Sigma \) is defined as

\[
\Gamma(\Sigma) = \text{Aut}_{\text{Teich}}(\Sigma),
\]

and the **pure mapping class group** of \( \Sigma \) is defined as the subgroup \( \Gamma'(\Sigma) \subset \Gamma(\Sigma) \) formed by the elements of \( \Gamma(\Sigma) \) which leave each of the individual points \( p_a \) fixed. Equivalently, \( \Gamma'(\Sigma) \) is the kernel of the natural homomorphism

\[
\Gamma(\Sigma) \rightarrow \text{Perm}(\{p_a\}) = \text{the group of permutations of the set } \{p_a\}
\]

Of course, if \( \partial \Sigma = \emptyset \), then \( \Gamma'(\Sigma) = \Gamma(\Sigma) \).
It is a simple exercise to check that the new
definition of $\Gamma'(\Sigma)$ is equivalent to the one
we gave on page 5 of these notes; more
precisely, the natural homomorphism
$\Gamma'_{old}(\Sigma) \rightarrow \Gamma'_{new}(\Sigma)$ is an isomorphism.

The groupoid Teich can be equipped with
another structure which encodes the gluing
operations for surfaces described above.
(In the end it will turn out that Teich is a
primary example of a “tower of groupoids,”
a notion we will introduce below; accordingly,
it will be renamed the “Teichmüller tower”.)

Let us first remark that intuitively one usually
thinks of gluing as an operation which takes
two surfaces (with some additional data) and
produces a third one. However, for various
technical reasons, it is more convenient to
view the operation described at the bottom of
page 6 as the main example of gluing. In fact,
if we combine that operation with the operation
of taking the disjoint union of surfaces (remember
that objects of Teich are allowed to be disconnected)
we obtain a collection of operations that includes
both the “usual” operations of gluing two surfaces
and other operations (such as gluing together two
ends of a cylinder to produce a torus).
let us now formalize what we mean by gluing. Consider an extended surface \((\Sigma, \{\partial_0\})\) such that \(\partial\Sigma\) has at least two components, and pick distinct elements \(x, y \in \pi_0(\partial\Sigma)\). Note that the circles \((\partial\Sigma)_x\) and \((\partial\Sigma)_y\) have orientations induced by that of \(\Sigma\). Pick an arbitrary orientation-reversing homeomorphism \(h: (\partial\Sigma)_x \xrightarrow{\cong} (\partial\Sigma)_y\) such that \(h(x) = y\), and define a new extended surface \(\Sigma'\) as follows:

\[
\Sigma' = (\Sigma, \{y \neq x\}, \Sigma),
\]

where \(\Sigma'\) is obtained from \(\Sigma\) by identifying every \(x \in (\partial\Sigma)_x\) with \(h(x) \in (\partial\Sigma)_y\), and the orientation on \(\Sigma'\) is induced by that on \(\Sigma\) in the obvious way.

**Exercises.** (1) Show that \(\Sigma'\) does not depend on the choice of \(h\), in the sense that a different choice leads to a result which is isomorphic to the original one by a uniquely determined morphism in the category \(\text{Teich}\).

(2) Using (1), show that the construction \(\Sigma'\) is functorial in the following sense: given a morphism \(\tilde{f}: \tilde{\Sigma} \to \tilde{\Sigma}\) in \(\text{Teich}\), we obtain a uniquely defined morphism

\[
\tilde{f} \circ h: \Sigma' \to \tilde{\Sigma}.
\]
As we already mentioned, we also have a much more elementary operation on Teich, namely, the disjoint union bifunctor

$$\Sigma : \text{Teich} \times \text{Teich} \to \text{Teich}$$

which makes Teich a symmetric monoidal category (the associativity constraint and the braiding are the obvious ones, and the unit object is the empty extended surface $\emptyset$).

Finally, it is useful to keep in mind the natural functor

$$A : \text{Teich} \to \text{Sets}$$

where (perhaps somewhat inconventionally) we denote by $\text{Sets}$ the symmetric monoidal groupoid whose objects are finite sets, whose morphisms are bijections of sets, and whose monoidal structure is again given by the disjoint union.

The functor $A$ is defined by $\Sigma \mapsto \pi_0(\partial \Sigma)$, or equivalently, $(\Sigma, \{p_\alpha\}) \mapsto \{p_\alpha\}$. It has an obvious structure of a strong symmetric monoidal functor.

Next we will define the notion of a "tower of groupoids". It will be obvious that Teich, together with the additional structures described on pp. 9-11, is a tower of groupoids.
Towers of groupoids (LBK, §5.6)

We will only explain carefully one viewpoint on towers of groupoids. For two other (equivalent) viewpoints we refer the reader to loc. cit.

Definition. A tower of groupoids is a 4-tuple
\[ \mathcal{T} = (\mathcal{T}, \mathbb{H}, A, G), \]
where:
- \( \mathcal{T} \) is a groupoid;
- \( \mathbb{H} : \mathcal{T} \times \mathcal{T} \to \mathcal{T} \) is a bifunctor equipped with additional structures needed to make \((\mathcal{T}, \mathbb{H})\) a symmetric monoidal category (we suppress the notations for these structures because they are not essential for this story);
- \( A \) is a strong symmetric monoidal functor \( A : \mathcal{T} \to \text{Sets} \) (see p.11) (once again, the notation for the strong monoidal structure on \( A \) is suppressed);
- \( G \) is a rule, called the "gluing operation", which to every object \( \Sigma \in \mathcal{T} \) and every unordered pair of distinct elements \( \alpha, \beta \in A(\Sigma) \) assigns an object \( G_x^y(\Sigma) \in \mathcal{T} \) and to every (iso)-morphism \( \mathcal{F} : \Sigma \to \Sigma' \) assigns an (iso)-morphism \( G_x^y : G_x^y(\Sigma) \to G_x^y(\Sigma') \), where
  \[ \alpha' = A(\mathcal{F})(\alpha), \quad \beta' = A(\mathcal{F})(\beta) \in A(\Sigma'). \]
Moreover, the gluing operation is equipped with extra structures and satisfies the extra properties described below.

1. **Compatibility with \( A \):** functorial isomorphisms
   \[
   A(G_{\alpha, \beta}((\Sigma))) \cong A(\Sigma) \setminus \{\alpha, \beta\}
   \]
   for every \( \Sigma \in \mathcal{C} \) and every unordered pair of distinct elements \( \alpha, \beta \in A(\Sigma) \).

2. **Compatibility with \( \Pi \):** for every \( \Sigma_1, \Sigma_2 \in \mathcal{C} \) and every unordered pair of distinct elements \( \alpha, \beta \in A(\Sigma_1) \), we need to specify an isomorphism
   \[
   (G_{\alpha, \beta}((\Sigma_1))) \Pi \Sigma_2 \cong G_{\alpha, \beta}((\Sigma_1 \Pi \Sigma_2));
   \]
   these isomorphisms should be functorial with respect to \( \Sigma_2 \) and the triple \((\Sigma_1, \alpha, \beta)\), and compatible with the associativity and right unit constraint for \( \Pi \) in the obvious sense.

3. **Associativity:** for each \( \Sigma \in \mathcal{C} \) and each quadruple of pairwise distinct elements \( \alpha, \beta, \gamma, \delta \in A(\Sigma) \), we need to specify a functorial isomorphism
   \[
   G_{\alpha, \beta}(G_{\gamma, \delta}((\Sigma))) \cong G_{\gamma, \delta}(G_{\alpha, \beta}((\Sigma)));
   \]
   compatible with the isomorphisms on (1).

4. **Functoriality:** with the notation on the previous page, \( G_{\phi_1 \circ \phi_2} = G_{\phi_1} \circ G_{\phi_2} \) and \( G_{\text{id}} = \text{id} \) whenever the composition \( \phi_1 \circ \phi_2 \) is defined.
Partial towers of groupoids

We will also need a variant of the definition of a tower of groupoids in which the gluing operation \( G_{x,y} \) is only defined for a subset of the set of all unordered pairs \( x, y \in \mathcal{A}(\Sigma) \).

The motivation for this comes from the notion of a 2-dimensional topological modular functor in genus 0. Note that if \( \Sigma \) is an extended surface, all of whose connected components have genus 0, and \( x, y \in \pi_0(\boldsymbol{\Sigma}) \) are distinct, then \( \Sigma \) will have the same property as \( \Sigma \) if and only if \( x \) and \( y \) lie in different connected components of \( \Sigma \).

In case we have such "partially defined" gluing operations, the axioms above have to be modified as follows. The isomorphism

\[
(G_x, \Sigma_1) \sqcup \Sigma_2 \cong G_{x, y} \left( \Sigma_1 \sqcup \Sigma_2 \right)
\]

and

\[
G_{x, y} \left( e, \delta(\Sigma) \right) \cong G_{y, x} \left( e, \delta(\Sigma) \right)
\]

only have to be specified when one of the two sides is defined, in which case we require that the other side be defined as well.

This modification leads to the notion of a partial tower of groupoids.
For us, the first main example of a partial tower of groupoids is given by the Teichmüller tower in genus zero, $\text{Teich}_0$. The underlying groupoid of $\text{Teich}_0$ is the full subcategory of $\text{Teich}$ formed by extended surfaces, all of whose components have genus 0.

At this point it is appropriate to recall a Definition. If $\Sigma$ is a connected compact oriented surface, possibly with boundary, the genus of $\Sigma$ is defined as the genus of the closed oriented surface obtained from $\Sigma$ by gluing a closed disc, along its boundary circle, to every boundary component of $\Sigma$.

For the purposes of this lecture, it is not unreasonable to think of the genus of $\Sigma$ in terms of the original (Riemann's?) definition, namely, as the maximal possible size of a collection $\{C_1, \ldots, C_n\}$ of pairwise disjoint simple closed curves $C_j \subset \Sigma$ so that $\Sigma \setminus (\bigcup C_j)$ is still connected.

There is a natural notion of a tower functor between two towers (or partial towers) of groupoids, which is not very hard to formulate. We refer the reader to Definition 5.6.7 in [BK].
Now that we have the notion of a tower of groupoids, we would also like to define the notion of a “representation” of a tower of groupoids (which, in a vague sense, should be analogous to the notion of a linear representation of a group). Then 2-dimensional topological modular functors (resp., those in genus 0) will be defined as representations of $\text{T}_2$ (resp., $\text{T}_{2,0}$).

However, it turns out that a tower of groupoids cannot “act” on a mere $k$-category: the latter must be equipped with an additional structure. We will now spell out the necessary definitions.

Let us fix a field $k$ and a $k$-category $M$.

**Definition.** A symmetric object in $M \otimes M$ is a pair $(R, \tau)$ consisting of an object $R \in M \otimes M$ and an isomorphism

$$\tau : R^{\text{rev}} \cong R$$

satisfying $\tau \circ \tau^{\text{rev}} = \text{id}$, where the functor

$$\text{rev} : M \otimes M \to M \otimes M$$

is defined by

$$(A \otimes B)^{\text{rev}} = B \otimes A.$$

(This definition is taken from §2.4 of [BK], where the notation “op” is used in place of “rev”.)

Next we define the “tower of automorphisms” of a given pair $(M, R)$ as above.
Definition. Let $\mathcal{M}$ be a $k$-category, and let $R \in \mathcal{M} \otimes \mathcal{M}$ be a symmetric object (by a usual abuse of notation, we often omit the symbol for the symmetric structure on $R$).

We define a tower of groupoids $\text{Fun}(\mathcal{M}, R)$ as follows (\cite{BKJ}, Example 5.6.11).

- The objects of $\text{Fun}(\mathcal{M}, R)$ are pairs $(S, F)$, where $S$ is a finite set and
  $$F: \mathcal{M} \otimes S \rightarrow k\text{-vect}$$
  is a $k$-linear functor.
- A morphism between pairs $(S_1, F_1)$ and $(S_2, F_2)$ as above is a pair $(\varphi, \xi)$, where $\varphi: S_1 \rightarrow S_2$ is a bijection of sets, $\xi: F_1 \rightarrow \varphi_* F_2$ is an isomorphism of functors, and $\varphi_* F_2$ is defined as the composition
  $$\mathcal{M} \otimes S_1 \xrightarrow{\varphi} \mathcal{M} \otimes S_2 \xrightarrow{F_2} k\text{-vect}.$$
- The boundary functor is given by $A(S, F) = S$, $A(\varphi, \xi) = \varphi$.
- The disjoint union bifunctor is given by
  $$(S_1, F_1) \sqcup (S_2, F_2) = (S_1 \sqcup S_2, F_1 \otimes F_2),$$
  and similarly for morphisms. The unit object is $\emptyset = (\emptyset, \text{Id}_k: k\text{-vect} \rightarrow k\text{-vect})$.
  (As usual, the convention is that $\mathcal{M} \otimes \emptyset = k\text{-vect}$, the category of finite dimensional vector spaces over $k$.)


Finally, let us describe the gluing operation. Let \((S, F) \in \text{Fun}(M, R)\), and fix distinct elements \(\alpha, \beta \in S\). We must define
\[
(S', F') = \bigoplus_{\alpha, \beta} (S, F) \in \text{Fun}(M, R)
\]
Of course, we put \(S' = S \setminus \{\alpha, \beta\}\). Next, let us write
\[
R = \bigoplus_{j=1}^n (V_j \boxtimes W_j) \quad V_j, W_j \in M.
\]
We define the functor
\[
F' : U(S \setminus \{\alpha, \beta\}) \rightarrow K \text{-} \text{vect}
\]
by the formula
\[
F'(((x_\gamma)_{\gamma \neq \alpha, \beta})) = \bigoplus_{j=1}^n F(\ldots x_\gamma, \ldots, V_j, \ldots, W_j, \ldots),
\]
where \(V_j\) and \(W_j\) are put in the places that correspond to the indices \(\alpha\) and \(\beta\), respectively.

The fact that \(R\) is a symmetric object in \(M \boxtimes M\) guarantees that the gluing operation does not depend on the order of \(\alpha\) and \(\beta\).

**Definition.** Let \(T\) be a (partial) tower of groupoids, let \(M\) be a \(k\)-category, and let \(R \in M \boxtimes M\) be a symmetric object. A representation of \(T\) in \((M, R)\) is a tower functor
\[
p : T \rightarrow \text{Fun}(M, R),
\]
where \(\text{Fun}(M, R)\) is the tower of groupoids constructed in the previous definition.
Modular functors (in dimension 2)

**Definition.** Let \((M, R)\) be a pair consisting of a \(k\)-category \(M\) and a symmetric object \(R \in M \otimes M\).

(a) An \((M, R)\)-extended 2-dimensional topological modular functor is a representation \(p\) of the Teichmüller tower \(\text{Teich}\) in \((M, R)\) satisfying the normalization condition

\[ p(S^2) = \text{Id} : k\text{-vect} \rightarrow k\text{-vect} \]

(b) Replacing \(\text{Teich}\) with \(\text{Teich}_0\), we arrive at the notion of an \((M, R)\)-extended 2D topological MF in genus 0.

**Remark.** Let \(p : \text{Teich} \rightarrow (M, R)\) be any representation. If \(\Sigma\) is a closed, connected, oriented surface, then \(\Sigma \in \text{Teich}\), and since \(\partial \Sigma = \emptyset\), we see that \(p(\Sigma)\) must be a \(k\)-linear functor \(k\text{-vect} \rightarrow k\text{-vect}\). However, any such functor is of the form \(W \mapsto V \otimes W\) for a uniquely determined (up to isomorphism) \(V \in k\text{-vect}\). In addition, since \(p\) itself is a functor, and since \(\text{Aut}_{\text{Teich}}(\Sigma) = \Gamma(\Sigma)\) (the mapping class group of \(\Sigma\)), we obtain an induced linear action of \(\Gamma(\Sigma)\) on \(V\).

Thus, 2D topological MFs give rise to finite-dimensional representations of mapping class groups of closed oriented surfaces.

\((\ast)\) I.e., compact and without boundary
Relation between 2D topological MF in genus $0$ and premodular categories

The rest of this lecture will be devoted to a discussion of the relationship between the notions of a 2D topological MF (resp., that in genus $0$) and of a modular (resp., premodular) category.

Due to the time constraints, we will only cover a relatively small portion of §§5.3–5.5 of [BK]. In particular, we will not explain how to construct a MF starting from a modular category.

We first focus on the genus $0$ situation. Note that every connected $\Sigma \in \text{Teich}_0$ is homeomorphic to a 2-dimensional sphere with finitely many open discs removed (where the discs have pairwise non-intersecting closures).

It will be convenient to fix a collection of "standard" objects of this sort:

\[ \Sigma_{0,n} : \text{ } (n=1,2,3,\ldots) \]

Now let $(M, R)$ be a pair consisting of a $k$-category $M$ and a symmetric object $R \in M$. Let us fix a 2D topological MF in genus $0$,

\[ \varphi : \text{Teich}_0 \rightarrow \text{Fun}(M, R). \]

**Correction!** We need to assume that $\varphi$ is nondegenerate in the sense defined on page 29.
We will see that $p$ can be used to make $U$ into an "almost premodular" category over $k$. The reason we say "almost" is that we can define all the structures on $U$ that a premodular category should have, but we do not know how to prove that the underlying monoidal category is rigid. (However, this is the only "obstruction" to $U$ being premodular.)

The basic principle of the construction is that the functor

$$p(\Sigma_0, n): U \boxtimes_k \rightarrow k \text{-vect}$$

should be isomorphic to the functor

$$(V_1, \ldots, V_n) \mapsto \text{Hom}_U(I, V_1 \boxtimes \cdots \boxtimes V_n)$$

The braiding and the balancing on $U$ will then arise from certain elements of the mapping class groups $\Gamma(\Sigma_{0,3})$ and $\Gamma(\Sigma_{0,2})$, respectively.

Let us explain some of the details. For notational convenience, let us write

$$\langle V_1, V_2, \ldots, V_n \rangle := p(\Sigma_0, n)(V_1, \ldots, V_n).$$

Also, let us choose a complete set of representatives $\{X_i\}_{i \in I}$ of the isomorphism classes of simple objects of $U$. 

We define a functor (called the "duality functor")

\[ M^* \longrightarrow M, \quad V \mapsto V^* \]

by the identity

\[ \text{Hom}_M(V^*, X) = \langle V, X \rangle. \]

Lemma (see [BK], Lemma 5.3.9). The duality functor \( V \mapsto V^* \) is an anti-auto-equivalence of \( M \). There exists an involution

\[ * : I \longrightarrow I, \quad i \mapsto i^* \]

such that

\[ \text{dim} \langle X_i, X_j \rangle = \delta_{i,j} \quad \forall \ i, j \in I, \]

and \( R \) is isomorphic (non-canonically) to

\[ \bigoplus_{i \in I} (X_i \boxtimes X_{i^*}). \]

Finally, \( X_i^* \cong X_i^* \) for all \( i \in I \) (non-canonically).

Next, we define a \( k \)-linear functor

\[ \otimes : M \boxtimes M \longrightarrow M \]

by the identity

\[ \langle X, V \boxtimes W \rangle = \langle X, V, W \rangle. \]

By the lemma above, this functor is well defined. We also define an object \( 1 \in M \) by

\[ \langle 1, V \rangle = \langle V \rangle. \]

It is not hard to define associativity and unit constraints for the functor \( \otimes \) and the object \( 1 \) using the gluing isomorphisms that arise from our 2D topological MF in genus 0, \( g \).

\[(4) \text{ This is where the nondegeneracy assumption on } p \text{ is used.} \]
The main point is that we have a natural isomorphism
\[ \Sigma_{0,m} \cong \Sigma_{0,m+n-2} \]
in Teich, where, by abuse of notation, we write
\[ \Sigma_{0,m} \cong \Sigma_{0,n} = \Sigma_{0, m+n}^{\Sigma_{0,n}} \]
with the understanding that the marked points of the disjoint union \( \Sigma_{0,m} \cup \Sigma_{0,n} \) are labeled by
\[ 1, 2, \ldots, m, 1', 2', \ldots, n' \]
coming from the labeling of \( \Sigma_{0,m} \) ditto for \( \Sigma_{0,n} \).

Now the associativity constraint for \( \otimes \) is defined as follows: we have isomorphisms
\[ \langle X, (U \otimes V) \otimes W \rangle \cong \langle X, U \otimes V, W \rangle \]
induced by gluing
\[ \varpi \otimes \bigoplus_{i \in I} \langle X, X_i, W \rangle \otimes \langle X_{i*}, U \otimes V \rangle \]
def \[ \cong \bigoplus_{i \in I} \langle X, X_i, W \rangle \otimes \langle X_{i*}, U \otimes V \rangle \]
induced by gluing
\[ \cong \langle X, U, V, W \rangle \]
Similarly, we have natural isomorphisms
\[ \langle X, U \otimes (V \otimes W) \rangle \cong \langle X, U, V, W \rangle \]
Thus we obtain functorial isomorphisms
\[ \delta_{U,V,W} : (U \otimes V) \otimes W \cong U \otimes (V \otimes W) \].
Similarly, we have

\[
\langle x, V \otimes 1 \rangle \overset{\text{def}}{=} \langle x, V, 1 \rangle
\]

\[
\simeq \bigoplus_{i \in I} \langle x, V, x_i \rangle \otimes_k \langle x_i^*, 1 \rangle
\]

\[
= \bigoplus_{i \in I} \langle x, V, x_i \rangle \otimes \langle x_i^* \rangle
\]

which induces functorial isomorphisms

\[
\rho_V : V \otimes 1 \xrightarrow{\sim} V \quad \forall V \in M
\]

and functorial isomorphisms

\[
\lambda_V : I \otimes V \xrightarrow{\sim} V \quad \forall V \in M
\]

can be defined similarly.

**Proposition.** The 6-tuple \((M, \otimes, I, \otimes, A, p)\) is a \(k\)-linear monoidal category.

**Remark.** The normalization condition

\[
p(S^2) = \text{Id} : k\text{-vect} \rightarrow k\text{-vect}
\]

that enters the definition of a 2D topological MF ensures that \(\dim_k \langle 1 \rangle = 1\), i.e., \(1\) is simple.

Next we describe the braiding and the balancing. First we define elements

\[
\text{br}_{23} \in \Gamma(\Sigma_{0,3}) \quad \text{tw}_2 \in \Gamma'(\Sigma_{0,2})
\]

where "br" stands for "braiding" and "tw" stands for "twist", as explained below.
To define $br_{23}$, it is convenient to visualize the extended surface $\Sigma_{0,3}$ as follows:

The punctured lines are only for illustration purposes. The meaning of this picture is explained in §§5.1 - 5.2 of [BK].

On the other hand, the element $tw_2$ is the Dehn twist around the second "puncture":

Now, for all $V, W \in M$, we define

$$p_{V,W} : V \otimes W \cong W \otimes V, \quad \Theta_V : V \cong V$$

as the isomorphisms corresponding to

$$p(br_{23}) : \langle X, V, W \rangle \cong \langle X, W, V \rangle$$

and

$$p(tr_2) : \langle X, V \rangle \cong \langle X, V \rangle$$

for all $X \in M$, where we recall that $p$ is the notation for our MF in genus 0.
To describe precisely the structure one gets on the category \( \mathcal{M} \), Bakalov and Kirillov introduce the following notion ([BKK], Def. 5.3.5).

**Definition.** A weakly ribbon category is a braided monoidal category \( \mathcal{M} \) equipped with an automorphism \( \Theta \) of the identity functor \( \text{Id}_\mathcal{M} \) satisfying

\[
\Theta_{V \otimes W} = (\Theta_V \otimes \Theta_W) \circ (\beta_{W,V} \circ \beta_{V,W}) \quad \forall V, W \in \mathcal{M},
\]

and such that the following properties hold.

(i) For every \( V \in \mathcal{M} \), the functor

\[
X \mapsto \text{Hom}_\mathcal{M}(\text{Id}_\mathcal{M}, V \otimes X)
\]

is representable.

Assuming that (i) holds, we let \( V^* \in \mathcal{M} \) denote an object representing this functor and call it a **weak left dual of** \( V \).

(ii) The functor

\[
\mathcal{M}^\text{op} \rightarrow \mathcal{M}, \quad V \mapsto V^*
\]

is an equivalence of categories.

(iii) We have

\[
\Theta_{V^*} = \Theta_V^* \quad \text{for all } V \in \mathcal{M}.
\]

For instance, every ribbon category is automatically weakly ribbon, and \( V^* \) is the left dual of \( V \) in the sense of Lecture 1.
The next result describes the relationship between modular functors in genus 0 and premodular categories. Its proof is contained in the proofs of Theorems 5.3.8 and 5.4.1 in [BK].

**Theorem.** Let $k$ be a field, let $M$ be a $k$-category, let $R \in M \otimes M$ be a symmetric object, and let $p : \text{Teich} \to \text{Fun}(M, R)$ be a 2D topological MF in genus 0. Let us equip $M$ with the structures $\otimes, I, \mathbb{I}, \mathbb{I}, p, f, \theta$ constructed on pp. 22-25. Then $M$ becomes a $k$-linear weakly ribbon category where $I$ is simple. In particular, if $M$ is rigid, it is premodular.

Moreover, the functor $p$ can more or less be recovered from $M$. However, we prefer not to state a precise converse of the theorem above, and refer the reader to Theorem 5.4.1 in [BK]. The formulation of the theorem stated above and of its converse is due to Moore and Seiberg. However, their proof contains a few serious gaps. The gaps were filled in in the paper "On the lego-Teichmüller game" by Bakalov and Kirillov.
Relation between MFs and modular categories

Let us first write down a precise statement.

Theorem. Let $k, M, R$ be as in the previous theorem, and let $p : \text{Teich} \to \text{Fun}(M, R)$ be a nondegenerate\(^{(5)}\) $(M, R)$-extended 2-dimensional topological modular functor.

Restricting $p$ to $\text{Teich}_0$, we obtain a $k$-linear weakly ribbon structure on $M$, as explained above. Assume that this structure is rigid (i.e., $M$ is premodular). Then $M$ is a modular category over $k$ whose Gauss sums are equal: $p^+ = p^-$. \[\]

[In particular, since $p^+ p^- = \text{dim}(M)$, we can take the rank of $M$ to be $D = p^+$ and then the multiplicative central charge, $\Xi(M)$, equals 1. The notation $p^+, \text{dim}(M), D, \Xi(M)$ is explained in §3.19 (pp. 24-26) of the notes for Lecture 3.]

Once again, in the situation of the last theorem, the modular functor $p$ can more or less be recovered from the modular category $M$. The result above, together with a precise converse statement, appear as Theorem 5.5.1 in [BK].

Let us explain the meaning of the term "nondegenerate".

\(^{(5)}\) This term is defined on the next page.
Definition. A 2D topological modular functor
\[ p: \mathcal{T}_{\text{top}} \rightarrow \text{Fun}(\mathcal{M}, \mathbb{R}) \]
is said to be nondegenerate if for every non-zero object \( X \in \mathcal{M} \), there exist \( \Sigma \in \mathcal{T}_{\text{top}} \) and objects \( \{ V_a \}_{a \in \pi_0(\Sigma)} \) of \( \mathcal{M} \) such that \( X = V_b \) for some \( b \in \pi_0(\Sigma) \) and
\[ p(\Sigma)(\{ V_a \}) \neq 0. \]

The definition of nondegeneracy for a 2D topological MF in genus 0 is analogous.

The proof of the theorem stated on the previous page is based on the following computation.

Lemma. Suppose that the assumption of the theorem above is satisfied. Let \( \Sigma_{1,0} \) denote the standard torus \( \mathbb{R}^2/\mathbb{Z}^2 \), and let \( \sigma, \tau \) be the elements of the mapping class group \( \Gamma(\Sigma_{1,0}) \) represented by the matrices
\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix} \in \text{SL}_2(\mathbb{Z}).
\]
The vector space \( p(\Sigma_{1,0}) \in \mathbb{R} - \text{vect} \) can be naturally identified with \( \mathbb{R}^I \) where \( I \) is the set of isomorphism classes of simple objects of \( \mathcal{M} \), so that the operators \( p(\sigma) \) and \( p(\tau) \) act by the matrices \( s, t \) defined at the end of the previous lecture.