$k$ is an algebraically closed field of characteristic 0.

Theorem (Nikshych & Ostrik). If $\mathcal{C}$ and $\mathcal{D}$ are fusion categories over $k$ such that $Z_1(\mathcal{C})$ and $Z_1(\mathcal{D})$ are equivalent as $k$-linear braided monoidal categories, then $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent.

Today we will explain how to prove this theorem. First we need some preparations.

1. Dimension theory

Let $X_1, \ldots, X_n$ be representatives of all the isomorphism classes of simple objects in a fusion category $\mathcal{C}$ over $k$. For any $X \in \mathcal{C}$, we have

$$X \otimes X_i \cong \bigoplus_{j=1}^{n} N_{ij}^X X_j$$

for some matrix $N^X = (N_{ij}^X)$ with nonnegative integral entries.

One can check, using rigidity, that $N^X$ is not nilpotent. Thus it has at least one nonzero (complex) eigenvalue.
Now $N^x$ has a positive real eigenvalue which dominates in absolute value all the other complex eigenvalues of $N^x$. This eigenvalue is called the Frobencius–Perron eigenvalue of $N^x$.

By definition, the Frobencius–Perron dimension, $FP\text{dim}(X)$, of $X$, is the Frobencius–Perron eigenvalue of $N^x$. Thus $FP\text{dim}(X) \in \mathbb{R} > 0$ if $X \neq 0$.

**Def.** If $\mathcal{C}$ is a fusion category over $k$, the Frobencius–Perron dimension of $\mathcal{C}$ is

$$FP\text{dim}(\mathcal{C}) = \sum_{i=1}^{n} (FP\text{dim}(X_i))^2$$

where the $X_i$'s are as above.

**Example.** If $G$ is a finite group, then

$$FP\text{dim}(\text{Rep}(G)) = |G|.$$ 

**Elementary properties.**

- $FP\text{dim}(X \oplus Y) = FP\text{dim}(X) + FP\text{dim}(Y)$
- $FP\text{dim}(X \otimes Y) = FP\text{dim}(X) \cdot FP\text{dim}(Y)$

for all $X, Y \in \mathcal{C}$

- If $M$ is an irreducible $\mathcal{C}$-module, then $FP\text{dim}(\mathcal{C}^*_M) = FP\text{dim}(\mathcal{C})$. 
In particular,
\[ \text{FPdim} \left( \mathcal{Z}_1(\mathcal{C}) \right) = \text{FPdim}(\mathcal{C})^2. \]

**Proposition.** Let \( \mathcal{C}, \mathcal{D} \) be fusion categories over \( \mathbb{k} \) such that \( \text{FPdim}(\mathcal{C}) = \text{FPdim}(\mathcal{D}) \), and let \( F: \mathcal{C} \to \mathcal{D} \) be a \( \mathbb{k} \)-linear monoidal functor. Let us say that \( F \) is **surjective** if every object of \( \mathcal{D} \) is a subobject (\( \cong \) direct summand) of an object of \( F(\mathcal{C}) \). (Perhaps the term “Karoubi surjective” is more appropriate?) If \( F \) is surjective, then it is an equivalence.

\[ \begin{array}{c}
\text{2. Commutative algebras in BTCs} \\
\text{(BTC = braided tensor category)}
\end{array} \]

Let \( \mathcal{C} \) be a BTC, and let \( A \) be an algebra (\( \cong \) a monoid) in \( \mathcal{C} \) with multiplication \( m: A \otimes A \to A \). We say that \( A \) is **commutative** if the diagram

\[ \begin{array}{ccc}
A \otimes A & \xrightarrow{\beta_{A, A}} & A \otimes A \\
\downarrow m & & \downarrow m \\
A & & A
\end{array} \]

commutes, where \( \beta \) is the braiding.
Note: $A \text{-mod}_C = \text{the category of right } A \text{-modules in } C$ is a tensor category, and the free module functor
\[ C \rightarrow A \text{-mod}_C \]
\[ X \rightarrow X \otimes A \]
is a surjective tensor functor.
(All of this is in case $A$ is commutative.)
and maybe $C$ should be fusion?
Conversely, suppose we are given a braided fusion category $E$ and a fusion category $D$, and a tensor functor $F: C \rightarrow D$.
It has a right adjoint, $I: D \rightarrow C$.
It has a right adjoint, $I: D \rightarrow C$.
Also, $D$ becomes a $C$-module category in the obvious way:
\[ (X, Y) \rightarrow F(X) \otimes Y \text{ for } X \in C, Y \in D. \]
If $F$ is essentially surjective, then we get an equivalence $D \simeq I(1) \text{-mod } C$.

Correction! We need to equip $F$ with a "central structure," i.e., a factorization
\[ C \xrightarrow{\text{braided tensor functor}} \mathcal{Z}_1(D) \xrightarrow{\text{forgetful functor}} D \]
\[ F \]
Example. If \( \mathcal{D} \) is any fusion category, we take \( \mathcal{C} = \mathbb{Z}_1(\mathcal{D}) \) and apply the construction above to the forgetful functor \( F: \mathbb{Z}_1(\mathcal{D}) \to \mathcal{D} \). The right adjoint \( I: \mathcal{D} \to \mathbb{Z}_1(\mathcal{D}) \) will be called the "induction functor."

Thus \( I(1_\mathcal{D}) \) becomes a commutative algebra in \( \mathbb{Z}_1(\mathcal{D}) \), and we get an identification \( \mathcal{D} \cong I(1_\mathcal{D}) - \text{mod} \mathbb{Z}_1(\mathcal{D}) \).

Special cases. (1) Take \( \mathcal{D} = \text{Rep}(G) \), for a finite group \( G \). We have
\[
\mathbb{Z}_1(\text{Rep}(G)) \cong \mathbb{Z}_1(\text{Vec}_G) \xrightarrow{F} \text{Rep}(G)
\]
If \( I \) is the right adjoint to \( F \), then \( I(1) \), as an equivariant sheaf on \( G \), is the sheaf with 1-dimensional stalks; as an algebra, \( I(1) = \text{Fun}(G) \), the algebra of functions \( G \to k \), with pointwise operations.

(2) Take \( \mathcal{D} = \text{Vec}_G \). Then \( I(1) = k[G] \), the group algebra. It is commutative because the braiding
\[
\beta: k[G] \otimes k[G] \to k[G] \otimes k[G]
\]
is not the obvious one, but, rather, is given by \( g \otimes h \mapsto h \otimes h^{-1} g \) \( \forall g, h \in G \).

Sketch of the proof of the theorem stated at the beginning:

Let \( C \) and \( D \) be fusion categories over \( k \), and let \( a: Z_1(C) \rightarrow Z_1(D) \) be a braided monoidal \( k \)-linear equivalence. Note that \( a \) takes commutative algebras to commutative algebras.

We have the picture:

\[
\begin{aligned}
&\xymatrix{ Z_1(C) \ar@/_/[d]_{I_C} \ar[r]^{a} & Z_1(D) \ar@/^/[d]^{I_D} \\
E \ar@/^/[u]^{F_E} & & & F_D \ar@/_/[u]_{F_D} \\
}
\end{aligned}
\]

Let \( L = a^{-1}(I_D(1_D)) \); it is a commutative algebra in \( Z_1(C) \), and we have \( D \cong (L - \text{mod } Z_1(C)) \).

By a slight abuse of notation, we also denote \( F_E(L) \) by \( L \); thus we can also view \( L \) as an algebra in \( C \).
Let us decompose

\[ L = \bigoplus_i L_i, \]

where the \( L_i \) are indecomposable algebras on \( E \) (so that the multiplication map for \( L \) vanishes on \( L_i \otimes L_j \) for \( i \neq j \)).

We would like to show that:

\[ \text{Li} \text{-bimod}_E \cong L \text{-mod} \cong \text{Der} \]

This would prove the theorem, because \( \text{Li} \text{-bimod}_E \) is the dual category to \( E \text{-Mod}_L \) with respect to the \( E \)-module \( L \text{-mod}_E \) (this was explained last time).

Consider the following commutative diagram of categories and tensor functors:

\[
\begin{array}{ccc}
\mathbb{Z}_1(E) & \xrightarrow{\sim} & \mathbb{Z}_1(L_i \text{-bimod}_E) \\
\downarrow \text{Z} & & \downarrow \text{forgetful} \\
\mathbb{Z} \otimes L & \xrightarrow{\pi_i} & \text{Li} \text{-bimod}_E \\
\end{array}
\]

(THE commutativity is easy to check.)
Under both compositions,
\[
Z \rightarrow Z \otimes L_i \in L_i\text{-}\text{bimod}_E
\]
thanks to $Z$ being central.

Let $F_i$ be the composition
\[
L\text{-mod}_{Z_i(\mathbb{C})} \xrightarrow{F_i} \bigoplus_j L_j\text{-}\text{bimod}_E
\]
One sees that $F_i$ is a surjective tensor functor. However,
\[
\text{FPdim} (L\text{-mod}_{Z_i(\mathbb{C})}) = \text{FPdim}(E)^2
\]
\[
= \text{FPdim} (L_i\text{-}\text{bimod}_E).
\]

By the Proposition on page 3, this implies that $F_i$ is an equivalence, which was what we wanted to show.

**Question.** In the situation above, is it true that $L_i\text{-}\text{mod}_E \cong L_j\text{-}\text{mod}_E$ for all $i$ and $j$?
Modular categories

Recall that a pivotal fusion category is a fusion category $\mathcal{C}$ equipped with an isomorphism of tensor functors

$$\psi: \text{Id}_\mathcal{C} \Rightarrow \text{the functor } X \mapsto X^{**}$$

This structure allows us to define the trace of any endomorphism $X \xrightarrow{f} X$ of an object $X \in \mathcal{C}$, namely, as the composition

$$\begin{array}{c}
1 \xrightarrow{\text{coev}_X} X \otimes X^* \xrightarrow{f \otimes \text{id}} X \otimes X^* \\
\xrightarrow{\psi \otimes \text{id}} X^{**} \otimes X^* \xrightarrow{\text{ev}_{X^*}} 1
\end{array}$$

which lives in $\text{End}(1) \cong k$.

We say that a pivotal structure $\psi$ on $\mathcal{C}$ is spherical if $\forall X \in \mathcal{C}$,

$$\dim(X) = \dim(X^*)$$

where $\forall Y \in \mathcal{C}$, we define $\dim(Y) = \text{tr}(\text{id}_Y)$.

Now suppose $\mathcal{C}$ is a braided fusion category. For any $X \in \mathcal{C}$, we can define a natural isomorphism $\psi_X: X^{**} \xrightarrow{\sim} X$. 
However, usually we do not have \( \psi_x \otimes \psi_y = \psi_{x \otimes y} \).

Let us recall the definition of \( \psi_x \):

\[
X^{**} \cong I \otimes X^{**} \xrightarrow{\text{id} \otimes \beta_{X^{**}}} X \otimes X^* \otimes X^{**} \xrightarrow{\text{id} \otimes \text{ev}_x} X \otimes I \cong X.
\]

One has: \( \psi_{x \otimes y} = \beta_{Y, X} \beta_{X, Y} (\psi_x \otimes \psi_y) \) for all \( x, y \in C \).

Next, suppose \( C \) is a fusion category which has both a spherical structure and a braiding.

**Def.** We put \( \Theta_X := \psi_{X^{-1}} \circ \psi_X : X \cong X \).

This defines a (non-tensor) automorphism of the identity functor \( \text{Id}_C \), called the balancing, satisfying the identities:

\[
\Theta_{X \otimes Y} = \beta_{Y, X} \beta_{X, Y} (\Theta_X \otimes \Theta_Y),
\]

and \( \Theta^*_X = \Theta_X \) for all \( X, Y \in C \).

**Remark.** Thus a ribbon fusion category is the same thing as a fusion category equipped with a braiding and a spherical structure. The compatibility
condition one usually requires for these two structures is automatic!

**Example.** \( A = \text{finite abelian group} \)
\[ \mathcal{C} = \text{Vec}_A \]

To give a braiding on \( \mathcal{C} \), is the same as to give a bi-multiplicative map
\[ B : A \times A \longrightarrow k^* \]

Then we get a natural balancing given by
\[ \mathcal{O}_{\delta_a} = B(a, a) \quad \forall a \in A, \]

where \( \delta_a \in \text{Vec}_A \) is the "delta-sheaf" at \( a \).

Let \( \mathcal{C} \) be a spherical fusion category over \( k \). Then \( \mathbb{Z}_1(\mathcal{C}) \) is also spherical with the same spherical structure, and thus \( \mathbb{Z}_1(\mathcal{C}) \) is naturally a ribbon category. This ribbon category turns out to be nondegenerate in the sense we will now try to explain.

Let \( \{X_1, \ldots, X_n\} \) be representatives of the isomorphism classes of simple objects in \( \mathcal{C} \). Form the matrix

\[ S = (s_{ij})_{i,j=1}^n \]

where
\[ s_{ij} = \text{tr} \left( X_i \otimes X_j \xrightarrow{B_{ij}} X_j \otimes X_i \xrightarrow{B_{ij}} X_i \otimes X_j \right) \]
Theorem (Müger). The matrix $S$ is invertible, i.e., $Z_1(\mathcal{E})$ is a modular category.

Question. Given a modular category $\mathcal{E}$, how can one tell whether there exists a (spherical) fusion category $\mathcal{D}$ such that $\mathcal{E} \cong Z_1(\mathcal{D})$ as braided fusion categories? (this word is irrelevant for various reasons)

Special situation. When is a given modular category $\mathcal{E}$ equivalent (as a braided fusion category) to $Z_1(\text{Vec}_G^\omega)$ for some finite group $G$ and a 3-cocycle $\omega : G \times G \times G \to k^*$.

Note that $Z_1(\text{Vec}_G^\omega)$ contains $\text{Rep}(G)$ as a symmetric subcategory, and the FP dimension of $\text{Rep}(G)$ is $\sqrt{\text{FPdim}(Z_1(\text{Vec}_G^\omega))}$.

Theorem. Let $\mathcal{E}$ be a modular category over $k$, containing a symmetric subcategory $\mathcal{L} \subset \mathcal{E}$ such that $\text{FPdim}(\mathcal{L}) = \sqrt{\text{FPdim}(\mathcal{E})}$.

Suppose also that $\mathcal{L} = \text{Rep}(G)$ for a finite group $G$. 

As braided fusion categories
Then $C \leq Z_1(\text{Vec}_G \wedge)$ for some 3-cocycle $\omega : G \times G \times G \to \mathbb{K}^\times$.

**Definition.** Subcategories $L \leq C$ having the properties listed above are called Lagrangian.

Let us explain the terminology.

Let $A = (\mathbb{F}_p)^n$ (as an additive group), and let $q$ be a nondegenerate quadratic form on $A$. Then $\text{Vec}_A$ is a modular category, and Lagrangian subcategories of $\text{Vec}_A$ correspond to Lagrangian subspaces of $A$.

**Sketch of the proof of the theorem.**

We want to find a suitable commutative algebra in $C$. In the category $\text{Rep}(G)$, we have the commutative algebra $A = \text{Fun}(G)$ of functions $G \to \mathbb{K}$, with pointwise operations. Of course, we can also view $A$ as a commutative algebra in $C$.

The functor

$$F : C \to A \text{-mod}_C$$

$$X \mapsto X \otimes A$$

has a natural central structure, i.e., a natural collection of isomorphisms

$$F(X) \otimes_A Y \cong Y \otimes_A F(X)$$
for all $X \in C, Y \in A_{\text{mod}_C}$, satisfying natural compatibility conditions.

In fact, the central structure is given by

$$\beta_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$$

Thus $F$ lifts to a functor

$$\mathcal{E} \rightarrow \mathcal{Z}_1(A_{\text{mod}_C}),$$

which turns out to be an equivalence.

Furthermore, $A_{\text{mod}_C}$ is faithfully graded by the group $G$:

$$A_{\text{mod}_C} = \bigoplus_{g \in G} (A_{\text{mod}_C})_g$$

and

$$(A_{\text{mod}_C})_g \otimes (A_{\text{mod}_C})_h \subset (A_{\text{mod}_C})_{gh}$$

On the other hand,

$$\text{FPdim} (A_{\text{mod}_C}) = 1$$

$$= \text{FPdim} (\mathcal{E})^{1/2} = |G|_2$$

which is only possible if each $(A_{\text{mod}_C})_g$ contains only one simple object. This is enough to identify $A_{\text{mod}_C} \cong \text{Vec}_G$ for some $G$. 

Remark. The proof sketched above shows a little bit more. Namely, from a Lagrangian subcategory \( \mathcal{L} \subseteq \text{Rep}(G) \) of \( \mathcal{C} \), we produced a canonical commutative algebra \( A \) in \( \mathcal{C} \) and a canonical equivalence \( \mathcal{C} \cong \mathcal{Z}_1(A\text{-mod}_\mathcal{C}) \).

We can therefore obtain another result. Consider the following equivalence relation on pairs \( (G, \omega) \), where \( G \) is a finite group and \( \omega \in H^3(G, K^\times) \):

\[
(G, \omega) \sim (G', \omega') \iff \mathcal{Z}_1(\text{Vec}_G) \cong \mathcal{Z}_1(\text{Vec}_{G'})
\]
as braided fusion categories.

This equivalence relation can be characterized in cohomological terms (Naidu, Nikshych).

Example. Let \( A \) be a finite abelian group, and let \( K \) be a finite group acting on \( A \) by automorphisms. Then

\[
\mathcal{Z}_1(\text{Vec}_{A \times K}) \cong \mathcal{Z}_1(\text{Vec}_{\hat{A} \times K}),
\]
even though \( A \times K \) and \( \hat{A} \times K \) could be non-isomorphic as groups.