

# EXAMPLES AND EXERCISES IN BASIC CATEGORY THEORY

## 1. CATEGORIES

1.1. **Generalities.** I've tried to be as consistent as possible. In particular, throughout the text below, categories will be denoted by capital script letters such as  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  (possibly with subscripts); objects of categories will be denoted by capital Roman letters such as  $X, Y, Z$ ; morphisms in categories will be denoted by lowercase Roman letters such as  $f, g, h$ ; functors between categories will be denoted by capital Greek letters such as  $\Phi, \Psi$ , and natural transformations between functors (in other terminology: morphisms of functors) will be denoted by lowercase Greek letters such as  $\alpha, \beta, \phi, \psi$ , etc.

Every non-obvious statement in each of the examples appearing below should be treated as an exercise, even if it is not called an exercise explicitly.

1.2. **Categories.** A *category* is a mathematical structure specified by giving the following data, satisfying a collection of axioms listed below.

Data needed to define a category  $\mathcal{C}$ :

- (a) A **class** (not necessarily a set) whose members are called *objects* of  $\mathcal{C}$ . In practice one often abuses notation by denoting the class of objects of  $\mathcal{C}$  by the letter  $\mathcal{C}$  as well. In particular, the notation  $X \in \mathcal{C}$  is to be understood as “ $X$  is an object of  $\mathcal{C}$ ”. I will follow this abuse of notation since the only reasonable alternative is to write something like  $Ob(\mathcal{C})$  for the class of objects of a category  $\mathcal{C}$ , consistently throughout all these notes, and this just takes too much time.
- (b) A (possibly empty!) **set**  $\text{Hom}_{\mathcal{C}}(X, Y)$  for any pair of objects  $X, Y \in \mathcal{C}$ . This set is called the set of *morphisms* from  $X$  to  $Y$  in the category  $\mathcal{C}$ . Elements of this set could be written as  $f : X \longrightarrow Y$  or as  $X \xrightarrow{f} Y$  (either of the last two formulas should be read as: “ $f$  is a morphism from  $X$  to  $Y$ ”). Often, when the category  $\mathcal{C}$  is clear from the context, one may omit  $\mathcal{C}$  from the notation and use  $\text{Hom}(X, Y)$  to denote the set of morphisms from  $X$  to  $Y$ . Some people also like to use the notation  $\text{Mor}(X, Y)$ . To me, however, the notation  $\text{Hom}(X, Y)$  is much more aesthetically pleasing.

If  $X \in \mathcal{C}$ , one often writes  $\text{End}(X)$  or  $\text{End}_{\mathcal{C}}(X)$  in place of  $\text{Hom}_{\mathcal{C}}(X, X)$ . The elements of the set  $\text{End}(X)$  are called *endomorphisms* of  $X$ .

- (c) A map of sets

$$\text{Hom}(Y, Z) \times \text{Hom}(X, Y) \longrightarrow \text{Hom}(X, Z)$$

for each triple of objects  $X, Y, Z \in \mathcal{C}$ . This map is called *composition of morphisms* and is often written as  $(g, f) \mapsto g \circ f$ . Some people like to write  $gf$  in place of  $g \circ f$ , but this could lead to unnecessary confusion. Some people also like to write composition

of morphisms in the opposite way (i.e.,  $fg$  in place of  $gf$ ). However, this is really a *terrible* idea. Morphisms should always be composed the following way:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \quad \Longrightarrow \quad X \xrightarrow{g \circ f} Z,$$

and never the other way around.

Axioms that have to be satisfied in order for the data above to define a category:

- (i) Composition of morphisms is associative. More precisely, given objects  $X, Y, Z, W$  of  $\mathcal{C}$  and morphisms  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ , we require that

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

In view of this axiom, whenever we have any composable<sup>1</sup> sequence  $f_1, \dots, f_n$  of morphisms in a category, the expression  $f_n \circ f_{n-1} \circ \dots \circ f_2 \circ f_1$  is unambiguous.

- (ii) Every object has an “identity endomorphism”. More precisely, if  $X \in \mathcal{C}$ , there exists an element  $\text{id}_X \in \text{End}(X)$  such that for every morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$ , we have  $f \circ \text{id}_X = f$ , and for every morphism  $g : Z \rightarrow X$  in  $\mathcal{C}$ , we have  $\text{id}_X \circ g = g$ .

**1.3. Exercise.** Show that for any category  $\mathcal{C}$  and any object  $X \in \mathcal{C}$ , there is only one endomorphism of  $X$  satisfying the defining property of  $\text{id}_X$ . Thus one can really speak of **the** identity endomorphism of  $X$ .

**1.4. First examples.** Unfortunately, even for the most basic examples of categories there is no universally accepted notation. I will use the following notation:

- (1) Sets denotes the category of sets. In other words, objects of Sets are sets; if  $X$  and  $Y$  are sets, then  $\text{Hom}(X, Y)$  is the set of all maps from  $X$  to  $Y$ ; and composition of morphisms in Sets is the usual composition of maps.

From now on, if the definition of the composition of morphisms in a category is obvious, I will not mention it.

- (2) sets denotes the category of *finite* sets.  
 (3) Groups denotes the category of groups, the morphisms being group homomorphisms.  
 (4) groups denotes the category of finite groups.  
 (5) Ab denotes the category of abelian groups.  
 (6) ab denotes the category of finite abelian groups.  
 (7) Vect<sub>k</sub> denotes the category of vector spaces over a given field  $k$  (if you wish, you can think of  $k = \mathbb{R}$  or  $\mathbb{C}$ ), the morphisms being linear maps between vector spaces.  
 (8) vect<sub>k</sub> denotes the category of finite dimensional vector spaces over  $k$ .  
 (9) Top denotes the category of topological spaces, the morphisms being continuous maps between topological spaces.

<sup>1</sup>Exercise: guess the definition of “a composable sequence of morphisms”!

Note that in each of the examples above, the objects of a given category are sets, possibly equipped with some extra structure and/or satisfying some extra properties. Morphisms are maps of sets that are in some sense compatible with the extra structure, and composition of morphisms is the usual composition of maps. However, there are other important examples of categories which do not look like that at all (cf. §1.9).

**1.5. Open subsets of a topological space.** The following less trivial example of a category is important in topology and geometry. Let  $X$  be a topological space. Define  $\underline{\text{Op}}(X)$  to be the following category.

The objects of  $\underline{\text{Op}}(X)$  are the open subsets of  $X$ . If  $U, V \in \underline{\text{Op}}(X)$ , we declare the set of morphisms from  $U$  to  $V$  to consist of exactly one element when  $U \subseteq V$ , and to be empty otherwise. [If you wish, in case  $U \subseteq V$ , you can think of the unique morphism from  $U$  to  $V$  as “the inclusion map”, but this is not necessary.]

Check that this leaves no freedom in defining the composition of morphisms in  $\underline{\text{Op}}(X)$ . Check that  $\underline{\text{Op}}(X)$  is a category, as claimed.

**1.6. Isomorphisms.** Let  $\mathcal{C}$  be a category and  $X \xrightarrow{f} Y$  a morphism in  $\mathcal{C}$ . We say that  $f$  is an *isomorphism*, or that  $f$  is *invertible*, if there exists a morphism  $g : Y \rightarrow X$  in  $\mathcal{C}$  such that  $g \circ f = \text{id}_X$  and  $f \circ g = \text{id}_Y$ . If this is the case,  $g$  is called an *inverse* of  $f$ .

Check that if  $f$  has an inverse, that inverse is unique. So, if  $f$  is an isomorphism, one can unambiguously denote its inverse by  $f^{-1}$ . Check that if  $g$  and  $f$  are composable morphisms and are both invertible, then  $g \circ f$  is also invertible, and  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ . What about the converse: if  $g \circ f$  is invertible, does it follow that  $f$  or  $g$  is invertible?

Is it true that in each of the examples in §1.4, a morphism in the given category is an isomorphism if and only if the underlying map of sets is a bijection? If not, then what are the examples in which this property does not hold?

If  $X, Y$  are objects of a category  $\mathcal{C}$  such that there exists an isomorphism  $f : X \rightarrow Y$ , we say that  $X$  and  $Y$  are *isomorphic*, and write  $X \cong Y$ . An isomorphism between  $X$  and  $Y$  is often denoted by an arrow like this:  $X \xrightarrow{\cong} Y$ . (Note that some people also use the notation  $X \xrightarrow{\sim} Y$ .) Usually, if an isomorphism between  $X$  and  $Y$  exists, it is non-unique. (When are two finite sets  $X$  and  $Y$  isomorphic in the category sets? If they are, how many different isomorphisms between  $X$  and  $Y$  are there?)

**1.7. Automorphisms.** If  $\mathcal{C}$  is a category and  $X \in \mathcal{C}$ , the set of all invertible morphisms from  $X$  to itself will be denoted by  $\text{Aut}(X)$ . Check that  $\text{Aut}(X)$  is a group under composition of morphisms. It is called the *group of automorphisms* of  $X$ .

**1.8. Groupoids.** A *groupoid* is a category in which every morphism is invertible. The notion of a groupoid is very important in algebraic topology. In particular, in §1.9 below I will define the *fundamental groupoid* of a topological space, which provides the correct way of thinking about fundamental groups. For now, let us note that you can get a groupoid from any given category (in particular, from any of the examples already mentioned) simply by discarding all non-invertible morphisms.

More precisely, let  $\mathcal{C}$  be any category, and let  $\mathcal{C}^\times$  be the category whose class of objects is that of  $\mathcal{C}$ , and where the morphisms are defined as follows. If  $X, Y$  are two objects, then  $\text{Hom}_{\mathcal{C}^\times}(X, Y)$  is the set of invertible elements of  $\text{Hom}_{\mathcal{C}}(X, Y)$ . The composition of morphisms in  $\mathcal{C}^\times$  is defined as the composition in  $\mathcal{C}$ . This is well defined by one of the exercises in §1.6. Check that  $\mathcal{C}^\times$  is a category and is a groupoid.

**1.9. The fundamental groupoid.** Let  $X$  be a topological space. We define a category  $\Pi(X)$  (which will turn out to be a groupoid) as follows.

- The objects of  $\Pi(X)$  are the points of  $X$ .

In order to define morphisms in  $\Pi(X)$  we need to recall the notion of homotopy of paths. Suppose  $x, y \in X$  and  $\gamma_0, \gamma_1 : [0, 1] \rightarrow X$  are continuous maps (where the closed interval  $[0, 1]$  is equipped with the usual topology) such that  $\gamma_0(0) = x = \gamma_1(0)$  and  $\gamma_0(1) = y = \gamma_1(1)$ . (One can say that  $\gamma_0$  and  $\gamma_1$  are (continuous) paths from  $x$  to  $y$ .)

We say that  $\gamma_0$  and  $\gamma_1$  are *homotopic* if there exists a continuous map  $H : [0, 1] \times [0, 1] \rightarrow X$ , called a *homotopy* between  $\gamma_0$  and  $\gamma_1$ , such that  $H(t, 0) = \gamma_0(t)$  and  $H(t, 1) = \gamma_1(t)$  for all  $t \in [0, 1]$ , and also  $H(0, s) = x$  and  $H(1, s) = y$  for all  $s \in [0, 1]$ . Observe that these conditions can be rephrased as follows. If we define  $H_s(t) = H(t, s)$ , then, for every fixed  $s \in [0, 1]$ ,  $H_s$  should be a (continuous) path from  $x$  to  $y$ , and, moreover, one should have  $H_0 = \gamma_0$  and  $H_1 = \gamma_1$ .

Check that being homotopic is an equivalence relation on continuous paths from  $x$  to  $y$ . This allows us to define, for every pair  $x, y \in X$ , the set of equivalence classes of continuous paths from  $x$  to  $y$  modulo homotopy.

- For two objects  $x, y$  of  $\Pi(X)$ , we define  $\text{Hom}(x, y) = \text{Hom}_{\Pi(X)}(x, y)$  to be the set of homotopy classes of continuous paths from  $x$  to  $y$ .

Finally, we need to define composition of paths. If  $\alpha : [0, 1] \rightarrow X$  and  $\beta : [0, 1] \rightarrow X$  are continuous maps such that  $\alpha(1) = \beta(0)$ , we can define a continuous map

$$\beta \cdot \alpha : [0, 1] \rightarrow X, \quad t \mapsto \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2}; \\ \beta(2t - 1), & \frac{1}{2} < t \leq 1. \end{cases}$$

Check that composition of paths respects homotopy. In other words, if  $\alpha_0, \alpha_1$  are two homotopic paths from  $x$  to  $y$ , and  $\beta_0, \beta_1$  are two homotopic paths from  $y$  to  $z$ , then the paths  $\beta_0 \cdot \alpha_0$  and  $\beta_1 \cdot \alpha_1$  from  $x$  to  $z$  are also homotopic.

- This allows us to define composition of morphisms in  $\Pi(X)$  unambiguously: if  $x, y, z \in X$ , then to define the composition map

$$\text{Hom}(y, z) \times \text{Hom}(x, y) \longrightarrow \text{Hom}(x, z),$$

let us pick equivalence classes  $f \in \text{Hom}(x, y)$ ,  $g \in \text{Hom}(y, z)$  and representatives  $\alpha : [0, 1] \longrightarrow X$ ,  $\beta : [0, 1] \longrightarrow X$  of  $f$  and  $g$ , respectively. Then we define  $g \circ f$  to be the equivalence class of  $\beta \cdot \alpha$ .

Verify the following statements:

- Composition of homotopy classes of continuous paths is associative. (There is something to think about, because composition of continuous paths, before passing to homotopy classes, is NOT associative!)
- The definitions above turn  $\Pi(X)$  into a category.
- $\Pi(X)$  is in fact a groupoid. Indeed, if  $f : x \longrightarrow y$  is a morphism in  $\Pi(X)$  and  $\alpha : [0, 1] \longrightarrow X$  is a representative of  $f$ , as before, check that the equivalence class of the path  $\alpha^{-1} : [0, 1] \longrightarrow X$  defined by  $\alpha^{-1}(t) = \alpha(1 - t)$  is an inverse of  $f$ .

One calls  $\Pi(X)$  the *fundamental groupoid* of the topological space  $X$ . If we fix a point  $x \in X$ , then, in particular, all morphisms from  $x$  to  $x$  in  $\Pi(X)$  form a group which we used to denote  $\text{Aut}_{\Pi(X)}(x)$ . In topology it has a different name: the *fundamental group of  $X$  at the point  $x$*  is defined to be

$$\pi_1(X, x) := \text{Aut}_{\Pi(X)}(x).$$

Check that this definition of the fundamental group is equivalent to the other one(s) you have seen. The proof will be essentially tautological. The definition of the fundamental groupoid is no more complicated than the definition of the fundamental group; however, for many purposes it is much more convenient to think in terms of the fundamental groupoid rather than the fundamental group.

For example, the definition of the fundamental groupoid is completely canonical, while the definition of the fundamental group depends on the choice of a base point. In particular, if  $X$  has several connected components, then  $\Pi(X)$  “keeps track” of all of them, while if we choose a base point  $x \in X$ , then  $\pi_1(X, x)$  does not know anything about the connected components of  $X$  that do not contain  $x$ . For instance, if  $X$  is the disjoint union of a circle and a line, and  $x \in X$  is a point lying on the line, then  $\pi_1(X, x)$  is the trivial group, while  $\Pi(X)$  looks like the “disjoint union” (you can try to think how to define the disjoint union of two categories in general – this is not difficult) of the fundamental groupoid of a circle and the fundamental groupoid of a line.

**1.10. Subcategories.** Let  $\mathcal{C}$  be a category. We say that a category  $\mathcal{D}$  is a *subcategory* of  $\mathcal{C}$  if: the class of objects of  $\mathcal{D}$  is a subclass of the class of objects of  $\mathcal{C}$  (i.e., every object of  $\mathcal{D}$  is also an object of  $\mathcal{C}$ ); for every pair  $X, Y$  of objects of  $\mathcal{D}$ , the set  $\text{Hom}_{\mathcal{D}}(X, Y)$  is a subset of  $\text{Hom}_{\mathcal{C}}(X, Y)$ ; and composition of two morphisms in  $\mathcal{D}$  is the same regardless

of whether it is computed in  $\mathcal{D}$  or in  $\mathcal{C}$ . We say that, moreover,  $\mathcal{D}$  is a *full subcategory* of  $\mathcal{C}$  if, for every pair of objects  $X, Y$  of  $\mathcal{D}$ , one has  $\text{Hom}_{\mathcal{D}}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y)$ .

Determine which of the categories listed in the examples in §1.4 can be viewed as subcategories of the other ones. Which of these subcategories are full?

## 2. FUNCTORS AND NATURAL TRANSFORMATIONS

**2.1. Functors.** Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be categories. A *covariant functor*  $\Phi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  is a rule which to every object  $X$  of  $\mathcal{C}_1$  assigns an object  $\Phi(X)$  of  $\mathcal{C}_2$ , and to every morphism  $f : X \rightarrow Y$  in  $\mathcal{C}_1$  assigns a morphism  $\Phi(f) : \Phi(X) \rightarrow \Phi(Y)$  in  $\mathcal{C}_2$  such that  $\Phi(g \circ f) = \Phi(g) \circ \Phi(f)$  whenever  $X \xrightarrow{f} Y \xrightarrow{g} Z$  are morphisms in  $\mathcal{C}_1$ , and such that  $\Phi(\text{id}_X) = \text{id}_{\Phi(X)}$  for all objects  $X$  of  $\mathcal{C}_1$ .

To get the notion of a *contravariant functor*  $\Psi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  one has to make the following changes:  $\Psi(f)$  should now be a morphism from  $\Psi(Y)$  to  $\Psi(X)$  (i.e.,  $\Psi$  “reverses the directions of all arrows”), and the first requirement in the definition of a functor has to be replaced by  $\Psi(g \circ f) = \Psi(f) \circ \Psi(g)$ .

Usually, the word “functor” without any adjectives refers to a covariant functor. I will also use this convention from now on.

**2.2. Examples of functors.** There are plenty.

- (1) For any category  $\mathcal{C}$  we have the identity functor  $\text{Id}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ .
- (2) If  $\mathcal{C}$  is a category and  $\mathcal{D} \subseteq \mathcal{C}$  is a subcategory, one has the obvious “inclusion functor”  $\mathcal{D} \hookrightarrow \mathcal{C}$ . In particular, we have inclusion functors  $\text{Vect}_k \hookrightarrow \text{Ab} \hookrightarrow \text{Groups} \hookrightarrow \text{Sets}$  and  $\text{ab} \hookrightarrow \text{groups} \hookrightarrow \text{sets}$ . These functors are usually called the “forgetful functors”.
- (3) For any set  $X$ , let  $\text{Fun}(X)$  denote the space of “functions” on  $X$ , i.e., maps  $X \rightarrow \mathbb{R}$ . Check that  $\text{Fun}(X)$  is naturally a vector space over  $\mathbb{R}$  (where addition and multiplication by scalars are defined pointwise). Check that one can make  $\text{Fun}$  into a **contravariant** functor  $\text{Sets} \rightarrow \text{Vect}_{\mathbb{R}}$  by defining it on morphisms in the following way. If  $f : X \rightarrow Y$  is a map of sets, we will denote the corresponding linear map  $\text{Fun}(f) : \text{Fun}(Y) \rightarrow \text{Fun}(X)$  by  $f^*$ , and it is defined by  $f^*(g) = g \circ f$ ; in other words, if  $g : Y \rightarrow \mathbb{R}$  is a function, then  $(f^*(g))(x) = g(f(x))$  for all  $x \in X$ .
- (4) Somewhat less trivially, if we restrict attention to finite sets, we can make  $\text{Fun}$  into a **covariant** functor as well. Check the following statement. If  $f : X \rightarrow Y$  is a map of finite sets, let define  $f_! : \text{Fun}(X) \rightarrow \text{Fun}(Y)$  by the formula<sup>2</sup>

$$(f_!(g))(y) = \sum_{x \in X, f(x)=y} g(x).$$

Then, if we define  $\text{Fun}(f) := f_!$ , we get a covariant functor  $\text{Fun} : \text{sets} \rightarrow \text{vect}_{\mathbb{R}}$ .

<sup>2</sup>The symbol  $f_!$  is pronounced “ $f$  lower shriek”.

- (5) Let  $X$  and  $Y$  be topological spaces, and let  $f : X \rightarrow Y$  be a continuous map. Show that the rule  $\alpha \rightarrow f \circ \alpha$  (where  $\alpha : [0, 1] \rightarrow X$  is a continuous path) can be used to define a functor between the fundamental groupoids (§1.9)  $\Pi(X) \rightarrow \Pi(Y)$ .
- (6) Let  $\underline{\text{Top}}_*$  denote the category of *pointed topological spaces*. The objects of  $\underline{\text{Top}}_*$  are pairs  $(X, x)$  consisting of a topological space  $X$  and a point  $x \in X$ . A morphism  $f : (X, x) \rightarrow (Y, y)$  in  $\underline{\text{Top}}_*$  is a continuous map  $f : X \rightarrow Y$  such that  $f(x) = y$ . Composition of morphisms is defined as the composition of maps in the usual sense. One has the obvious forgetful functor  $\underline{\text{Top}}_* \rightarrow \underline{\text{Top}}$ .
- (7) More interestingly, the fundamental group is a functor from  $\underline{\text{Top}}_*$  to  $\underline{\text{Groups}}$ . More precisely, check that if  $f : (X, x) \rightarrow (Y, y)$  is a morphism in  $\underline{\text{Top}}_*$ , then one can use  $f$  to define a group homomorphism  $\pi_1(X, x) \rightarrow \pi_1(Y, y)$ , and this yields a functor

$$\pi_1 : \underline{\text{Top}}_* \rightarrow \underline{\text{Groups}}.$$

**2.3. The dual category.** Let  $\mathcal{C}$  be any category. The *dual category* of  $\mathcal{C}$  (some people also use the term *opposite category*) is the category  $\mathcal{C}^\circ$  defined as follows. The objects of  $\mathcal{C}^\circ$  are the objects of  $\mathcal{C}$ . If  $X, Y \in \mathcal{C}$ , we define  $\text{Hom}_{\mathcal{C}^\circ}(X, Y) := \text{Hom}_{\mathcal{C}}(Y, X)$ . If  $X \xrightarrow{f} Y \xrightarrow{g} Z$  are morphisms in  $\mathcal{C}$ , then we can view  $f$  and  $g$  as morphisms  $Y \rightarrow X$  and  $Z \rightarrow Y$  in  $\mathcal{C}^\circ$ , and the composition  $f \circ g$  in  $\mathcal{C}^\circ$  is defined as the composition  $g \circ f$  in  $\mathcal{C}$ . Sometimes, to avoid confusion, if  $f : X \rightarrow Y$  is a morphism in  $\mathcal{C}$ , it is convenient to denote the corresponding morphism in  $\mathcal{C}^\circ$  by  $f^\circ : Y \rightarrow X$ . With this notation, the definition of the composition of morphisms in  $\mathcal{C}^\circ$  becomes  $f^\circ \circ g^\circ = (g \circ f)^\circ$ .

Informally speaking,  $\mathcal{C}^\circ$  is obtained from  $\mathcal{C}$  by “reversing all the arrows”.

Check that  $\mathcal{C}^\circ$  is indeed a category. Check that the rule  $X \mapsto X, f \mapsto f^\circ$  defines a contravariant functor  $\mathcal{C} \rightarrow \mathcal{C}^\circ$ . If  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are two categories, check that a covariant functor  $\Phi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  can also be thought of as a contravariant functor  $\mathcal{C}_1^\circ \rightarrow \mathcal{C}_2$ , or a contravariant functor  $\mathcal{C}_1 \rightarrow \mathcal{C}_2^\circ$ , or a covariant functor  $\mathcal{C}_1^\circ \rightarrow \mathcal{C}_2^\circ$ . The same holds if we switch “covariant” and “contravariant” throughout the last sentence.

**2.4. Natural transformations.** Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be categories, and let  $\Phi, \Psi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  be functors. A *morphism of functors*, or a *natural transformation*,  $\alpha : \Phi \rightarrow \Psi$ , is a rule which to every object  $X$  of  $\mathcal{C}_1$  assigns a morphism  $\alpha_X : \Phi(X) \rightarrow \Psi(X)$  such that for any morphism  $X \xrightarrow{f} Y$  in  $\mathcal{C}_1$ , the following diagram commutes:

$$\begin{array}{ccc} \Phi(X) & \xrightarrow{\Phi(f)} & \Phi(Y) \\ \alpha_X \downarrow & & \downarrow \alpha_Y \\ \Psi(X) & \xrightarrow{\Psi(f)} & \Psi(Y) \end{array}$$

We say that the collection  $(\alpha_X)$  is an *isomorphism* (of functors) between  $\Phi$  and  $\Psi$  if each morphism  $\alpha_X$  is invertible. Check that in this case the collection  $(\alpha_X^{-1})$  defines a morphism of functors from  $\Psi$  to  $\Phi$ . We call this collection  $\alpha^{-1}$ .

**2.5. Compositions of functors and natural transformations.** Functors between categories can be composed. Namely, let  $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$  be categories and let  $\Phi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ ,  $\Psi : \mathcal{C}_2 \rightarrow \mathcal{C}_3$  be functors. Invent the definition of the composed functor  $\Psi \circ \Phi : \mathcal{C}_1 \rightarrow \mathcal{C}_3$ .

Similarly, if  $\mathcal{C}$  and  $\mathcal{D}$  are categories,  $\Phi_1, \Phi_2, \Phi_3 : \mathcal{C} \rightarrow \mathcal{D}$  are three functors, and  $\phi : \Phi_1 \rightarrow \Phi_2$ ,  $\psi : \Phi_2 \rightarrow \Phi_3$  are natural transformations, invent the definition of the composition  $\psi \circ \phi : \Phi_1 \rightarrow \Phi_3$ . In fact, modulo some set-theoretical issues (which should be ignored at this point), one can define the *category of functors*  $\text{Funct}(\mathcal{C}, \mathcal{D})$  whose objects are functors from  $\mathcal{C}$  to  $\mathcal{D}$  and whose morphisms are natural transformations.

If  $\alpha : \Phi \rightarrow \Psi$  is an isomorphism between two functors, then the natural transformation  $\alpha^{-1}$  defined in §2.4 has the property that  $\alpha \circ \alpha^{-1}$  (respectively,  $\alpha^{-1} \circ \alpha$ ) is the identity natural transformation on  $\Psi$  (respectively, on  $\Phi$ ).

**2.6. Equivalence and isomorphism of categories.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be two categories. An *isomorphism* between  $\mathcal{C}$  and  $\mathcal{D}$  is a functor  $\Phi : \mathcal{C} \rightarrow \mathcal{D}$  such that there exists a functor  $\Psi : \mathcal{D} \rightarrow \mathcal{C}$  for which the compositions  $\Psi \circ \Phi$  and  $\Phi \circ \Psi$  are **equal** to the identity functors  $\text{Id}_{\mathcal{C}}$  and  $\text{Id}_{\mathcal{D}}$ , respectively. It is easy to check that the functor  $\Psi$  (if it exists) is determined uniquely by this condition; it is called the *inverse* of  $\Phi$ , denoted by  $\Phi^{-1}$ .

A functor  $\Phi : \mathcal{C} \rightarrow \mathcal{D}$  is said to be an *equivalence* of categories if there exists a functor  $\Psi : \mathcal{D} \rightarrow \mathcal{C}$  such that the compositions  $\Psi \circ \Phi$  and  $\Phi \circ \Psi$  are **isomorphic** to the identity functors  $\text{Id}_{\mathcal{C}}$  and  $\text{Id}_{\mathcal{D}}$ , respectively. Any such functor  $\Psi$  is called a *quasi-inverse* of  $\Phi$ . It is usually highly non-unique.

Any isomorphism of categories is clearly also an equivalence, but the converse is far from true. An instructive example appears in §2.7. In practice, the notion of an equivalence of categories is much more flexible than the notion of an isomorphism of categories, and equivalences occur much more often than isomorphisms. (However, the notion of an isomorphism of categories is NOT entirely useless, contrary to what some authors claim!)

**2.7. Instructive example of an equivalence of categories.** Let us define a category  $\mathcal{C}$  as follows. The objects of  $\mathcal{C}$  are the nonnegative integers. If  $m, n \in \mathcal{C}$ , then  $\text{Hom}_{\mathcal{C}}(m, n)$  is the set of all  $n \times m$  matrices<sup>3</sup> with real entries (note that the order of  $m$  and  $n$  has been switched!). If  $k, m, n \in \mathcal{C}$ , then composition of morphisms

$$\text{Hom}_{\mathcal{C}}(m, n) \times \text{Hom}_{\mathcal{C}}(k, m) \rightarrow \text{Hom}_{\mathcal{C}}(k, n)$$

is defined as the usual multiplication of matrices. Check that  $\mathcal{C}$  is indeed a category.

We claim that  $\mathcal{C}$  is equivalent to the category  $\text{vect}_{\mathbb{R}}$  of finite dimensional vector spaces over  $\mathbb{R}$ . (Note that  $\mathcal{C}$  has no chance to be isomorphic to  $\text{vect}_{\mathbb{R}}$ ; for instance,  $\mathcal{C}$  has only countably many objects.) I will define functors  $\Phi : \mathcal{C} \rightarrow \text{vect}_{\mathbb{R}}$  and  $\Psi : \text{vect}_{\mathbb{R}} \rightarrow \mathcal{C}$ , and leave it to you to check that they are quasi-inverse to each other.

<sup>3</sup>If either  $m = 0$  or  $n = 0$ , this could cause confusion. In this case we simply define  $\text{Hom}_{\mathcal{C}}(m, n)$  to be the set with just one element, thought of as the “zero matrix”.

For every nonnegative integer  $n$ , define  $\Phi(n) = \mathbb{R}^n$ , the “standard”  $n$ -dimensional vector space over  $\mathbb{R}$ . For every  $n \times m$  matrix  $A$  with real entries, define  $\Phi(A)$  to be the linear map  $\mathbb{R}^m \rightarrow \mathbb{R}^n$  represented by the matrix  $A$ .

The definition of  $\Psi$  is slightly more complicated. For each  $V \in \underline{\text{vect}}_{\mathbb{R}}$ , let us choose, once and for all, an *ordered* basis  $(v_1, \dots, v_m)$  of  $V$ . Implicitly, of course, it is understood that  $m$  and the vectors  $v_j$  depend on  $V$ . If  $V$  consists only of the zero vector, then, by definition,  $m = 0$  and the basis of  $V$  is empty.

With this notation, we define  $\Psi(V) = m$ , and for any linear map  $T : V \rightarrow W$  of finite dimensional vector spaces over  $\mathbb{R}$ , we define  $\Psi(T)$  to be the matrix of  $T$  with respect to our chosen bases  $(v_1, \dots, v_m)$  of  $V$  and  $(w_1, \dots, w_n)$  of  $W$ .

**2.8. Representable functors.** Let  $\mathcal{C}$  be a category, and let  $\tilde{\mathcal{C}} = \underline{\text{Funct}}(\mathcal{C}^\circ, \underline{\text{Sets}})$ . In other words,  $\tilde{\mathcal{C}}$  is the category of **contravariant** functors from  $\mathcal{C}$  to the category of sets. If  $X \in \mathcal{C}$  is any object, we define a functor  $h_X : \mathcal{C}^\circ \rightarrow \underline{\text{Sets}}$  by the rule

$$h_X(Y) = \text{Hom}_{\mathcal{C}}(Y, X);$$

$$\begin{aligned} \text{and if } f \in \text{Hom}_{\mathcal{C}}(Y, Z), \quad \text{then } h_X(f) : \text{Hom}_{\mathcal{C}}(Z, X) &\rightarrow \text{Hom}_{\mathcal{C}}(Y, X) \\ \text{is defined by } \quad h_X(f)(g) = g \circ f \quad \forall g \in \text{Hom}_{\mathcal{C}}(Z, X). \end{aligned}$$

A functor  $F : \mathcal{C}^\circ \rightarrow \underline{\text{Sets}}$  is said to be *representable* if it is isomorphic to a functor of the form  $h_X$  for some object  $X \in \mathcal{C}$ . If this is the case, then the object  $X$  is determined up to a natural isomorphism. The notion of a representable functor provides a convenient (and correct!) way of thinking about various “universal properties” that appear in algebra.

**2.9. Categories with one object.** For a good understanding of basic categorical notions, it is important to test them on the examples of categories that have only one object. (In particular, it is important to realize that categories with one object do provide non-trivial examples of many notions!)

Open-ended exercise: play with categories that have only one object. For example, can you give a more concrete interpretation of a category with one object? What about a groupoid with one object? In these concrete terms, what is a functor between two categories (respectively, groupoids), each of which has only one object? What is a natural transformation between such functors? If  $\mathcal{G}$  is a groupoid with one object, try to understand in concrete terms the category  $\underline{\text{Funct}}(\mathcal{G}, \underline{\text{Sets}})$  of functors from  $\mathcal{G}$  to the category of sets. Does this category have a different name?

Prove that if  $X$  is a path-connected space, then the fundamental groupoid  $\Pi(X)$  is equivalent (as a category) to a groupoid with one object. Hint: choose a point  $x \in X$  and look at the groupoid with one object defined by the fundamental group  $\pi_1(X, x)$ . This exercise explains why for path-connected spaces one often studies the fundamental group rather than the fundamental groupoid.