TOPICS IN ALGEBRA: HOMEWORK 1

SUMMER REU 2007

The list of exercises below includes those given during my first lecture, sometimes with slight modifications. However, I also added many new exercises. There are two groups of exercises; the first one has to do with vector spaces of functions on sets, and the second one has to do with tensor products and duals of vector spaces.

1. FUNCTIONS ON SETS

1.1. Recall that throughout this series of lectures we will mostly be working with the field $\mathbb{C}$ of complex numbers, and with vector spaces over $\mathbb{C}$. However, if you know what a field is and what a vector space over a field is, you should replace $\mathbb{C}$ by an arbitrary field; in particular, none of the exercises in this problem set uses any special properties of $\mathbb{C}$.

1.2. Recall that if X is any (not necessarily finite) set, we write $\operatorname{Fun}(X)$ for the vector space of all functions $X \longrightarrow \mathbb{C}$, where the vector space operations (i.e., addition and multiplication by scalars) are defined pointwise. Given a map $\phi : X \longrightarrow Y$ of sets, we define the pullback by ϕ as the map

$$\phi^* : \operatorname{Fun}(Y) \longrightarrow \operatorname{Fun}(X)$$

given by $\phi^*(f) = f \circ \phi$; in other words,

$$(\phi^* f)(x) = f(\phi(x)) \qquad \forall\, f \in \operatorname{Fun}(Y),\ x \in X.$$

If X and Y are arbitrary sets and $f \in \operatorname{Fun}(X)$, $g \in \operatorname{Fun}(Y)$, we define

$$f \boxtimes g \in \operatorname{Fun}(X \times Y)$$

by the formula $(f \boxtimes g)(x, y) = f(x) \cdot g(y)$.

Now let $\phi : X \longrightarrow Y$ be a map of *finite* sets. We define

$$\phi_! : \operatorname{Fun}(X) \longrightarrow \operatorname{Fun}(Y), \qquad (\phi_! f)(y) = \sum_{x \in f^{-1}(y)} f(x).$$

If X and Y are finite sets and $K \in \operatorname{Fun}(Y \times X)$, we define

$$T_K : \operatorname{Fun}(X) \longrightarrow \operatorname{Fun}(Y), \qquad (T_K f)(y) = \sum_{x \in X} K(y, x) f(x).$$

1.3. Remark. Note that I slightly changed the notation from the one I used in the first lecture. First of all, the order of the factors X and Y is interchanged. This is of course immaterial, but makes the formulas look somewhat nicer. Also, I am denoting the function on $Y \times X$ by the letter K, as opposed to F. The reason is that the operator T_K does have a standard name: it is called the integral operator defined by the "kernel" K.

1.4. Exercises. Most of these are completely straightforward, but they are worth doing nonetheless, because many of these exercises provide toy models for more complicated and important concepts in algebraic topology and algebraic geometry. Feel free to skip some of the exercises if their statements use terms that are unfamiliar to you: e.g., if you are not familiar with the language of categories and functors. I will review this language later in my lectures.

(1) Check that, with the notation above, ϕ^*, $\phi_!$ and T_K are $\mathbb{C}$-linear maps of vector spaces.

(2) Check that the map $\phi_!$ is still well defined even if one (or both) of the sets X, Y is infinite, but ϕ has *finite fibers*. We will call such a map ϕ proper. The name comes from topology: if we equip X and Y with the discrete topologies, then they both become Hausdorff topological spaces, every map $\phi : X \longrightarrow Y$ is continuous, and ϕ has finite fibers if and only if it is proper in the usual topological sense (the inverse image of any compact subset of Y under ϕ is compact).

(3) Let $\mathscr{S}$ denote the category of all sets. Use the pullback maps ϕ^* to turn the assignment $X \longmapsto \operatorname{Fun}(X)$ into a functor from $\mathscr{S}$ to the category $\mathbb{C}-\mathrm{Vect}$ of vector spaces over $\mathbb{C}$. Is this functor covariant or contravariant?

(4) Let $\mathscr{S}'$ denote the subcategory of $\mathscr{S}$ whose objects are all sets, but whose morphisms are proper maps of sets. Is it a full subcategory? Use the direct image maps $\phi_!$ to turn the assignment $X \longmapsto \operatorname{Fun}(X)$ into a functor from $\mathscr{S}'$ to the category $\mathbb{C}-\mathrm{Vect}$. Is this functor covariant or contravariant?

(5) Prove that if $\phi : X \longrightarrow Y$ is a bijection between sets, then the maps $\phi^* : \operatorname{Fun}(Y) \longrightarrow \operatorname{Fun}(X)$ and $\phi_! : \operatorname{Fun}(X) \longrightarrow \operatorname{Fun}(Y)$ are inverse to each other.

(6) If $f, g \in \operatorname{Fun}(Y)$, we denote by $f \cdot g \in \operatorname{Fun}(Y)$ the pointwise product of the functions f and g. If $\phi : X \longrightarrow Y$ is a map of sets, show that $\phi^*(f \cdot g) = \phi^*(f) \cdot \phi^*(g)$. Is the analogous statement true for $\phi_!$ (when ϕ is a proper map)?

(7) Let $\phi : X \longrightarrow Y$ be a proper map of sets (i.e., a map with finite fibers). Prove that if $f \in \operatorname{Fun}(X)$ and $g \in \operatorname{Fun}(Y)$, then

$$\phi_!\big(f \cdot \phi^*(g)\big) = \phi_!(f) \cdot g.$$

This is called the "Projection Formula".

(8) Let X, Y be sets, and let $p : X \times Y \longrightarrow X$, $q : X \times Y \longrightarrow Y$ be the "natural projections", i.e., $p(x, y) = x$ and $q(x, y) = y$. Prove that if $f \in \operatorname{Fun}(X)$ and $g \in \operatorname{Fun}(Y)$, then $f \boxtimes g = p^*(f) \cdot q^*(g)$. Thus the "external product" $\boxtimes$ of functions can be expressed in terms of pointwise products and pullbacks.

(9) Conversely, the pointwise product can be expressed in terms of external products and pullbacks. Namely, let X be any set, and consider the map $\Delta_X : X \longrightarrow X \times X$ given by $\Delta_X(x) = (x, x)$. This map is called "the diagonal", for obvious reasons. If $f, g \in \operatorname{Fun}(X)$, show that $f \cdot g = \Delta_X^*(f \boxtimes g)$.

(10) Prove that if X and Y are finite sets, p and q are the natural projections as above, and $K \in \operatorname{Fun}(Y, X)$, then for every $f \in \operatorname{Fun}(X)$, we have $T_K(f) = q_!(K \cdot p^*(f))$.

(11) Prove that if X and Y are finite sets, then every $\mathbb{C}$-linear map $\operatorname{Fun}(X) \longrightarrow \operatorname{Fun}(Y)$ has the form T_K for a unique $K \in \operatorname{Fun}(Y \times X)$.

(12) Let X, Y and Z be finite sets. Define the <u>convolution</u> operation

$$\operatorname{Fun}(Z, Y) \times \operatorname{Fun}(Y, X) \longrightarrow \operatorname{Fun}(Z, X), \qquad (L, K) \longmapsto L * K,$$

by the formula

$$(L * K)(z, x) = \sum_{y \in Y} L(z, y) \cdot K(y, x).$$

Formulate and prove the associativity property of convolution (hint: you will need four finite sets for this). Prove that by taking $X = Y = Z$, you get an operation on $\operatorname{Fun}(X \times X)$ which turns it into a monoid (in fact, a $\mathbb{C}$-algebra, if you know what this term means). What is the unit element in this monoid?

(13) With the same notation, prove that

$$T_{L*K} = T_L \circ T_K : \operatorname{Fun}(X) \longrightarrow \operatorname{Fun}(Z).$$

Can you use this result to solve the previous exercise in a different way?

(14) Let X and Y be two sets, and assume that each of them has (strictly) more than 1 element (in particular, they could be infinite). Prove that the map

$$\operatorname{Fun}(X) \times \operatorname{Fun}(Y) \longrightarrow \operatorname{Fun}(X \times Y), \qquad (f, g) \longmapsto f \boxtimes g, \tag{1.1}$$

is NOT surjective.

(15) Show that if either X or Y is finite, the image of the map (1.1) spans the vector space $\operatorname{Fun}(X \times Y)$. Show that this is false if both X and Y are infinite.

(16) For any $n \in \mathbb{N}$ and any $1 \leq p \leq \infty$, we will use the standard analysis notation $L^p(\mathbb{R}^n)$. Show that if $f, g \in L^p(\mathbb{R})$, then the function $f \boxtimes g : \mathbb{R}^2 \longrightarrow \mathbb{C}$ lies in $L^p(\mathbb{R}^2)$.

(17) Show that not every function $h \in L^p(\mathbb{R}^2)$ can be written as a *finite* linear combination[1] of functions of the form $f \boxtimes g$, where $f, g \in L^p(\mathbb{R})$.

(18) However, prove that if $1 \le p < \infty$, then the set of finite linear combinations of such functions is *dense* in $L^p(\mathbb{R}^2)$, with respect to the usual L^p norm. What if $p = \infty$?

2. TENSOR PRODUCTS AND DUALITY

2.1. Let V, W, U be vector spaces over $\mathbb{C}$. A map $B : V \times W \longrightarrow U$ is said to be *bilinear* if for each fixed $v \in V$, the map $w \longmapsto B(v, w)$ is linear as a map $W \longrightarrow U$, and for each fixed $w \in W$, the map $v \longmapsto B(v, w)$ is linear as a map $V \longrightarrow U$. Exercise: check that if B is bilinear and $T : U \longrightarrow U'$ is any *linear* map, where U' is another vector space, then the composition $T \circ B : V \times W \longrightarrow U'$ is also a bilinear map.

2.2. We say that a bilinear map $B : V \times W \longrightarrow U$ *satisfies the universal property of tensor products* if the following condition holds. For every vector space U' (over $\mathbb{C}$, as usual) and every bilinear map $B' : V \times W \longrightarrow U'$, there exists *a unique* linear map $T : U \longrightarrow U'$ such that $B' = T \circ B$. Exercise: formulate and prove the appropriate uniqueness (up to unique isomorphism) property of such a universal pair (U, B).

2.3. If a given bilinear map $B : V \times W \longrightarrow U$ does satisfy the universal property of tensor products, we say that (U, B) is a *tensor product* of V and W over $\mathbb{C}$. In practice, one usually ignores the map B and calls U "the" tensor product of V and W; then one writes $U = V \otimes_{\mathbb{C}} W$. However, it is important to remember that it is only the pair (U, B), and not just the vector space U, that is uniquely determined up to unique isomorphism.

2.4. Exercises.

(1) Show that the tensor product of any pair of vector spaces always exists, by completing the outline of the proof explained in the second lecture.

(2) If X and Y are arbitrary sets, show that the map (1.1) is bilinear. Prove that it satisfies the universal property of tensor products if either X or Y is finite. Prove that this is false when both X and Y are infinite.

(3) If V and W are vector spaces over $\mathbb{C}$, we will write $\mathrm{Hom}_{\mathbb{C}}(V, W)$ for the space of linear maps $V \longrightarrow W$. Show that it is a vector space with respect to pointwise addition and multiplication by scalars. If $W = \mathbb{C}$ with the obvious vector space structure, we write $V^* = \mathrm{Hom}(V, \mathbb{C})$ and call it the <u>dual</u> of V.

(4) If V and W are vector spaces over $\mathbb{C}$, construct a natural (in the colloquial sense) bilinear map $V^* \times W \longrightarrow \mathrm{Hom}(V, W)$.

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[1] The word "finite" is only added for emphasis. In algebra, whenever we talk about linear combinations, they are always assumed to be finite: one cannot make sense of the sum of an infinite number of (nonzero) elements in a vector space (or an abelian group) *unless* an additional topological structure is present.

(5) By the universal property of tensor products, you get a *linear* map $V^* \otimes_{\mathbb{C}} W \longrightarrow \operatorname{Hom}(V, W)$. Show that it is an isomorphism when V is finite dimensional. What happens when V is infinite dimensional?

(6) If V and W are vector spaces over $\mathbb{C}$, use the universal property of tensor products to construct a natural linear map $V^* \otimes_{\mathbb{C}} W^* \longrightarrow (V \otimes_{\mathbb{C}} W)^*$. When is this map an isomorphism?

3. POSTLUDE: SOME PHILOSOPHICAL COMMENTS

3.1. The understanding of the comments that follow is by no means essential to following my lectures. These comments are meant to place the topics I discuss during my lectures into a more general framework, so that those of you with more advanced backgrounds will not get bored out of your minds. Moreover, you should only read these comments after attempting most of the exercises above.

3.2. In standard linear algebra courses one usually encounters one of the following two approaches to the study of vector spaces.

In the first approach, the space $\mathbb{C}^n$ of vectors with n coordinates is the main object of study. An arbitrary vector space of dimension n is identified with $\mathbb{C}^n$ by means of choosing a basis. Linear maps between vector spaces are interpreted as matrices, and so on. This approach has an unquestionable computational advantage.

In the second approach, one works with the abstract concept of a vector space. The notion of a basis is only used as a technical tool (for example, in the proof of the following statement: if $T : V \longrightarrow W$ is an *injective* linear map of vector spaces, then there exists a linear map $S : W \longrightarrow V$ such that $S \circ T = \mathrm{id}_V$). This approach is the most convenient one for the study of linear maps between vector spaces (e.g., spectral theory; classification of linear operators on a finite dimensional vector space up to conjugation, and so on).

The approach to vector spaces that I outlined in my first lecture lies between these two; it is closer to the first one than to the second one. Namely, if X is a finite set, the "delta-functions" δ_x, for $x \in X$, form a natural basis of the vector space $\operatorname{Fun}(X)$. (Check that this is no longer true if X is infinite!) It should also be easy to see that the correspondence between $\operatorname{Fun}(Y \times X)$ and linear maps $\operatorname{Fun}(X) \longrightarrow \operatorname{Fun}(Y)$ is equivalent to the usual description of linear maps via matrices, and that the convolution operation $\operatorname{Fun}(Z, Y) \times \operatorname{Fun}(Y, X) \longrightarrow \operatorname{Fun}(Z, X)$ corresponds to the usual matrix multiplication.

However, the approach I took has many ideological advantages. For instance, it is more invariant than the approach via bases. More importantly, it makes the passage to infinite dimensional vector spaces and "matrices of infinite size" psychologically easier. Still more importantly, as we will see later this summer, many vector spaces that we will be working with will actually be *defined* as spaces of functions on various sets, and the games we were playing during the first lecture will turn out to be quite useful.

3.3. In particular, recall that with my approach, tensor products of (finite dimensional) vector spaces arise not as objects satisfying a certain universal property, but rather as the answer to the (arguably natural) question of how the space $\mathrm{Fun}(X \times Y)$ is related to the spaces $\mathrm{Fun}(X)$ and $\mathrm{Fun}(Y)$, when X and Y are finite sets. We will see that this is in fact precisely the reason why tensor products of vector spaces arise in TQFT. Furthermore, for many purposes it is better to *forget* that tensor products of vector spaces are usually defined by a certain universal property, and realize instead that the category of vector spaces with their tensor product is merely an example of a *symmetric monoidal category* (this notion will be discussed later this summer). There are many interesting and important examples of symmetric monoidal categories where the "tensor product" functor has nothing to do with any sort of universal properties.

3.4. The result that $\mathrm{Fun}(X \times Y) \cong \mathrm{Fun}(X) \otimes_{\mathbb{C}} \mathrm{Fun}(Y)$ for finite sets X, Y deserves a name; it is called the Künneth formula. If you know something about algebraic topology, you will understand the reason why. Namely, if X and Y are, say, compact manifolds (or finite CW complexes), and we let $H^*(\cdot)$ denote (for example) the singular cohomology with coefficients in $\mathbb{C}$, then the Künneth formula in algebraic topology states that

$$H^n(X \times Y) \cong \bigoplus_{p+q=n} H^p(X) \otimes_{\mathbb{C}} H^q(Y) \qquad \forall\, n \geq 0.$$

Now if X is a finite set, we can equip it with the discrete topology, and then it is trivial to check that $H^0(X)$ can be naturally identified with $\mathrm{Fun}(X)$, and $H^p(X) = 0$ for all $p \neq 0$. Thus "our" Künneth formula becomes a special case of the Künneth formula in topology.

3.5. The reason for playing with the pullback and direct image maps ϕ^*, $\phi_!$ is that these games provide toy models for various important constructions in geometry. For instance, one can easily define pullbacks on cohomology with respect to continuous maps of topological spaces. From a different perspective, constructions like ϕ^* and $\phi_!$ make sense for sheaves in various contexts (on manifolds, on algebraic varieties, and so on), where they become *functors* on the appropriate categories of sheaves.

3.6. The "integral operators" that I defined are toy models both for integral operators that one studies in analysis (by replacing sums with appropriate integrals; for instance, a function $K \in L^2(\mathbb{R}^2)$ can be used to define an operator $L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ called the integral operator with kernel K), and for certain important functors between categories of sheaves that are defined by "kernels" (e.g., the Fourier-Mukai transforms for sheaves on abelian varieties).