1. More on reduced norms & traces:

Reminder: If $A$ is a C.S.A. over a field $K$, we can define a map $N_{N/A}^\text{red} : A \rightarrow K$ the reduced norm.

$\text{Tr}^\text{red}_{N/A} : A \rightarrow K$ the reduced trace.

where $N_{N/A}^\text{red}$ is a homog. poly. map of degree $n = (\dim_K A)^{1/2}$.

Property: Let $R$ be a commutative $K$-alg. and $\phi : A \rightarrow \text{Mat}_n(R)$ a splitting of $A$ over $K$; i.e. a $K$-alg. homo. s.t. the induced map $R \otimes_K A \rightarrow \text{Mat}_n(R)$ is an isomorphism.

Then the following diagram commutes:

\[
\begin{array}{ccc}
A & \rightarrow & \text{Mat}_n(R) \\
N_{N/A}^\text{red} \downarrow & & \downarrow \text{*det} \\
K & \rightarrow & R
\end{array}
\]

and similarly for the trace $\text{Tr}^\text{red}$.

Remark: In fact, we can define a map $\text{Pr}^\text{red} : A \rightarrow K[\text{Et}]$. 
So that $V$ as above, the diagram

\[ A \rightarrow \text{Mat}_n(R) \]

\[ P \rightarrow \det(tI-M) \rightarrow \text{commutes} \]

\[ \text{Ker} \rightarrow R[t] \]

**Question**: How do we compute these things?

Let $D$ be a $p.d.$ central division algebra over $K$. For any $a \in D$, define $\rho_a : D \rightarrow D$ by $x \mapsto ax$.

is a $K$-linear map. Let $P_{\text{art}}(a) := \det(t \cdot Id_D - \rho_a)$.

**Exercise**: $P_{D/k}(\text{art}) = P_{D/k}^{\text{red}}(\text{art})^n$ when $n = (\dim_K D)^{1/2}$.

In particular, $\text{tr}(\rho_a) = n \cdot \text{Tr}_{D/k}^{\text{red}}(\rho_a)$.

This explains why when $\text{char}(K) > 2$, $\text{Tr}_{D/k}^{\text{red}}$ is much more useful than $\text{Tr}_{D/k}$.

2
Prop: Let \( L \subseteq D \) be any maximal (w.r.t. inclusion) commutative \( R \)-subalgebra of \( D \). (This implies that \( L \) is a field extension of \( K \) of degree \( n \), and \( L \otimes_R D \cong \text{Mat}_n(L) \)).

Then, \( \forall \alpha \in L \), \( \text{P}_{D/K}(\alpha, t) = \text{P}_{L/K}(\alpha, t) \).

Cor: (Exercise) \( \forall \alpha \in D \),

\[
\text{P}_{D/K}(\alpha, t) = \left( \text{min poly of } \alpha \text{ over } K \right)^n_{[K(\alpha) : K]}
\]

Lemma: If \( \text{P}(t) \in K[t] \) is monic and \( \text{P}(\alpha) = 0 \) for some \( \alpha \in D \),

then \( \text{P}_{D/K}(\alpha, t) = 0 \).

Proof of prop: Since \( D \) is a division ring, for every monic \( p(t) \in K[t] \),

we have \( p(\alpha) \cdot x = 0 \) for some \( x \in D \), \( \Rightarrow p(\alpha) = 0 \).

Thus, \( \text{P}_{D/K}(\alpha, t) \) is a power of the minimal poly. of \( \alpha \) over \( K \), and so is \( \text{P}_{L/K}(\alpha, t) \), and by the previous exercise,

so is \( \text{P}_{D/K}(\alpha, t) \). But \( \text{P}_{L/K}(\alpha, t) \) and \( \text{P}_{D/K}(\alpha, t) \)

are monic of deg \( n \), so we're done.
Cor: In the situation of the prop., if $D \neq K$, then $\exists x \in D_K$

such that $x$ is sep. over $K$. (Jacobson-Nother Theorem)

pf: Suppose this is not the case. Then $\forall x \in D_K$, the

minimal polynomial $f(x)$ has the form $t^p - a$ for some $n \in \mathbb{N}$

and some $a \in K$, where $p = \text{char}(K) > 0$. \[ a \]

By the last corollary, $T_{D/K}^x(t) = 0$ for all $x \in D_K$.

Extending scalars to any field extension $L/K$ over which

$D$ splits yields a contradiction.

Cor: In the same situation, $\exists$ a finite Galois extension of

$K$ which splits $D$.

2. Example of $G$-fields

Recall: A field $K$ is $G$ if $\forall$ hom. poly. $f \in K[x_1, \ldots, x_n]$

with $\deg(f) = d$, $1 \leq d < n$, then $\exists x \in K^n \setminus \{a\}$ s.t.

$f(x) = 0$. 

4
We proved that if $K$ is a $C_1$ field, then $Br(K) = 0$.

2.a. Finite fields are $C_1$.

**Theorem (Chevalley-Warning):**

Consider $f_1, \ldots, f_m \in \mathbb{F}_q[x_1, \ldots, x_n]$ s.t. $\sum_{j=1}^m \text{deg}(f_j) < n$. Then the number of $x \in \mathbb{F}_q^n$ such that $f_j(x) = 0$ for all $1 \leq j \leq m$ is $\equiv 0 \pmod{\text{char}(\mathbb{F}_q)}$.

**Proof:** Consider $p(x) = \prod_{j=1}^m (1 - f_j(x))^{q-1}$. Call this $X$.

Then $x \in X \Rightarrow p(x) = 1$.

$x \in \mathbb{F}_q^n \setminus X \Rightarrow p(x) = 0$.

So $|X| = \sum_{x \in \mathbb{F}_q^n} p(x) \equiv 0 \pmod{\text{char}(\mathbb{F}_q)}$.

But $p(x)$ is a linear combo of monomials $x_1^{d_1} \ldots x_n^{d_n}$ of total degree $< n(q-1)$. So at least one $d_i < q-1$, so their sum equals zero.
2b. Function fields in one variable over an alg. closed field.

\[ \text{are } C_1 \text{.} \]

**Theorem:** (Tsen-Laue) Let \( K \) be an alg. closed field and \( K \) a finite ext. of \( K(a) \). Then \( K \) is \( C_1 \).

**Lemma:** Any finite ext. of a \( C_1 \) field is \( C_1 \).

**Proof of the lemma**

The proof is given in the next lecture.

(i) Enough to deal with two cases:

1) \( K \) is finite & sep.
2) \( \text{char}(K) = p > 0 \) and \( K \) is finite & purely inseparable.

**Lemma:** If \( K \) is alg. closed, then \( K(a) \) is \( C_1 \).

**Proof:** Let \( F(x) \in K(a)[x_1, \ldots, x_n] \) be homogeneous of degree \( d \), where \( 1 \leq d \leq n \). We will show that \( F \) has a non-trivial zero in \( K[a]^n \).

Clearly, we may assume that \( F \) has coeffs. in \( K[a] \), and takes \( K[a]^n \) into \( K[a] \).

Therefore **6**
Idea: Restrict $F$ to a suitable finite dim. sub. of $K[t]^n$ and use the following.

**Theorem:** If $k$ is alg. closed and $\phi: X \to Y$ is a morphism of irreducible alg. var. over $k$, then the non-empty fibres of $\phi$ have dim $\leq \dim(X) - \dim(Y)$.

For $N \in \mathbb{N}$, write $V_N = \{ f(t) \in K[t] \mid \deg f < N \}.$ Let $D = \max$ of the deg. of the coefficients (w.r.t. $t$) of $F(t)$.

If $f_1(t), \ldots, f_n(t) \in V_N$, then $F(f_1(t), \ldots, f_n(t)) \in \sqrt{D + d(N-1) + 1}$.

So $F(V^n) \subseteq \sqrt{D + d(N-1) + 1}$.

Exercise: $F(\sqrt{V}_N^n)$ is a poly map with coeff in $t$.

But $\dim(V^n_N) = n. N > D + d(N-1) + 1$

whenever $N > D$. So by the Alg. geom. result, $F$ has a non-trivial zero on $V^n_N.$
Exercise: From the Riemann–Roch theorem in general by completing the following outline.

- \( K \) = field of rational functions on some smooth proj. curve \( X \).
- Fix \( x_0 \in X(K) \) and define \(( \forall N \in \mathbb{N})\)
  \[ V_N = \{ f \in K \mid f \text{ is regular away from } x_0, \text{ and the order of the pole of } f \text{ at } x_0 < N \} \]
  \[ V_N = \mathcal{P}(X, O((N-1)x_0)) \]
- In other words, for all \( \frac{1}{x_0} \in \mathbb{P}^1 \), \( \frac{1}{x_0} \in \mathbb{P}^1 \).
  \[ \dim V_N = N - g(X) \]
  where \( g(X) \) is the genus of \( X \). Proceed as above.