Construction of the bar resolution

Recall: $k$ is a fixed comm. ring.

we have the notion of dga over $k$.

**Lemma:** If $B = (B, d)$ is a dga over $k$ s.t. $\exists \epsilon \in B^{-1}$ with $d \epsilon = 1$, then $H^n(B) = 0$, $\forall n \in \mathbb{Z}$.

**Proposition:** Fix a $k$-alg. $A$.

(a) If a $k$-alg. $B$ together with a $k$-alg. homo. $A \rightarrow B$

and a disting. element $\epsilon \in B$ s.t. $\forall k$-alg. $R$, $\forall$ homo. $\psi: A \rightarrow R$

$\exists R$, $\exists!$ extension of $\psi$ to a homo. $\psi: B \rightarrow R$

$b \rightarrow x$

(b) There's a unique grading on $B$ s.t. $A = B^0$ and $\epsilon \in B^{-1}$.

Moreover, with this grading, $B^0 = A$, and $B^n = 0$, $\forall n \neq 0$.

(c) With respect to the grading in (b), there's a unique superdifferential

$d: B \rightarrow B$ of deg 1 with $d(1) = 1$.

(d) If $A$ is a proj. $k$-module, then $\forall n \geq 1$, $B^{-n}$ is a proj. $A^e$-module. ($A^e = A \otimes A^*$)
Terminology: B = B^0(A) is called the bar resolution of A.

Proof: Consider the following graded R-module:

\[ C^* = \bigoplus_{n \geq 0} C^{-n} \]

where \( C^{-n} = \mathbb{R}^n \otimes A \)

We have a natural graded R-module structure on \( C^* \). Namely, define

\[ C^p \otimes C^q \to C^{p+q} \]

It's clear that this gives an associative module on \( C^* \). Moreover, it agrees with the given module on \( C^* = A \).

Define \( e \in C^1 \) to be \( 1 \in \mathbb{R} \otimes A \).

Claim: the pair \( (C, e) \) satisfies the universal property.

First note that \( A \leq e \) as an algebra. Indeed,

\[ C^0 \otimes \cdots \otimes \otimes \otimes A = e \leq e \otimes \cdots \otimes \otimes A \]

Remark: Classically, the element \( e \otimes \cdots \otimes \otimes A \) of \( C^0 \) is denoted by \([e_0 \otimes \cdots \otimes A]_n\). This is where the name "bar resolution" came from.
With our approach, the element $e$ plays the role of
the vertical line in the classical notations.

Now given $\omega: A \to R$ and $x \in R$, we want to find an
extension of $\omega$ to $\psi: C \to R$

Uniqueness follows from the previous comment. For existence,

define $\psi: C \to R$ by

$$\psi(a_0 \circ \cdots \circ a_p) \to \omega(a_0) \times \omega(a_1) \times \cdots \times \omega(a_p) \in R$$

This is a well-defined algebra homomorphism, and it is trivial
to check that $\psi$ is an algebra homomorphism.

For part (b), the existence of such a grading follows from
the construction of $C$, and the uniqueness follows from the fact
that $A$ and $e$ generate $C$.

For (c), if $d$ exists it must be given by

$$d(a_0 \circ \cdots \circ a_p) = d(a_0 \circ a_1 \circ \cdots \circ a_p)$$

$$= \omega(a_0 \circ \cdots \circ a_p) - \omega(a_0) \circ a_1 \circ \cdots \circ a_p$$
\[ = a_0 \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_p - a_0 \otimes (a_1 \otimes a_2) \otimes a_3 \otimes \ldots \otimes a_p \]

\[ + (-1)^{p-1} a_2 \otimes \ldots \otimes a_p \otimes (a_1 \otimes a_p) \]

Conversely, it is easy to check that this works.

For part (d): Note that \( \forall n \geq 1, \ C^{-n} \cong A \otimes_\mathbb{Z} (A \otimes_\mathbb{Z} \ldots \otimes_\mathbb{Z} A) \)

as an \( A_\mathbb{Z} \)-module.

\[ (\text{Reduced bar resolution}) \]

Consider a \( R \)-algebra \( A \) and let \( B^*(A) \) be its bar resolution.

Define \( B^*(A) = B^*(A)/(\epsilon^2) \), where \( (\epsilon^2) \) is the two-sided ideal of \( B^*(A) \) generated by \( \epsilon^2 \).

Note that \( B^*(A) \) is also a ring over \( k \), because \( (\epsilon^2) \) is a homog. ideal and \( d(\epsilon^2) = 0 \).

We call \( B^*(A) \) the reduced bar resolution of \( A \).

By abuse of notation, we'll write \( \epsilon \) for the image of \( \epsilon \)

in \( B^*(A) \). We still have \( d(\epsilon) = 1 \) in \( B^*(A) \), so

\[ H^i(B^*(A)) = 0 \quad \forall \; i \in \mathbb{Z}. \]
Explicit description of $\overline{B}(A)$:

\[
\begin{cases}
\overline{B^n}(A) = A \\
\overline{B^{-1}}(A) = A \otimes_k A.
\end{cases}
\]

Introduce $\overline{A} = A/_{k,1}$ as a $k$-module.

Claim: For all $n \geq 2$, the quotient map

\[
\overline{B}^n(A) \longrightarrow \overline{B}^{-n}(A).
\]

descends to an isomorphism

\[
A \otimes_k \overline{A} \otimes_k \cdots \otimes_k \overline{A} \otimes_k A \longrightarrow \overline{B}^{-n}(A).
\]

With this identification, the multiplicity $\overline{B}^i(A)$ is defined by the same formula as for $B^i(A)$. (Exercise: prove this.)

Cor: If $\overline{A}$ is proj. as a $k$-module, then $\overline{B}(A)$ gives a resol. of $A$ by proj. $A^\otimes$-modules.
Exercise: Given a left $A$-module $M$ which is proj as $K$-module, write down the formulas defining the complexes

$B^{-n}(A) \otimes_A M$, and $B^{-n}(A) \otimes_A M$.

Note that by what we discussed last time, these complexes give resolutions of $M$ by proj $A$-modules.

Examples: $G$ = any group

$A = \mathbb{K} G$

Consider $k$ as an $A$-module, where $G$ acts trivially.

Note that $k, A, A \otimes k, \ldots, A \otimes \cdots \otimes k A$ are proj (in fact, free) $k$-modules. So the precise discussion applies.

What is the complex $B^{-n}(A) \otimes_k k$?

$B^{-n}(A) \otimes_k k = k$

$B^{-n}(A) \otimes_k k = A \otimes_k \cdots \otimes_k A$

$n$-copies of $A$
The diff. on $B^k(A) \otimes_A 1$ is given by

$$d(g_0 \otimes \cdots \otimes g_n) = (g_0 g_1) \otimes g_2 \otimes \cdots \otimes g_n$$

$$+ (-1)^{n-2} g_0 \otimes (g_1 g_2) \otimes \cdots \otimes g_n$$

$$+ (-1)^{n-1} g_0 \otimes \cdots \otimes g_{n-2}$$

\[\forall g_0, \ldots, g_n \in G.\]

This implies that for any $kG$-module $M$, the groups $\text{Ext}_k^i(k, M)$ can be computed as the cohomology of the complex

$$C^i(G, M) = \text{Hom}_{kG}(B^i(kG) \otimes_{kG} k, M)$$

Explicitly, since $\forall n \geq 0$, $B^{n-1}(kG) \otimes_{kG} k$ has a basis consisting of all elements of the form $1 \otimes g_0 \otimes \cdots \otimes g_n$ ($g_i \in G$).

So we can identify $C^n(G, M)$ with the $k$-module of all functions $G \rightarrow M$ (called $n$-cochains of $G$ with coefficients in $M$).

The diff. $d: C^n(G, M) \rightarrow C^{n+1}(G, M)$ is given by

$$\text{d}(d_{g_0, \ldots, g_n})$$
\((d\omega)(g_1, \ldots, g_{n+1}) = g_1(\omega(g_2, \ldots, g_{n+1})) - \omega(g_1g_2, \ldots, g_{n+1}) + (-1)^n \omega(g_1, g_2, \ldots, g_n)\)

Exercise: Do the same computation replacing \(B^*(A)\) with \(\overline{B}^*(A)\). Check that the resulting complex \(\overline{C}^*(G,M)\)

is the subcomplex of \(C^*(G,M)\) formed by the "normalized cochains"; where for \(n \geq 1\), a cochain \(\omega: G \to M^n\)

is said to be normalized if

\(\omega(g_1, \ldots, g_n) = 0\) provided \(g_i = 1\) for some \(1 \leq i \leq n\).

Example:

\[C^1(G,M) = \left\{ \text{functions } G \to M \right\}\]

\[d: C^1(G,M) \to C^2(G,M)\]

is given by \(d\omega(g_1, g_2) = g_1(\omega(g_2)) - \omega(g_1g_2) + \omega(g_1)\)

\[C^0(G,M) = M\] and \(i: C^0(G,M) \to C^1(G,M)\)

is given by \(i(m)(g) = g(m) - m\). So taking \(k = \mathbb{Z}\)

we see that \(H^1(G,M) = \left\{ \omega \in C^1(G,M) \mid \omega(gh) = \omega(h) + \omega(g) \right\}\)

functions of the form \(g \to g(m) - m\).
It is clear that this definition of $H^1$ agrees with the one given in the (not necessarily abelian) situation (via torsors).

In particular, note that if $M$ is a comm. and $G$ acts trivially on $M$, then $H^1(G, M) = \text{Hom}_G(G, M)$.

Exercise: Check that formula for $H^2$ agrees with the one given a few lectures ago.