§0. Introduction

Throughout, $R$ will denote a complete DVR with field of fractions $K$ and residue field $k$. We will assume that $k$ is perfect.

Our aim will be to prove:

**Theorem 1:** There is a short exact sequence:

$$0 \to \text{Br}(k) \to \text{Br}(K) \to \text{Hom}_{\text{cont}}(\text{Gal}(\bar{K}/k), \mathbb{Q}/\mathbb{Z}) \to 0$$

The proof of this will be based on two propositions.

**Proposition 1:** Every element of $\text{Br}(K)$ splits over an unramified finite Galois extension of $K$.

**Proposition 2:** For every finite Galois extension $E/F$, we have

$$\text{Br}(F/E) \cong H^2(\text{Gal}(F/E), F^\times)$$

We proved both of these earlier in the lecture series.
§1. Useful Computation

Let \( L \) be a finite unramified Galois extension of \( K \). We are going to compute \( H^2(\text{Gal}(L/K), L^\times) \).

Let \( R_L \) denote the integral closure of \( R \) in \( L \), so \( R_L \) is a complete DVR. Write \( G = \text{Gal}(L/K) \) and let \( \mathfrak{r}_L \) be the residue field of \( R_L \). Then \( R_L \) is \( G \)-stable, so we get an induced action on \( \mathfrak{r}_L \), and because \( L/K \) is unramified, this gives an isomorphism \( G \cong \text{Gal}(\mathfrak{r}_L/k) \).

We are now ready to start computing. We have an exact sequence of \( G \)-modules:

\[
1 \rightarrow R_L^\times \rightarrow L^\times \xrightarrow{\text{val}} \mathbb{Z} \rightarrow 1
\]

(1)

Note that this sequence splits (non-canonically, of course) by choosing a uniformizer \( \pi \in R \) (in \( R \), not in \( R_L \), because we need it to be \( G \)-invariant). This implies that the long exact cohomology sequence associated to (1) breaks up into split short exact sequences:

\[
0 \rightarrow H^0(G, R_L^\times) \rightarrow H^0(G, L^\times) \rightarrow H^0(G, \mathbb{Z}) \rightarrow 0
\]

(2)

In particular, this is true where \( q = 2 \).

We next compute \( H^2(G, \mathbb{Z}) \).
Consider the exact sequence of trivial $G$-modules:
\[ 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0 \quad (3) \]
Since $|G|$ acts invertibly on $\mathbb{Q}$, we have $H^j(G, \mathbb{Q}) = 0$ for all $j \geqslant 1$, so we obtain $H^1(G, \mathbb{Q}/\mathbb{Z}) \cong H^2(G, \mathbb{Z})$. On the other hand, since $G$ acts trivially on $\mathbb{Q}/\mathbb{Z}$, we have $H^1(G, \mathbb{Q}/\mathbb{Z}) \cong \text{Hom}(G, \mathbb{Q}/\mathbb{Z})$.

Next, we compute $H^2(G, R^*_L)$. Note that $R^*_L$ has a natural filtration $R^*_L \supset 1 + M_L \supset 1 + M_L^2 \supset \ldots$, where $M_L$ of course denotes the maximal ideal of $R_L$. We have natural isomorphisms compatible with the $G$-action:
\[ R^*_L/(1 + M_L) \cong \mathbb{R}^*_L \quad \text{and} \quad (1 + M_L^k)/(1 + M_L^{k+1}) \cong \mathbb{R}_L \quad (4) \]

Let us assume for now the additive part of Hilbert's Theorem 90:
\[ \text{Theorem 2 (Hilbert)}: \quad H^q(G, R_L) = 0 \quad \forall q \geqslant 1. \]
We will explain the proof of this later.
Claim: $H^q(G, 1 + M_L) = 0 \quad \forall q \geqslant 1$. 

To prove this claim, we will use the description of cohomology in terms of the standard complex and the fact that $HM_2$ is complete w.r.t. the filtration $\mathfrak{F}^iHM_2$.

Consider

$$C^2(G, \mathbb{H}M_2) \xrightarrow{d} C^1(G, \mathbb{H}M_2) \xrightarrow{d} C^0(G, \mathbb{H}M_2).$$

Let $\phi \in C^0(G, \mathbb{H}M_2)$ with $d\phi = 0$. The image of $\phi$ in $C^0(G, \mathbb{H}M_2)$ is a coboundary by Theorem 2 and (4).

Hence, $\exists \psi \in C^1(G, \mathbb{H}M_2)$ such that $\phi - \text{d}\psi$ takes values in $\mathbb{H}M_2^\perp$. Applying the same argument to $C^1(G, \mathbb{H}M_2^\perp)$ and so on, we construct $\psi_j$ such that $\phi - \sum_{j=1}^{\infty} \text{d}\psi_j$ takes values in $\mathbb{H}M_2^\perp$, for $\psi_j \in C^0(G, \mathbb{H}M_2^\perp)$. By completeness, we may put $\psi = \sum_{j=1}^{\infty} \psi_j$, so $d(\psi) = 0$. This proves the claim.

Now, we have a short exact sequence of $G$-modules:

$$1 \rightarrow \mathbb{H}M_2 \rightarrow R_2^* \rightarrow R_2^* \rightarrow 1$$

And by the previous claim we have $H^2(G, R_2^*) \cong H^2(G, \mathbb{H}M_2^\perp)$.

In the language of Brauer groups, the results of this section combined with (2) prove:

**Proposition 3:** For any unramified finite Galois extension $L/K$, there is a short exact sequence:

$$0 \rightarrow \text{Br}(L/K) \rightarrow \text{Br}(L/K) \rightarrow \text{Hom}(\text{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) \rightarrow 0$$

Moreover, any choice of uniformizer in $K$ gives a splitting of this SES.
§2. Main Theorem

Proof of Theorem 1: Choose a maximal unramified extension $K''$ of $K$. The residue field of $K''$ is $\overline{k}$, the algebraic closure of $k$. There is a natural (continuous) isomorphism

$$\text{Gal}(K''/K) \cong \text{Gal}(\overline{k}/k)$$

In particular, there is a canonical order-preserving bijection between $I := \{\text{finite Galois extensions of } K \text{ in } K'' \}$ and the set $\{\text{finite Galois extensions of } k \text{ in } \overline{k} \}$. Clearly $I$ is a directed set as ordered by inclusion.

We have already shown that the natural maps:

$$\lim_{I \in \text{I}} \text{Br}(L/K) \to \text{Br}(K)$$

are isomorphisms. Moreover, by definition,

$$\lim_{I \in \text{I}} \text{Hom}(\text{Gal}(L/k), \mathbb{Q}/\mathbb{Z}) \to \text{Hom}_{\text{cont}}(\text{Gal}(\overline{k}/k), \mathbb{Q}/\mathbb{Z})$$

Thus, taking limits in Proposition 3 yields the theorem.

Example: Suppose $K$ is locally compact, i.e., $k$ is finite. Then $\text{Br}(k) = 0$ and $\text{Gal}(\overline{k}/k)$ has a canonical topological generator $x \mapsto x^q$ for $q := q(k)$. Hence, $\text{Hom}_{\text{cont}}(\overline{k}/k, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$ canonically.
Thus, we get a canonical isomorphism $\text{inv}_K : \text{Br}(K) \to \mathbb{Q}/\mathbb{Z}$.

**Definition:** If $a \in \text{Br}(K)$, then $\text{inv}_K(a)$ is the **Hasse invariant** of $a$.

§3. Proof of Hilbert's Theorem 90 (additive version)

The aim of this section is to (finally, finally) prove Theorem 2. How will we do this?

First, consider $G$, any group. We have the forgetful functor from $\mathbf{ZG-mod} \to \mathbf{Z-mod}$.

**Lemma 1:** $F$ has a right adjoint, called the coinduction functor.

**Proof:** For any abelian group $N$, consider $(\mathcal{C}(N)) := \{ \text{functions } G \to N \}$ under pointwise addition and with $G$-action given by $(g \cdot p)(h) = p(hg)$. Clearly this gives a functor $C: \mathbf{Z-mod} \to \mathbf{ZG-mod}$. Let's check that $C$ is right adjoint to $F$, i.e., $\text{Hom}_{\mathbf{ZG-mod}}(F(M), N) \cong \text{Hom}_{\mathbf{Z-mod}}(M, C(N))$ functionally with respect to $M \in \mathbf{ZG-mod}$.

Indeed, take $\alpha \in \text{Hom}_{\mathbf{ZG-mod}}(F(M), N)$. We get a homomorphism $\tilde{\alpha} : M \to C(N)$ defined by $\tilde{\alpha}(m)(g) = \alpha(g \cdot m)$, as one trivially checks is well defined.

It's easy to check that this map $\alpha \mapsto \tilde{\alpha}$ is a
bijection, so we have proved the lemma.

**Proposition 4:** In the notation of Lemma 1, for any \( N \in \text{Z-mod} \), we have \( H^q_\mathfrak{G}(G, (CN)) = 0 \) for \( q \geq 1 \).

**Proof:** Choose any injective resolution of \( N \) in \( \text{Z-mod} \)

\[
0 \to N \to I^0 \to I^1 \to \ldots
\]

The functor \( C \) is exact by construction, so we get an exact sequence of \( \text{ZG-modules} \)

\[
0 \to (CN) \to (CI^0) \to (CI^1) \to \ldots
\]  \( (6) \)

Using the fact that \( C \) has an exact left adjoint, we check easily that \( C \) takes injective objects to injective objects. Hence, we can use \( (6) \) to compute

\[
H^q_\mathfrak{G}(G, (CN)) := \text{Ext}^q_\mathfrak{G}(\mathbb{Z}, (CN)),
\]

and using our resolution we get

\[
\text{Ext}^q_\mathfrak{G}(\mathbb{Z}, (CN)) = H^q(\text{Hom}_\mathfrak{G}(\mathbb{Z}, (CI^1))) = H^q(\text{Hom}_\mathfrak{G}(\mathbb{Z}, CI^1)) \cong \text{Ext}^q_\mathbb{Z}(\mathbb{Z}, N),
\]

This is zero for \( q \geq 1 \) because \( \mathbb{Z} \) is a projective \( \mathbb{Z} \)-module.

We are now in a sufficiently nice place to offer:

**Proof of Theorem 2:** By the normal basis theorem, \( L \cong K[G] \) as \( K[G] \)-modules. This means \( L \) is comodulated by \( K \), i.e., \( L \cong \text{C}(K) \). Now apply the previous proposition.

\( \square \)

Fin!