Tensor product of V.S., continued

Def: Let \( U, V, W \) be V.S. over \( \mathbb{C} \). A bilinear map

\[ B : V \times W \rightarrow U \]

satisfies the universal property of a tensor product if given any V.S. \( U' \) and a bilinear map \( B' \)

\[ B' : V \times W \rightarrow U' \]

there's a unique linear map \( T : U \rightarrow U' \) such that \( B' \circ T = B \).

\[ \begin{array}{ccc}
V \times W & \xrightarrow{B} & U \\
\downarrow & & \downarrow \\
& \circ T & \\
v \times w & \xrightarrow{B'} & U'
\end{array} \]

Lemma: If \( X \) and \( Y \) are finite sets, the map

\[ \text{Fun}(X) \times \text{Fun}(Y) \rightarrow \text{Fun}(X \times Y) \]

\[ (f,g) \mapsto f \circ g \]

is bilinear and satisfies the universal property of a tensor product. (Note that the bilinearity of the map \( (f,g) \mapsto f \circ g \) is completely obvious)
Proof: Consider any bilinear map $B' : \text{Fun}(X) \times \text{Fun}(Y) \to U$. We need a linear map $T : \text{Fun}(X \times Y) \to U'$ such that $B'(f,g) = T(f \circ g) \quad \forall f \in \text{Fun}(X)$, where $f \circ g$ denotes composition of functions.

If such $T$ exists, then we must have

$$T(S_{x,y}) = T(S_x \otimes S_y) = B'(S_x, S_y) = B'(\delta_x, \delta_y) \quad \forall (x,y) \in X \times Y.$$ 

Since $X$ and $Y$ are finite, $\forall f \in \text{Fun}(X \times Y)$, we have

$$h = \sum_{(x,y) \in X \times Y} h(x,y) \cdot S_{x,y}.$$ 

So, we must have

$$T(h) = \sum_{(x,y) \in X \times Y} h(x,y) \cdot T(S_{x,y}) = \sum_{(x,y) \in X \times Y} h(x,y) B'(\delta_x, \delta_y).$$

So $T$ is unique if it exists.

Conversely, define $T$ by the above formula and check that it works (exercise).
Existence of tensor products in general

For any pair of V.S. $V, W$, there exists a bilinear map $B: V \times W \rightarrow U$ which satisfies the universal property of a tensor product.

*Standard notation* $U = V \otimes W$ or $V \otimes_{F} W$.

**Construction:** Given $V, W$ V.S., define $V \otimes W$ to be the vector space whose elements are finite formal linear combinations of symbols $v \otimes w$ when $v \in V$, $w \in W$.

[So elements of $V \otimes W$ are lin. comb. $\sum_{i=1}^{n} \alpha_{i} (v_{i} \otimes w_{i})$.]

Consider the subspace $R \subset V \otimes W$ spanned by elements of the form:

- $(v_{1} + v_{2}) \otimes w - v_{1} \otimes w - v_{2} \otimes w$,
- $v \otimes (w_{1} + w_{2}) - v \otimes w_{1} - v \otimes w_{2}$,
- $\lambda(v \otimes w) - v \otimes (\lambda w)$,
We will take \( U = (V \otimes W) / R \).

**Idea:** We have a natural map of sets
\[
V \times W \rightarrow V \otimes W, \quad (v, w) \rightarrow v \otimes w
\]

\( R \) exactly contains all the obstructions to this map being bilinear.

In particular, the composite
\[
V \times W \rightarrow V \otimes W \rightarrow (V \otimes W)_R
\]
is bilinear.

**Exercise:** Check the universal property.

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**General Outline**

May's course is about the following result:

(a 2-dim) is the same as (a commutative Frobenius algebra) over \( \mathbb{C} \).

Frobenius algebras came up in representation theory of finite groups.

If \( G \) is a finite group, a representation of
\( G \) is a \( \mathbb{C} \)-vector space \( V \) and for each \( g \in G \), an \( \mathbb{C} \)-linear transform
\( T_g : V \rightarrow V \) s.t.

\[ T_{gh} = T_g T_h \quad \forall g, h \in G. \]
Remark: There are Frobenius algebras which do not come from rep. theory. However, rep. theory of finitegps is a good motivation.

There's also a notion of comm. Frobenius algebra. But the non-comm. one, in some respects, more interesting.

\[
\begin{align*}
\text{incoming} & \quad \text{boundary components} \\
\text{level set of } f & \quad \text{at time } t \\
\text{outgoing boundary components} & \quad (\text{possibly disconnected}) \\
\end{align*}
\]

\[f \text{ "time" function} \]

\[\begin{align*}
\text{initial point of time} & \quad t \quad \text{final point in time} \\
\text{Circles = cloud strings} \\
\text{For all but finitely many "times" } t \in \mathbb{R}, \text{ the level set } f^{-1}(t) \text{ is a disjoint union} \\
\text{of circles, representing the "state of our system of strings" at time } t. \\
\end{align*}\]
Collisions of strings

at this point of time we have created a string.

These are pictures of what the level sets of forms "attract" locally near critical points. In other words, these are the level sets which are

were "interesting" from just descript views of cones.

Things that could occur with string...
Next week we will define and study more functions. They are useful because:

Fundamental Observation:

Morse theory on surfaces can be used to prove their classification and at the same time to explain the physical motivation behind looking at pictures of these kinds.

Why fields? (in TQFT)

Why tensor product?
Fields are "things" that live on manifolds, which can be defined locally. 

Example: The assignment \( M \rightarrow C^\infty(M) \) is a field theory.

More relevant is the assignment

\[ M \rightarrow \text{set of covering spaces of } M. \]

In QFT, you can about functions defined on the space of fields.

In TQFT, pictures of the type we discussed, should give rise to maps

\[ \text{Fun}(F(\text{incoming part of })) \rightarrow \text{Fun}(F(\text{outgoing part })) \]

Note that \( F(C_1, \ldots, C_n) = F(C_1) \times \cdots \times F(C_n). \)

Now \[ \text{Fun}(F(C_1 \cup C_2)) = \text{Fun}(F(C_1) \times F(C_2)) \]
\[ = \text{Fun}(F(C_1)) \otimes \text{Fun}(F(C_2)). \]

This is the reason we care about tensor products!