Two different viewpoints on Category theory:

1) Categories formalize abstract properties of mathematical structures

2) Categories, and categories with extra structure, arise by means of "categorification".

Let us consider a table of analogies:

<table>
<thead>
<tr>
<th>Category theory</th>
<th>First point of view</th>
<th>Second point of view</th>
</tr>
</thead>
<tbody>
<tr>
<td>Category</td>
<td>a collection of math. obj. of a specified type</td>
<td>set</td>
</tr>
<tr>
<td>Morphisms</td>
<td>a map between math obj. preserving this structure</td>
<td></td>
</tr>
<tr>
<td>Functor</td>
<td>map between sets</td>
<td></td>
</tr>
<tr>
<td>natural transf.</td>
<td>related to formulating</td>
<td>Technical tools which make categorification useful</td>
</tr>
<tr>
<td>equiv. of catego.</td>
<td>universal properties</td>
<td></td>
</tr>
<tr>
<td>adj. pair of functors</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Examples / philosophy of categorification:

Roughly, categorification is a procedure of the following type:

1) Start with an alg. structure
   (sets, groups, abelian groups, rings, modules)

defined by a set with some maps, and some distinguished elements, satisfying certain equations.

(Example: a set is a set

A group: \( G \) is a set, plus a \( \cdot \) element

\[ G \times G \to G \]

\[ (g_1, g_2) \mapsto g_1 g_2 \]

"Structure maps" \[
\begin{align*}
g & \mapsto g^{-1} \\
g \in G & \quad \text{is invertible} \\
g & \mapsto g \\
1 & \in G \\
g^{-1} & \in G, g^{-1} = (g \cdot g^{-1})^{-1} \quad \text{inverse} \\
g \cdot g^{-1} & = 1 \quad \text{identity}
\end{align*}
\]

If we add the requirement \( gh = hg \quad \forall g, h \in G \)
we get an abelian group.
(2) Replace sets with categories

Replace maps of sets with functors between the appropriate categories.

Replace distinguish elements by distinguished objects.

Learn to be smart and find the correct replacement for equalities.

General Philosophy: Often, it is bad to require two functors to be equal. (This is too strong) and does

One should replace equalities by isomorphisms: But now you might get something too weak.

The tricky part is to find just the right kind of structure and properties here.

Example: (a) The category of the notion of a set is the notion of a category.

(1) How do you categorify the notion of a monoid?

(Result: monoidal category)
Comment about 2-D TQFT

Def of Peter May:

2-D TQFT is a Symm. monoidal functor

\[ 2 \text{-} \text{Cob} \rightarrow \text{Vect}_k \]

(Each of these categories is symmetric monoidal.)

- Symmetric monoidal categories
- Something which formalizes the properties of Vect_k
- Categorification of the notion of a monoidal commutative

Key point: In the sentence

"2-Cob is a Symm. mon. cat."

You have to think of this notion of an SMC as coming from categorification.

Reason: From the point of view of TQFT itself, and for the purpose of deriving the axioms of TQFT...
physical motivation, the notion of SMC is a red herring.

On the other hand, suppose you care about

\[ \text{Theorem:} \quad 2 \text{-D TQFT over a field } K \text{ is "the same thing"} \]
\[ \text{as a commutative Frob. alg. over } K. \]

Then you do need to talk about SMC. But (1) you should think of SMC from the point of view of TQFT.

Reason: To prove the theorem, one finds an explicit presentation of this category by "generators and relations."

Digression: How can we construct lin. rep. of groups?

Two different ways:

1. For every element of the group, write down the corresponding lin. transformation.

2. Specify the action of the suitable set of generators such that the relations work out.
Fun Example

$$\text{PSL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}(2, \mathbb{Z}) \mid ad - bc = 1 \right\} / \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$$

This group is generated by the images of
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and the relations are $h^2 = 1$ and $(gh)^3 = 1$ for some $h$.

something like this:

A lin. rep. of $\text{PSL}_2(\mathbb{Z})$ is a triple $(V, T, S)$

where $V$ is a v.s., $T, S : V \rightarrow V$ are lin. operators,

s.t. $S^2 = 1$, $(TS)^3 = 1$.

End of Discussion

Go back to 2-D TQFT:

2-cob can be described by gen. & relations.

a gen. for the objects is a circle

to which morphism should you take to generate the morphism?

Now, under $2$-cob $\rightarrow$ Vector objects of a TQFT form

our circle generator goes to a vector space $V$, and the
generator of morphism, induce some extra structure on \( V \).

satisfying certain axioms. The end result is precisely a Frobenius algebra.

Back to the question of categorifying a monoid?

\[ M = \text{a set} \]

\[ M \times M \to M \text{ a map} \]

\[ (a, b) \mapsto ab \]

\( 1 \in M \text{ distinct element} \)

Equations:

\[ \text{asso: } (ab)c = a(bc) \quad \forall a, b, c \in M \text{ (associativity)} \]

\[ 1 \cdot a = a = a \cdot 1 \quad \forall a \in M \]

Categorify:

\[ M = \text{a cat.} \]

\[ \otimes : M \times M \to M \text{ a functor} \]

\[ \text{obj: } (Y, X) \mapsto X \otimes Y \]

\[ \text{morph: } \left( \begin{array}{c}
X \\
Y
\end{array} \right) \mapsto X \otimes Y \quad \text{with} \quad \mathcal{l} : f \mapsto \mathcal{l}f \quad \text{and} \quad \mathcal{r} : f \mapsto \mathcal{r}f \]
Distinguished obj \( 1 \) (The unit object)

Question: What should replace the axioms \((ab)c = a(bc) \) & \( 1 \cdot a = a \cdot 1 \)?

Naive answer: Require

\[(x \otimes y) \otimes z = x \otimes (y \otimes z) \quad \forall x, y, z \in M \]

\[(f \circ g) \circ h = f \circ (g \circ h) \quad \forall f, g, h \in \text{Mor}(M)\]

\[1 \otimes x = x = x \otimes 1 \quad \forall \text{obj. } x \in M\]

\[\text{id}_1 \otimes f = f = f \otimes \text{id}_1 \quad \forall \text{morphisms in } M\]

and arrive at the definition of \textit{strictly associative} and \textit{strictly initial monoidal category}.

Example: \( C = \) any small cat.

Let \( M = \text{Fun}(C \to C) \)

\[
\begin{bmatrix}
\text{obj } M = \text{functors } C \to C \\
\text{morphisms in } M = \text{natural transformations} \\
\times = \text{composition of functors} \\
1 = \text{identity functor } C \to C
\end{bmatrix}
\]
In many other situations, strict associativity is too strong to be useful.

E.g.: \((\text{Vect}_C, \otimes, \mathbb{C})\)

Then,

\[
\text{U} \otimes (\text{V} \otimes \text{W}) \not\cong U \otimes (\text{V} \otimes \text{W})
\]

Instead, there's a natural isomorphism

\[
(\text{U} \otimes \text{V}) \otimes \text{W} \cong U \otimes (\text{V} \otimes \text{W})
\]

Stupid answer:

1. \((X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)\)

2. \(\text{I} \otimes X \cong X \cong X \otimes \text{I} \)

Little less stupid:

Require 1 and 2 to be functorial in \(X, Y, Z\) (resp. \(X\))
This means that

\[ \lambda_{x,y,z} : (x \otimes y) \otimes z \rightarrow x \otimes (y \otimes z) \]

\[ (f \circ g) \circ h \]
\[ \downarrow \]
\[ f \circ (g \circ h) \]

\[ \lambda_{x',y',z'} : (x' \otimes y') \otimes z' \rightarrow x' \otimes (y' \otimes z') \]

But it's NOT ENOUGH.

You need to impose further relations!

This will be the topic next time.

Example: C = any small cat. \[ \forall x \in X \Rightarrow x \otimes x \]