The Cosmological Constant

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Abstract

The Cosmological constant (Λ) was introduced by Einstein in the theory of General Relativity to obtain a static solution for the evolution of the universe. However, since the universe is expanding, and current theories estimate extremely small values for the upper bound of the constant ($\sim 10^{-33}s^{-2}$), most of the work done in the field of relativity has been done assuming $\Lambda = 0$. Specifically, the bending of light rays in a Schwarzchild field and precession of the perihelion of Mercury have not been solved with a non-zero Λ. In this paper we investigate the effects of non-zero Λ on both these phenomena. Given empirical results, this also allows us to estimate an upper bound for Λ. Additionally, we clarify the physical meaning of Λ by investigating the above solutions with values of Λ much larger than realistic ones.

1. Introduction

After deriving the gravitational field equations for space-time, Einstein solved them in the context of the whole universe and obtained a solution which described an expanding universe. The equations did not yield a static solution. However, Einstein firmly believed in a static universe, and hence proceeded to introduce within his field equations an extra term in order to obtain a static solution. The coefficient attached to this extra term came to be known as the Cosmological constant (generally written as Λ). Although it was found later that the universe is expanding, due to conflicting results from quantum mechanics and theories in cosmology, it is not clear whether or not the constant is zero.

One of the first predictions that Einstein made based on his work in general relativity was the bending of a light ray in a gravitational field. This prediction was later validated for light rays in the gravitational field of the sun. A more significant achievement of the theory of relativity was the computation of the precession of perihelion for the planet Mercury. Until then, physicists had not been able to account for a shift of approximately $43''$ per century of the precession, after making all the possible corrections due to other objects in the solar system. The significance of this computation was that not only did the theory of relativity account for this correction, but the computed value agreed with the discrepancy to within the uncertainty of $0.1''$ per century associated with the observation ([2],406). However, neither of these computations have been done assuming a non-zero Λ in Einstein's equations. This paper computes the same with the cosmological constant as a non-zero parameter in the field equations. Given the empirical results, such a computation also enables us determine an upper bound for the value of Λ. More importantly, this helps obtain a more conceptual understanding of Λ in the field equations.

We begin with an introduction to the various tools and concepts that we will use in our computations. We assume that space-time is Riemannian (section 2.1) and proceed to solve Einstein's equations to compute the characteristics of this Riemann space-time in a spherically symmetric, static case (Schwarzchild solution). The goal of section 2 (Preliminaries) is to arrive at the line element from the Schwarzchild solution (in empty space), assuming $\Lambda = 0$. Section 3 (The Cosmological Constant) introduces the constant in the field equations mathematically, and derives the new line element assuming a non-zero Λ. Finally, in section 4 (Consequences of the modified line element within the solar system), we shall use this line element to solve the field equations in the context of the solar system.

2. Preliminaries

In this section, we present an introduction to the various mathematical and physical structures and notations used in general relativity. This section draws heavily from chapters 1,2,3,5 and 6 in [1].

2.1 Tensors

In general relativity, we are often required to work in different coordinate frames. Hence, we would like to work with mathematical structures that enable invariance of forms under coordinate transformations. Tensors are structures which follow such transformation laws.

Since we shall always be dealing with a four-dimensional space, we shall label a vector quantity A as A^α (where the index α runs from 0 to 3). Given a change in coordinate system from x^α to $\bar{x}^\alpha$, we find that the differentials dx^α transform in the following manner:

$$d\bar{x}^\alpha = \sum_{\beta=0}^{3} \frac{\partial \bar{x}^\alpha}{\partial x^\beta} dx^\beta. \tag{2.1}$$

Such a transformation is known as a contravariant transformation. More generally, a vector ξ^β transforms *contravariantly* if it obeys the following law of transformation:

$$\bar{\xi}^\alpha = \sum_{\beta=0}^{3} \frac{\partial \bar{x}^\alpha}{\partial x^\beta} \xi^\beta. \tag{2.2}$$

Similarly, a vector η_β transforms *covariantly* if it obeys the following law of transformation:

$$\bar{\eta}_\alpha = \sum_{\beta=0}^{3} \frac{\partial x^\beta}{\partial \bar{x}^\alpha} \eta_\beta. \tag{2.3}$$

By convention, we denote a contravariant vector by writing its index as a superscript, and a covariant vector by writing its index as a subscript. Further, in writing equations (as above) where a summation is performed over an index, we notice that the index always occurs twice. Hence, we use the Einstein summation convention ([1],20) where the summation sign is suppressed (but is implicitly present) when an index appears twice in a term. Therefore, using this convention, the covariant transformation equation is written as:

$$\bar{\eta}_\alpha = \frac{\partial x^\beta}{\partial \bar{x}^\alpha} \eta_\beta \tag{2.4}$$

where the repeated appearance of the index β in the right hand side indicates that the term is summed over the index β. Now, consider a form with several such vectors, or a multilinear form P:

$$P = (T^{i_1 i_2 \cdots i_a}_{j_1 j_2 \cdots j_b})(\xi^{j_1}_{(1)} \xi^{j_1}_{(2)} \cdots \xi^{j_b}_{(b)})(\eta^{(1)}_{i_1} \eta^{(2)}_{i_2} \cdots \eta^{(a)}_{i_a}) \tag{2.5}$$

where $\xi^{j_1}_{(1)} = j_1$th component of an arbitrary contravariant vector labeled (1), etc.; $\eta^{(1)}_{i_1} = i_1$th component of an arbitrary covariant vector labeled (1), etc.; and where $T^{i_1 i_2 \cdots i_a}_{j_1 j_2 \cdots j_b}$ is a set of n^{a+b} elements with a upper and b lower indices which will be called contravariant and covariant, respectively. By definition, we say that the quantities $T^{i_1 i_2 \cdots i_a}_{j_1 j_2 \cdots j_b}$ form the components of a tensor if, for an arbitrary change of coordinates under which the vectors ξ and η transform according to aforementioned laws, they transform in such a way that P remains unchanged (is a scalar). $T^{i_1 i_2 \cdots i_a}_{j_1 j_2 \cdots j_b}$ is called a tensor of rank $(a + b)$ with a contravariant indices and b covariant indices.

It is worth mentioning some important properties of tensors. The sum, difference and product of two tensors is a tensor. A tensor of rank 0 is a scalar, of rank 1 is a vector. If a tensor is zero in one coordinate system, then it is zero in *all* coordinate systems. Finally, we mention a process by which new tensors can be created from given ones. Starting with a tensor $T^{i_1 i_2 \cdots i_a}_{j_1 j_2 \cdots j_b}$ of rank $(a + b)$, we set $i_a = j_b = \sigma$. This gives us $T^{i_1 i_2 \cdots i_{a-1}\sigma}_{j_1 j_2 \cdots j_{b-1}\sigma}$, which by our summation notation simply translates to a sum of all the terms over the index σ, giving us a new tensor, of rank $(a + b) - 2$ (simply due to the fact that there are only $(a + b) - 2$ indices now). This process is known as contraction.

We shall assume that space-time is a Riemann space ([1],17), that is a space in which the 'distance' between two points is given by

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu \tag{2.6}$$

where the coefficient matrix $g_{\mu\nu}$ forms a covariant tensor known as the metric tensor. It is this quantity that characterizes the behaviour of an object in space-time. Note that the metric tensor is symmetric in its indices ($g_{\mu\nu} = g_{\nu\mu}$).

2.2 Christoffel Symbols

On a flat surface, the translation of a vector from one point to another does not change the direction of the vector. However, this is not true in general. If we denote by δA_γ the change produced in the vector A_γ by an infinitesimal parallel transport dx^β, we find that δA_γ must be of the general form

$$\delta A_\gamma = -\left\{ {\alpha \atop \beta\,\gamma} \right\} A_\alpha dx^\beta \tag{2.7}$$

where the $4 \times 4 \times 4$-component object $\{ {\alpha \atop \beta\gamma} \}$ is some function of position. It turns out that $\{ {\alpha \atop \beta\gamma} \}$ is related to the derivatives of the metric tensor (of the surface) ([1],43):

$$\left\{ {\alpha \atop \beta\,\gamma} \right\} = \frac{1}{2} g^{\alpha\mu} \left(\frac{\partial g_{\beta\mu}}{\partial x^\gamma} + \frac{\partial g_{\gamma\mu}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\mu} \right) \tag{2.8}$$

where $g_{\beta\gamma}$ is the metric tensor and $g^{\beta\gamma}$ is the contravariant metric tensor (can be thought of as the matrix inverse of $g_{\beta\gamma}$). The coefficients $\{ {\alpha \atop \beta\gamma} \}$ are known as Christoffel symbols. These will be used extensively in later sections.

2.3 Geodesics

In ordinary Euclidean space, the definition of a straight line follows our intuition; however, we are forced to look deeper into the definition of a 'straight line' in a Riemann Space. We find that a straight line is characterized by the property that an arbitrary tangent vector along it remains parallel to itself when displaced along the 'line'. Such a generalized straight line is a geodesic. Consider a curve $x^\mu(q)$ parametrized by q, and $\xi^\mu(q)$, an arbitrary tangent vector to the curve, then recalling the definition of the Christoffel symbols $\{ {\alpha \atop \beta\gamma} \}$, we find that the equation

$$\frac{d\xi^\mu}{dq} + \left\{ {\mu \atop \alpha\,\beta} \right\} \frac{dx^\alpha}{dq} \xi^\beta = 0 \tag{2.9}$$

expresses mathematically the fact that $\xi^\mu(q)$ remains parallel to itself along the curve $x^\mu(q)$. If we were to take a general tangent vector of the form $\lambda(q)\frac{dx^\mu}{dq}$, where $\lambda(q)$ is an arbitrary function of q, replacing $\xi^\mu(q)$ in the above equation by $\lambda(q)\frac{dx^\mu}{dq}$, and making the substitution $dp = \frac{dq}{\lambda(q)}$, we get the equation of a geodesic in "normal form":

$$\frac{d^2x^\mu}{dp^2} + \left\{ {\mu \atop \alpha\,\beta} \right\} \frac{dx^\alpha}{dp}\frac{dx^\beta}{dp} = 0. \tag{2.10}$$

2.4 The Riemann Curvature Tensor

The Riemann curvature tensor gives us information about the curvature of space-time. Loosely speaking, the curvature tensor is obtained by observing the change produced in the direction of a vector as it is displaced along the surface parallel to it. Given a contravariant vector field $\xi^\alpha(x^\mu)$ defined on the space, let us then study how ξ^α varies when moved by an infinitesimal distance dx^μ. To do this, we need to compare the vectors $\xi^\alpha(x^\mu + dx^\mu)$ and $\xi^{\alpha *}(x^\mu + dx^\mu)$ where $\xi^{\alpha *}(x^\mu + dx^\mu)$ is the (parallel) transport of the vector $\xi^\alpha(x^\mu)$ to the point $(x^\mu + dx^\mu)$. Using Taylor series expansion, we have

$$\xi^\alpha(x^\mu + dx^\mu) = \xi^\alpha(x^\mu) + \frac{\partial \xi^\alpha}{\partial x^\beta} dx^\beta + O(dx^\beta)^2. \tag{2.11}$$

By the definition of Christoffel symbols,

$$\xi^{\alpha *}(x^\mu + dx^\mu) = \xi^\alpha(x^\mu) - \left\{ {\alpha \atop \beta\,\gamma} \right\} \xi^\gamma dx^\beta. \tag{2.12}$$

Therefore, the difference between the vectors is

$$\xi^\alpha(x^\mu + dx^\mu) - \xi^{\alpha *}(x^\mu + dx^\mu) = \left[\frac{\partial \xi^\alpha}{\partial x^\beta} + \left\{ {\alpha \atop \beta\,\gamma} \right\} \xi^\gamma\right] dx^\beta + O(dx^\beta)^2. \tag{2.13}$$

The expression $\left[\frac{\partial \xi^\alpha}{\partial x^\beta} + \left\{ {\alpha \atop \beta\gamma} \right\} \xi^\gamma\right]$ above (which is a tensor) is known as the covariant derivative of the contravariant vector ξ^α ([1],67). Here, we introduce some new notation:

$$\xi^\alpha_{|\beta} \equiv \frac{\partial \xi^\alpha}{\partial x^\beta} \qquad\qquad \xi^\alpha_{||\beta} \equiv \left[\frac{\partial \xi^\alpha}{\partial x^\beta} + \left\{ {\alpha \atop \beta\,\gamma} \right\} \xi^\gamma\right] \tag{2.14}$$

which gives us the relationship

$$\xi^\alpha_{||\beta} = \xi^\alpha_{|\beta} + \left\{ {\alpha \atop \beta\,\gamma} \right\} \xi^\gamma. \tag{2.15}$$

We similarly define the covariant derivative of a covariant vector η_α as

$$\eta_{\alpha||\beta} = \eta_{\alpha|\beta} - \left\{ {\gamma \atop \alpha\,\beta} \right\} \eta_\gamma. \tag{2.16}$$

We now define

$$\xi^\alpha_{||\beta||\gamma} - \xi^\alpha_{||\gamma||\beta} = R^\alpha_{\eta\beta\gamma}\xi^\eta \tag{2.17}$$

where ξ^α is an arbitrary contravariant vector defined over the surface. The tensor $R^\alpha_{\eta\beta\gamma}$ above is known as the Riemann curvature tensor ([1],149). In terms of the metric tensor, the curvature tensor turns out to be:

$$R^\alpha_{\eta\beta\gamma} = \left\{ {\alpha \atop \eta\,\beta} \right\}_{|\gamma} - \left\{ {\alpha \atop \eta\,\gamma} \right\}_{|\beta} + \left\{ {\alpha \atop \tau\,\gamma} \right\}\left\{ {\tau \atop \eta\,\beta} \right\} - \left\{ {\alpha \atop \tau\,\beta} \right\}\left\{ {\tau \atop \eta\,\gamma} \right\}. \tag{2.18}$$

We call a space *flat* if its Riemann tensor vanishes. We find, for example, that the Lorentz metric in special relativity

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.19}$$

results in a zero Riemann tensor; that is to say, a Lorentz metric defines a flat space.

We further define two new quantities: the Ricci tensor ($R_{\beta\gamma}$), obtained by contracting (section 2.1) two of the indices of the Riemann tensor, and the Ricci scalar (R), obtained by contracting all four indices of the Riemann tensor. The Ricci tensor can be thought of as a weaker form of the Riemann Tensor, while the Ricci scalar is simply the average curvature of space-time.

2.5 The Energy-Momentum Tensor

Poisson's equation relates the behaviour of a potential φ in space to its matter density ρ ($\nabla^2\varphi = \rho$). However, in the theory of relativity, such a description is incomplete due to the equivalence of matter and energy as shown by Einstein ($E = mc^2$). Hence, we need a more general quantity which describes the energy content and flow of space, which includes matter, radiant energy, elastic energy etc. This is done using a rank 2 tensor known as the energy-momentum tensor ($T^{\mu\nu}$) ([1],330).

2.6 The Gravitational Field Equations

Following his insight into the inextricable link between the curvature of space-time and the distribution of matter in space, Einstein arrived at the conclusion that the presence of matter curves space-time and this curvature defines the trajectory of matter. Theoretically, this gives the following general form for the gravitational field equation:

$$(Tensor\ representing\ geometry\ of\ space) = (Tensor\ representing\ energy\ content\ of\ space)$$

which translates mathematically to setting the Ricci tensor and the matter tensor equal to one another. However, we find that the Ricci tensor has non-zero divergence. Hence, we compute the divergenceless form of the Ricci tensor, known as the Einstein tensor ($G^{\mu\nu}$) and set that equal to the matter tensor, to obtain Einstein's gravitational field equations:

$$G^{\mu\nu}(= R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R) = T^{\mu\nu}. \tag{2.20}$$

2.7 The Schwarzchild solution

We now solve Einstein's equations to obtain the metric for the simplest case, in the matter-free field of a spherically symmetric, static central body. This is known as a Schwarzchild field. Due to symmetry, the metric tensor can be assumed to be of the form:

$$g_{\alpha\beta} = \begin{pmatrix} e^{\nu} & 0 & 0 & 0 \\ 0 & -e^{\lambda} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{pmatrix} \tag{2.21}$$

where ν and λ are functions of r. This gives us a line element of the form:

$$ds^2 = e^{\nu}c^2dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2(\theta)d\varphi^2). \tag{2.22}$$

We then compute all the Christoffel symbols in terms of $\nu(r)$ and $\lambda(r)$ ([1],189). Now, using $R_{\alpha\beta} = 0$ (for matter-free space), we solve for ν and λ to obtain:

$$e^{\nu(r)} = e^{-\lambda(r)} = \left(1 - \frac{2m}{r}\right) \tag{2.23}$$

which gives us the metric:

$$g_{\alpha\beta} = \begin{pmatrix} \left(1 - \frac{2m}{r}\right) & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{2m}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{pmatrix}. \tag{2.24}$$

From the above metric, we obtain the Schwarzchild line element:

$$ds^2 = \left(1 - \frac{2m}{r}\right)c^2dt^2 - \left(1 - \frac{2m}{r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2(\theta)d\varphi^2) \tag{2.25}$$

where $m = \frac{\kappa M}{c^2}$ is a constant of integration, $\kappa = 6.67 \times 10^{-8}$ dyne-cm^2/g^2 is the gravitational constant, M is the mass of the central, spherically symmetric body, and c the speed of light.

3. The Cosmological Constant

3.1 Introduction of the constant in Einstein's equations

We now turn to the original problem of the Cosmological constant. It was seen that the Einstein tensor ($G^{\mu\nu}$) was obtained as a divergenceless form of the Ricci tensor ($R^{\mu\nu}$). However, this is only a particular form; the most general divergenceless second-rank tensor ($B^{\mu\nu}$) that can be constructed from the Ricci tensor is a linear combination of the Einstein tensor and the metric tensor:

$$B^{\mu\nu} = G^{\mu\nu} + \Lambda g^{\mu\nu} \tag{3.1}$$

where Λ is an arbitrary constant. Hence the gravitational equations become:

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = CT^{\nu\mu} \tag{3.2}$$

where C is a constant. By the postulate that the field equations possess the classical equations as a limit case, we can compute the constant C ([1],347):

$$C = -\frac{8\pi\kappa}{c^2} \tag{3.3}$$

where κ is the gravitational constant and c the speed of light. Hence we obtain the general form of Einstein's field equations:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu} = -\frac{8\pi\kappa}{c^2}T^{\nu\mu}. \tag{3.4}$$

3.2 The modified Schwarzchild solution

Our first step after deriving the general form of Einstein's equations is to look for the modified line element in matter-free space. As before, we assume a metric tensor of the form (2.26) due to static, spherical symmetry, and proceed to compute all the Christoffel symbols. Finally, using $R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu} = 0$, we arrive at the following solution ([1],398):

$$e^{\nu(r)} = e^{-\lambda(r)} = \Big(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big). \tag{3.5}$$

Comparing this to the previous solution (2.23), we find that the only change is the addition of an extra $-\frac{1}{3}\Lambda r^2$ term, giving us the line element:

$$ds^2 = \Big(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)c^2dt^2 - \Big(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)^{-1}dr^2 - r^2(d\theta^2 + \sin^2(\theta)d\varphi^2). \tag{3.6}$$

We shall now discuss the effects of this new line element in the context of our solar system.

4. Consequences of the modified line element within the solar system

In this section, we look at how the presence of the cosmological constant in Einstein's equations affects different trajectories of objects in the solar system. We base our approach to the problem on the assumption that the trajectory of any object in space-time is always a geodesic. This is the generalization of Newton's laws which state that an object moves with constant velocity in a straight line when there is no external force acting on it; since a straight line in Euclidean space corresponds to a geodesic on a manifold in general. Hence, to obtain the equations of motion of an object, we simply solve the equations of geodesics, as given in (2.10).

4.1 The trajectory of a light ray in a Schwarzchild field

In special relativity, the path of a light ray in space-time is given by $c^2dt^2 = dx^2 + dy^2 + dz^2$. That is, the path of a light ray is characterized by the null line element $ds^2 = c^2dt^2 - dx^2 + dy^2 + dz^2 = 0$. We assume that the same holds in general relativity. Hence the path of a light ray in a Schwarzchild field (static, spherically symmetric) is characterized by the null line element in (3.6),

$$ds^2 = \Big(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)c^2dt^2 - \Big(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)^{-1} dr^2 - r^2(d\theta^2 + \sin^2(\theta)d\varphi^2) = 0. \tag{4.1}$$

Now, to find the trajectory of a light ray, we use Lagrange's method to compute the Christoffel symbols, and consequently the equations of geodesics (see Appendix A). This leads us to the following equation of motion:

$$u'' + u = 3mu^2 \tag{4.2}$$

where $u(\varphi) = 1/r(\varphi)$, and m is the same as before (2.25). However, this equation of the trajectory is the same as in the original Schwarzchild solution ([1],215), giving us the first order solution to the trajectory:

$$u = \frac{1}{r_0}\cos\varphi - \frac{\epsilon}{3r_0^2}\cos^2\varphi + \frac{2\epsilon}{3r_0^2} \tag{4.3}$$

where r_0 is the closest approach of the light ray to the sun, and $\epsilon = 2m$. This gives us a total deflection of

$$\Delta = \frac{4\kappa M}{c^2 r_0} \tag{4.4}$$

where κ and c are the same as before (2.25) and M is the mass of the sun. *Hence we find that the* *presence of Λ in Einstein's equations and consequently in the line element of a Schwarzchild field has no effect on the trajectory of a light ray through such a field.*

4.2 The precession of the perihelion of Mercury around the Sun

It has been known for a long time that the perihelion of the planet Mercury precesses around the sun with time. This is due primarily to the presence of the other planets in the solar system, and hence the angle of precession was computed using Newtonian mechanics. However, the correction obtained theoretically fell short of the observed value by approximately $43''$ per century. This discrepancy was finally accounted for by the theory of general relativity, after Einstein computed the precession to within the uncertainty of the observed value from his field equations. However, it is unclear whether the presence of the cosmological constant in Einstein's field equations would alter the predicted precession. Any such alteration would have to be within the uncertainty of observation, and would enable us to obtain an upper bound for Λ.

Hence, we proceed to solve the equations of geodesics with the modified line element as in (3.6) (see Appendix B), and obtain the following equation of motion for Mercury around the sun:

$$u'' + u = \frac{m}{h^2} + 3mu^2 - \frac{2}{3h^2}\frac{\Lambda}{u^3} \tag{4.5}$$

where $u(\varphi) = \frac{1}{r(\varphi)}$, m is the same as in (2.25), and h the same as in $(B.7)$. It is instructive to compare the above to the equations of motion obtained from Newtonian mechanics and from Einstein's correction without the constant Λ (see Appendix C). However, the above equation has no closed form solution. Hence, we approach the problem numerically. We find that to first order approximation, the solution to the above equation is of the form

$$u(\varphi) = A + B\cos(\varphi - \varepsilon\varphi) \tag{4.6}$$

where A and B are constants as defined in appendix C. Comparing this to the orbit for a complete ellipse as obtained from Newtonian mechanics:

$$u(\varphi) = A + B\cos(\varphi), \tag{4.7}$$

we find that the difference is given by

$$B\cos(\varphi - \varepsilon\varphi) - B\cos(\varphi)$$

$$= B\cos(\varphi)\cos(\varepsilon\varphi) - B\sin(\varphi)\sin(\varepsilon\varphi) - B\cos(\varphi)$$

which, upon assuming that ε is a small number, reduces to

$$B\varepsilon\varphi\sin(\varphi). \tag{4.8}$$

We plot the difference obtained in equation (4.8) as a function of φ, and focus our attention on the maximas of the graph which occur at the points where $\sin(\varphi) = 1$. A line through these points would have the equation $B\varepsilon\varphi$ obtained from (4.8) by substituting $\sin(\varphi) = 1$. This is a straight line with slope $B\varepsilon$. Hence, knowing the slope of this line allows us to compute ε which gives us the precession associated with the orbit. This method of computation is detailed in Appendix D.

The difference between the precession obtained by assuming a zero Λ and the observed precession is smaller than the uncertainty associated with the observed value. Therefore any shift in the perihelion due to Λ must also be within the uncertainty of observation. This allows us to compute an upper bound for Λ. Given the observed value of $43.1 \pm 0.1''$ per century ([2],406) for the precession of perihelion of Mercury, we compute an upper bound for Λ given by:

$$|\Lambda| \leq 7 \times 10^{-25} s^{-2}. \tag{4.9}$$

Although this upper bound is larger than the upper bound of $10^{-33} s^{-2}$ obtained from cosmology, it is still very small, and in the range of expected values. Further, by varying Λ, we obtain as a function of Λ the precession of the orbit, shown in figure 1. As shown in the graph, the uncertainties associated with observation give us the aforementioned upper bound on Λ.

Figure 1. Graph of precession of the orbit of Mercury as a function of $\log(\Lambda)$ (Λ is in units of c^2). The three horizontal lines represent the observed precession (in the middle) along with upper and lower limits of uncertainty.

5. Conclusions

We found that the presence of Λ in the field equations does not affect the trajectory of a light ray in a Schwarzchild field. However, the introduction of Λ does affect the precession of the perihelion of Mercury. Since any such effect must be within the uncertainty of observation of the precession, this allows us to compute an upper bound for Λ. We found that $|\Lambda| \leq 7 \times 10^{-25} s^{-2}$. Although this value is larger than the upper bound of $\sim 10^{-33} s^{-2}$ predicted by various cosmological models, it is still in the same range of magnitudes as determined by the various models. We also found that in the neighbourhood of the computed upper bound, computations suggest a linear dependence of the precession on Λ. This may be true only in the regime of small Λ however, since several approximations made in our calculations assume a small Λ.

The above results confirm our expectations of Λ being a very small quantity. However, it is still unclear as to whether or not Λ is zero. Although the constant adds to the precession, it cannot be a simple intrinsic curvature of space-time, since that would affect the trajectory of a light ray as well. The above analysis demonstrates the power of this approach to understand Λ. Further analysis of the impact of Λ on other systems such as binary pulsars and neutron stars could help gain a better feel for the value and the physical meaning of the constant.

[**1**] R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity*, 2nd ed. (McGraw Hill, New York, 1975).

[**2**] H.C. Ohanian and R. Ruffini *Gravitation and Spacetime*, 2nd ed. (W.W. Norton & Company, New York, 1994).

[**3**] S. Weinberg, *Gravitation and Cosmology*, (John Wiley & Sons Inc., New York, 1972).

Appendix A: Solving for the trajectory of a light ray in a Schwarzchild field

Here we present the details of the computation leading to the equation of the trajectory of a light ray in a Schwarzchild field. Using the Euler-Lagrange formulation in (4.2) and the Euler-Lagrange equations of motion:

$$\frac{d}{ds}\Big(\frac{\partial F}{\partial \dot{x}^{\alpha}}\Big) = \frac{\partial F}{\partial x^{\alpha}} \tag{A.1}$$

for θ, φ, and t, we get:

$$\frac{d}{ds}(r^2\dot{\theta}^2) = r^2 \sin\theta\cos\theta\dot{\varphi}^2, \tag{A.2}$$

$$\frac{d}{ds}(r^2\sin^2\theta\dot{\varphi}) = 0, \tag{A.3}$$

$$\frac{d}{ds}\Big[(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)\dot{t}\Big] = 0. \tag{A.4}$$

We note that for a null geodesic, $ds^2 = 0$, so that from (3.6) we obtain:

$$0 = \Big(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)c^2\dot{t}^2 - \Big(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)^{-1}\dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2(\theta)\dot{\varphi}^2). \tag{A.5}$$

As in classical mechanics, the orbit remains in a plane, hence by an appropriate choice of axes, we can set $\theta = \frac{\pi}{2}$, and $\dot{\theta} = 0$, so that from $(A.2)$, it follows that for all s,

$$\theta = \frac{\pi}{2}. \tag{A.6}$$

Substituting this result in $(A.3)$ and integrating, we get:

$$r^2\dot{\varphi} = h \quad (= \text{const}). \tag{A.7}$$

Similarly, $(A.4)$ integrates to

$$\Big(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)\dot{t} = l \quad (= \text{const}). \tag{A.8}$$

Substituting $(A.6)$, $(A.7)$, and $(A.8)$ into $(A.5)$, we get:

$$0 = \Big(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)^{-1}c^2l^2 - \Big(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)^{-1}\dot{r}^2 - \frac{h^2}{r^2}. \tag{A.9}$$

To simplify the problem, we write r as a function of φ instead of a function of s. We denote differentiation with respect to φ by a prime, hence:

$$r' = \frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}}, \tag{A.10}$$

that is

$$\dot{r} = \dot{\varphi} r' = \frac{h}{r^2} r'. \tag{A.11}$$

The differential equation $(A.9)$ then becomes:

$$c^2 l^2 - \dot{r}^2 - \left(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\right)\frac{h^2}{r^2} = 0. \tag{A.12}$$

Introducing a new variable to simplify the equation:

$$u(\varphi) = \frac{1}{r(\varphi)}, \tag{A.13}$$

$(A.12)$ becomes:

$$c^2 l^2 - h^2 u'^2 - \left(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\right) h^2 u^2 = 0 \tag{A.14}$$

$$\Rightarrow \qquad c^2 l^2 - h^2 u'^2 - h^2 u^2 + 2mh^2 u^3 + \frac{1}{3}\Lambda h^2 = 0. \tag{A.15}$$

In order to solve the above, we differentiate with respect to φ to get:

$$-2h^2 u' u'' - 2h^2 u u' + 6mh^2 u^2 u' = 0 \tag{A.16}$$

$$\Rightarrow \qquad u'(-2h^2 u'' - 2h^2 u + 6mh^2 u^2) = 0. \tag{A.17}$$

Discarding the trivial solution $u' = 0$ and simplifying, we arrive at

$$u'' + u = 3mu^2 \tag{A.18}$$

which is the same as the solution of the original Schwarzchild problem.

Appendix B: Solving for the orbit of Mercury around the sun

In this appendix, we present the method used to obtain the equation of motion for the planet Mercury around the sun from the theory of relativity. Proceeding in a similar manner as in appendix A, we solve:

$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{x}^\alpha}\right) = \frac{\partial F}{\partial x^\alpha} \tag{B.1}$$

for θ, φ, and t. This gives us:

$$\frac{d}{ds}(r^2 \dot{\theta}^2) = r^2 \sin\theta \cos\theta \dot{\varphi}^2, \tag{B.2}$$

$$\frac{d}{ds}(r^2 \sin^2\theta\dot{\varphi}) = 0, \qquad (B.3)$$

$$\frac{d}{ds}\Big[(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)\dot{t}\Big] = 0. \qquad (B.4)$$

Dividing (3.6) by ds^2, we obtain:

$$1 = \Big(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)c^2\dot{t}^2 - \Big(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)^{-1}\dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2(\theta)\dot{\varphi}^2). \qquad (B.5)$$

As before, the orbit remains in a plane, hence by an appropriate choice of axes, we can set $\theta = \frac{\pi}{2}$, and $\dot{\theta} = 0$, so that from $(B.2)$, it follows that for all s,

$$\theta = \frac{\pi}{2}. \qquad (B.6)$$

Substituting this result in $(B.3)$ and integrating, we get:

$$r^2\dot{\varphi} = h \quad (= \text{const}). \qquad (B.7)$$

Similarly, $(B.4)$ integrates to

$$\Big(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)\dot{t} = l \quad (= \text{const}). \qquad (B.8)$$

Substituting $(B.6)$, $(B.7)$, and $(B.8)$ into $(B.5)$, we get:

$$1 = \Big(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)^{-1}c^2l^2 - \Big(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)^{-1}\dot{r}^2 - \frac{h^2}{r^2}. \qquad (B.9)$$

As with the trajectory of a light ray, to simplify the problem, we write r as a function of φ instead of a function of s. We denote differentiation with respect to φ by a prime, hence:

$$r' = \frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}}, \qquad (B.10)$$

that is

$$\dot{r} = \dot{\varphi}r' = \frac{h}{r^2}r'. \qquad (B.11)$$

The differential equation $(B.9)$ then becomes:

$$c^2l^2 - \dot{r}^2 - \Big(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)\frac{h^2}{r^2} = \Big(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big). \qquad (B.12)$$

Introducing a new variable to simplify the equation:

$$u(\varphi) = \frac{1}{r(\varphi)}, \tag{$B.13$}$$

$(B.12)$ becomes:

$$c^2l^2 - h^2u'^2 - \Big(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)h^2u^2 = \Big(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big) \tag{$B.14$}$$

$$\Rightarrow \qquad c^2l^2 - h^2u'^2 - h^2u^2 + 2mh^2u^3 + \frac{1}{3}\Lambda h^2 - 1 + 2mu + \frac{1}{3}\frac{\Lambda}{u^2} = 0. \tag{$B.15$}$$

In order to solve the above, we differentiate with respect to φ to get:

$$-2h^2u'u'' - 2h^2uu' + 6mh^2u^2u' + 2mu' - \frac{2}{3}\frac{\Lambda}{u^3}u' = 0 \tag{$B.16$}$$

$$\Rightarrow \qquad u'(-2h^2u'' - 2h^2u + 6mh^2u^2 + 2m - \frac{2}{3}\frac{\Lambda}{u^3}) = 0. \tag{$B.17$}$$

Discarding the trivial solution $u' = 0$ and simplifying, we arrive at

$$u'' + u = \frac{m}{h^2} + 3mu^2 - \frac{2}{3h^2}\frac{\Lambda}{u^3} \tag{$B.18$}$$

which is the same as (4.5).

Appendix C: Comparing the various solutions for the orbit of Mercury around the sun

In this appendix, we compare the different solutions obtained for the orbit of Mercury around the sun: from Newtonian mechanics, Einstein's correction, and the inclusion of Λ. The comparison illustrates very clearly the differences between the three solutions.

When solved using Newtonian mechanics, the equation governing the motion of the planet is given by:

$$u'' + u = \frac{1}{A} \tag{$C.1$}$$

where $u(\varphi) = \frac{1}{r(\varphi)}$. The solution to the above equation is a standard ellipse (figure 1), of the form:

$$u = \frac{1}{A} + B\cos(\varphi) \tag{$C.2$}$$

Figure 2. Newtonian orbit

where A is the semi-latus rectum as shown in the figure, and $B = \frac{e}{A}$ where e is the eccentricity of the orbit.

Einstein's correction to the orbit results in the following equation of motion:

$$u'' + u = \frac{1}{A} + 3mu^2 \tag{C.3}$$

where m is the same as before (2.25). We note that the correction is due to the $3mu^2$ term, which results in a precessive ellipse for the orbit (figure 2). The solution is of the form:

$$u = \frac{1}{A} + B\cos(\varphi - \varepsilon\varphi) + O(\varepsilon^2). \tag{C.4}$$

Figure 3. Einstein's correction/Introduction of Λ: orbit is a precessing ellipse with angle of precession $= \varepsilon$. Precession has been exaggerated, and shown units are arbitrary.

Ignoring the order ε^2 terms, we find that the orbit precesses by an angle of ε radians each revolution, which enables us to compute the angle of precession. For the planet Mercury, this gives us a precession of approximately $43.01''$ per century.

Finally, the introduction of Λ in Einstein's equations adds another term to the equation of motion which now becomes:

$$u'' + u = \frac{1}{A} + 3mu^2 - \frac{2}{3h^2}\frac{\Lambda}{u^3} \tag{C.5}$$

where h is a measure of the angular momentum, given in $(B.7)$. For small Λ, the solution to first order is still of the form $(C.4)$. Hence, the resultant orbit is similar to the previous one (figure 2). It is interesting to note that the modification due to Λ is proportional to r^3 while Einstein's correction is proportional to $\frac{1}{r^2}$.

Appendix D: Computation of the angle of precession using numerical methods

We found that assuming a small ε, the difference Δu between the solutions mentioned in section 4.2 is equal to $B\varepsilon\varphi\sin(\varphi)$. Upon solving the equations numerically, we plotted Δu vs. φ to obtain a graph similar to the one shown below.

Figure 4. Extracting the angle of precession from the numerical solution. In the above case, $\Lambda = 10^{-33}m^{-2}$ and $B\varepsilon\varphi\sin(\varphi)$ has been multiplied by a factor of 10^{12} for easier presentation.

From the graph above, the slope of the straight line shown is computed. We note that this slope is equal to $B\varepsilon$, which gives us the value of the precession ε.

The numerical solutions were generated in *Mathematica*† using a working precision of 25 decimal places. This corresponds to an agreement of 25 decimal places between the exact and numerical

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solutions for the Newtonian orbit after 10 revolutions.