

# RESEARCH STATEMENT

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## Introduction

My current research interests lie primarily in the field of algebraic geometry, although strongly influenced by mathematical physics. Most of my work [14, 15, 16] has focused on the study of geometric objects known as Higgs bundles. More recently, I have become interested in the relationship of these objects with the phenomenon of Mirror Symmetry, and hope to use Higgs bundles to answer some of the questions in this field.

## Higgs bundles

Higgs bundles were first introduced in a well-known paper by Hitchin [8]. Subsequently, the general theory behind these objects was developed by several people, most notably Corlette [5], Donaldson [7] and Simpson [19, 20]. Higgs bundles are extremely interesting for a variety of reasons; they come naturally endowed with many rich geometric structures. For instance, given any reductive group  $G$ , the space  $\mathcal{H}$  of  $G$ -Higgs bundles over a compact Kähler manifold  $X$  is a quasi-projective complex symplectic manifold, with a natural hyperkähler structure. It is the nonabelian analogue of the first cohomology of  $X$ , i.e.,  $\mathcal{H}$  may be interpreted as  $H^1(X, G)$ .  $\mathcal{H}$  also has a foliation by Lagrangian submanifolds, and provides an interesting example of a completely integrable system.

To motivate the definition of these objects, we recall a deep result following from the work of Narasimhan and Seshadri [17] and Atiyah and Bott [1], which identifies, under suitable conditions, holomorphic vector bundles (geometric objects) on a Riemann surface  $X$  with *local systems* (topological objects) on  $X$  with group  $U(r)$  (i.e., representations of the fundamental group of  $X$  into the unitary group  $U(r)$ ). In such a setup, Higgs bundles may be interpreted as the geometric objects that generalize the above correspondence when we replace  $U(r)$  with  $GL(r)$  (or more generally, with an arbitrary reductive group  $G$ ). More precisely, a *Higgs bundle* on a Riemann surface  $X$  is a pair  $(E, \theta)$ , where  $E$  is a holomorphic vector bundle (of rank  $r$  and degree  $d$ ) on  $X$  and  $\theta$ , often referred to as a Higgs field, is a holomorphic global section of  $\text{End } E \otimes \Omega_X^1$ . This definition is for  $GL(r)$  and an analogous definition for general  $G$  is easily given using principal bundles.

As in the case of vector bundles, it is possible to construct a moduli space parametrizing all stable\* Higgs bundles over  $X$ . We shall denote this moduli space by  $\mathcal{H}(r, d)$ .

## Early work

When  $r$  and  $d$  are coprime,  $\mathcal{H}(r, d)$  is known to be a smooth, non-compact, quasi-projective variety. From Deligne's work on Hodge theory, it follows that the cohomology of  $\mathcal{H}$  carries a natural Mixed Hodge Structure (MHS). A priori, since  $\mathcal{H}$  is non-compact, we expect some 'mixing' in this Hodge structure. However, using the results of Hausel and Thaddeus [10] and Markman [13], I showed that this Hodge structure is pure of expected weight [14]. This gives us a non-trivial example of a non-compact variety having pure Hodge structure. A simple consequence is that the natural map from the cohomology of any compactification of  $\mathcal{H}(r, d)$  surjects onto the cohomology of

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\*As in Geometric Invariant Theory, in order to ensure that the resulting space is Hausdorff, it is constructed using the open subset of stable bundles, as opposed to all bundles.

$\mathcal{H}(r, d)$ . A more interesting consequence is that this allows us to use the Weil conjectures to compute geometric information about  $\mathcal{H}(r, d)$ , as done by Harder and Narasimhan in the case of the moduli space of vector bundles [9], without the need to worry about cancellations in Grothendieck's formula.

### Work in Ph.D. thesis

In my thesis [15] I construct a birational map between moduli spaces of Higgs bundles of ranks differing by one, which allows us to study these spaces using induction on rank.

I begin by constructing moduli spaces of triples  $(E, \theta, \phi)$  over a Riemann surface  $X$ , where  $(E, \theta)$  is a Higgs bundle and  $\phi$  is a holomorphic section of  $E$  such that  $\theta(\phi) = 0$  i.e.,  $\phi$  is a 'section of the Higgs bundle'. Due to a choice of linearization in the GIT construction of these triples, the stability criteria for these objects depend on a real parameter  $\tau$ . My first result is a Hitchin-Kobayashi correspondence for these triples, which determines certain preferred metrics on such a triple.

1. *If a triple  $(E, \theta, \phi)$  is  $\tau$ -stable, then the  $\tau$ -vortex equation*

$$i\Lambda(F_H + [\theta, \theta^{*H}]) + \phi \otimes \phi^{*H} = \frac{\tau}{2}\mathbf{I}$$

*considered as an equation for the Hermitian metric  $H$  on  $E$  has a unique smooth solution. Conversely, if the above equation has a solution, then the triple  $(E, \theta, \phi)$  is either  $\tau$ -stable or a direct sum of  $\tau$ -stable triples.*

Here  $\Lambda$  is the adjoint of the Lefschetz operator and  $F_H$  is the curvature of the metric connection on  $E$ . This result also follows from the recent work of Bradlow, Garcia-Prada and Mundet i Riera [4], and is used to construct the birational equivalence. The moduli space  $\mathcal{L}_\tau$  of  $\tau$ -stable triples happens to be non-empty precisely when  $\tau \in [\frac{d}{r}, \frac{d}{r-1}]$ . I construct a master space consisting of  $\tau$ -stable triples for *all* values of  $\tau$ . This space carries a natural  $S^1$  action, whose moment map gives us a Morse function on the space, with level sets corresponding precisely to the moduli spaces of stable triples. Using this correspondence and the Morse flow on the master space, I obtain the birationality result.

2. *For all but finitely many values of  $\tau$  in the interval  $[\frac{d}{r}, \frac{d}{r-1}]$ , the spaces  $\mathcal{L}_\tau$  of  $\tau$ -stable triples are birational.*

The condition  $\theta(\phi) = 0$  becomes important now, as it ensures that the spaces  $\mathcal{L}_\tau$  at the ends of the interval above are closely related to the moduli spaces of Higgs bundles of ranks  $r$  and  $r - 1$ . In this manner an explicit geometric relationship between these moduli spaces is established.

Similar results in the case of vector bundles were obtained by Bradlow, Daskalopoulos and Wentworth [3]. They obtain a birational map between the moduli spaces of vector bundles of rank  $r$  and  $r - 1$  over a Riemann surface. Their result was also obtained in rank 2 by Thaddeus, who used it in his well-known paper [22] to compute the Poincaré polynomial of the moduli space of vector bundles of rank 2 over a Riemann surface.

### Current work and future plans

*Higgs bundles on curves  $X/\mathbb{F}_q$ .* One of my current projects is to study Higgs bundles on curves  $X/\mathbb{F}_q$ . The study of the moduli of vector bundles on curves over  $\mathbb{F}_q$  proved useful in understanding the corresponding moduli space on  $X/\mathbb{C}$ . Harder and Narasimhan use this approach in [9] to prove the Siegel formula for vector bundles on  $X/\mathbb{F}_q$ . This formula relates the number of vector bundles on  $X/\mathbb{F}_q$  to the Zeta function of the curve  $X$ , allowing one to count the number of stable vector bundles

on  $X/\mathbb{F}_q$ . Using this information and the Weil conjectures, they were able to compute the Poincaré polynomial of the moduli space of vector bundles on  $X/\mathbb{C}$ . They also showed that the action of the discrete subgroup  $\Gamma$  (of line bundles of order  $r$ ) of the Jacobian on the  $l$ -adic cohomology of the moduli space on  $X/\mathbb{F}_q$  is trivial, which was subsequently used to conclude a similar result about the action of  $\Gamma$  on  $H^*(\mathcal{M}(r, d), \mathbb{Q})$  on  $X/\mathbb{C}$ . The idea behind this current project is to do the same with Higgs bundles. In [16] (joint work with V. Baranovsky), we are working on a Siegel formula for Higgs bundles, which would allow us to count the number of stable Higgs bundles on a curve over  $\mathbb{F}_q$ . Similarly, understanding the action of  $\Gamma$  on the rational cohomology of the moduli space of Higgs bundles would be useful because this is closely related to the cohomology of the mirror dual (see below) of this moduli space.

When working with curves over  $\mathbb{C}$ , Higgs bundles correspond to representations of (a central extension of) the fundamental group of the curve into  $GL(r)$ . Another reason to investigate Higgs bundles on curves over  $\mathbb{F}_q$  is to see if there exists a meaningful version of this correspondence on curves in finite characteristic.

*Mirror Symmetry.* Mirror Symmetry is a discovery made many years ago in string theory as a duality between families of 3-dimensional Calabi-Yau manifolds. A *Calabi-Yau* manifold is a complex Kähler manifold with trivial canonical bundle. Mathematically, given a family of such manifolds, Mirror Symmetry essentially conjectures the existence of a ‘mirror’ family of Calabi-Yaus with certain duality relations between the deformations of their Kähler and complex structures. Several different formulations of these relations have been suggested, and few examples are known.

One formulation is due to Strominger-Yau-Zaslow (SYZ) [21]. In this approach, two Calabi-Yaus are said to be mirror partners if they both fibre over the same real base such that the fibres over regular values are special Lagrangian tori which are dual to each other. An observation first made by Hitchin, and proved recently by Hausel and Thaddeus [11], is that moduli spaces of  $SL(r)$ -Higgs bundles and  $PSL(r)$ -Higgs bundles are SYZ mirror partners. A different formulation using homological algebra was given by Kontsevich [12], where two Calabi-Yaus are said to be Mirror dual if the derived category constructed from the Fukaya category of the first is equivalent to the derived category of coherent sheaves on the other. Recently Seidel [18] proved this formulation of the conjecture for a quartic surface in  $\mathbb{P}^3$ . However, in both cases, the geometry behind the construction of the dual is not very well understood.

My goal here is to understand the geometry behind this apparent duality. I believe that the rich structure in the moduli space of Higgs bundles, a Calabi-Yau, can provide a deeper understanding of this mirror phenomenon. To this end, I plan to use the work in my thesis to compute Hodge numbers of moduli spaces of Higgs bundles (since the SYZ formulation of Mirror symmetry essentially reduces to showing a numerical relationship between these numbers). I am also learning about Fukaya categories and Picard-Lefschetz theory and wish to investigate these objects in the context of the moduli space of Higgs bundles. Finally, I am an active participant in a weekly seminar on Mirror Symmetry run jointly under the auspices of the Max Planck Institute and the University of Leipzig, and am currently working in collaboration with M. Schwarz on this subject.

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