Variation and Share-Weighted Variation Swaps on Time-Changed Lévy Processes

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Abstract

For a family of functions G, we define the *G-variation*, which generalizes power variation; G-variation *swaps*, which pay the G-variation of the returns on an underlying share price F; and *share-weighted* G-variation swaps, which pay the integral of F with respect to G-variation. For instance, the case $G(x) = x^2$ reduces these notions to, respectively, quadratic variation, variance swaps, and gamma swaps.

We prove that a multiple of a log contract prices a G-variation swap, and a multiple of an $F \log F$ contract prices a share-weighted G-variation swap, under arbitrary exponential Lévy dynamics, stochastically time-changed by an arbitrary continuous clock having arbitrary correlation with the Lévy driver, under integrability conditions.

We solve for the multipliers, which depend only on the Lévy process, not on the clock.

In the case of quadratic G and continuity of the underlying paths, each valuation multiplier is 2, recovering the standard no-jump variance and gamma swap pricing results. In the presence of jump risk, however, we show that the valuation multiplier differs from 2, in a way that relates (positively or negatively, depending on the specified G) to the Lévy measure's skewness.

In three directions this work extends Carr-Lee-Wu, which priced only variance swaps. First, we generalize from quadratic variation to G-variation; second, we solve for not only unweighted but also share-weighted payoffs; and third, we apply these tools to analyze and minimize the risk in a family of *hedging* strategies for G-variation.

1 Introduction

Assuming continuous underlying price paths, the standard theory (Neuberger [16], Dupire [10], Carr-Madan [5], Derman et al [8]) finds that a variance swap has the same value as two log contracts on the underlying F, and implies ([5],[15]) that a gamma swap has the same value as two $F \log F$ contracts. The former valuation result has become a standard reference point for volatility traders, and underpins the well-known volatility indicators VIX, VXN, and VSTOXX. However,

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empirical studies of equity markets reject the continuity assumption. Carr-Geman-Madan-Yor [3] and Broadie-Jain [1] do allow jump risk, but instead of using the information in log contracts, they take the "parametric" approach of imposing fully specified models (including CGMY in [3], Merton, Heston, Bates in [1]), and pricing variance contracts in terms of the model parameters.

Returning to the log contract approach, Carr-Lee-Wu [4] generalized Neuberger and Dupire by incorporating jump risk, in the form of "arbitrary time-changed exponential Lévy processes (under integrability conditions), where the background Lévy process may have jumps of arbitrary distribution, and where the stochastic time-change, an arbitrary continuous clock, may have arbitrary dependence or correlation with the Lévy process. This allows stochastic volatility, stochastic jump intensity, volatility clustering, and leverage effects." They found that a variance swap still admits pricing in terms of a log contract times a multiplier (not necessarily 2) which depends on the driving Lévy process but *not* on the arbitrary stochastic clock. This "semi-parametric" approach is thereby robust to misspecification or miscalibration of stochastic volatility or jump intensity.

In the same general semi-parametric framework of time-changed Lévy dynamics, with the same robustness feature, we extend Carr-Lee-Wu [4] in three directions. First, we generalize to G-variation (including but not limited to pth-power variation); second, we allow the G-variation to be share-weighted; and third, we consider hedging in the general setting. The first extension, to G-variation, allows us to price a family of variability statistics relevant in financial derivatives and portfolio management, such as one-sided variance, capped-jump variance, and total variation; the latter two are of particular relevance in the aftermath of market events of 2008-09, as sellers of volatility have sought contracts with less tail-risk exposure than the variance swap's quadratic exposure. The second extension, to share-weighting, allows us to price G-variation contracts which have gamma-swap-like features, such as directional exposure to the underlying level, and dampening of downside variability. The third extension analyzes and minimizes the risk in approximately hedging G-variation by a strategies which exactly attain H-variation for some $H \neq G$.

This paper is organized as follows. Section 2 defines G-variation and share-weighted G-variation. In the time-changed Lévy setting, section 3 prices the G-variation swap, in terms of a log contract times a multiplier that depends *only* on G and the Lévy driver X. Then section 4 gives financial examples of G-variation. Section 5 prices the share-weighted G-variation swap, in terms of an $F \log F$ contract times a dual multiplier that again has no dependence on the stochastic clock; and section 6 gives financial examples. Section 7 finds explicit multiplier and dual-multiplier formulas for some examples of X: a two-jump-size process (for which, moreover, we find perfect hedges of unweighted and share-weighted G-variation swaps), and the extended CGMY process, including Variance Gamma. For general X, section 7.4 shows that the multiplier differs from 2, in a way that relates (positively or negatively, depending on the contract) to the Lévy measure's skewness. Section 7.5 computes numerical multipliers for three distinct G functions, with three distinct weighting schemes, under three distinct time-changed Lévy processes, using parameters empirically calibrated in [2]. Section 8 analyzes the risk in hedges of G-variation, and exhibits numerical examples of optimal hedges, within a three-parameter family of trading strategies, under two distinct risk measurement criteria, for hedging three distinct G contracts. Section 9 concludes.

2 G-variation

Let us define the G-variation of a semimartingale Y, for G which belong to the following family.

Definition 2.1 (The family $\mathbb{V}$)**.** *Let Y be a semimartingale with respect to a probability measure $\mathbb{P}$. Let Y^{c} denote its continuous local martingale part, and let ν_Y denote its jump compensator. Let*

$$I_Y := \{p \geq 0 : \int_{(0,t]\times\mathbb{R}} (|x|^p \wedge 1)\mathrm{d}\nu_Y < \infty \text{ for all } t > 0\}. \tag{2.1}$$

In the case $1 \in I_Y$, let

$$Y_t^{\mathrm{d}} := Y_t - Y_0 - Y_t^{\mathrm{c}} - \sum_{0<s\leq t} (\Delta Y_s). \tag{2.2}$$

In any case, let us say that $G : \mathbb{R} \to \mathbb{R}$ satisfies

$$G \in \mathbb{V}(Y, \mathbb{P}) \tag{2.3}$$

(or simply $G \in \mathbb{V}(Y)$ when the measure is understood), if

$$G(x) = \alpha_G|x| + \beta_G x + \gamma_G x^2 + \Lambda_G(x) \tag{2.4}$$

for some constants α_G and β_G and γ_G such that

$$\alpha_G = 0, \text{ or } Y \text{ has finite variation a.s.} \tag{2.5}$$

and some

$$\begin{aligned}\Lambda_G \in \mathbb{J}(Y) := \{&\text{Locally bounded functions } \Lambda : \mathbb{R} \to \mathbb{R} \text{ which are } \nu(\omega; \mathbb{R}_+ \times \mathrm{d}x)\text{-a.e.} \\ &\text{continuous for } \mathbb{P}\text{-almost all } \omega \text{ and satisfy at least one of (2.7-2.10)}\}.\end{aligned} \tag{2.6}$$

The conditions referenced in the definition of $\mathbb{J}(Y)$ are

$$\Lambda(x) = o(x^2), \tag{2.7}$$

$$\Lambda(x) = O(|x|^p) \text{ and } p \in I_Y \cap (1, 2] \text{ and } Y^{\mathrm{c}} = 0 \tag{2.8}$$

$$\Lambda(x) = o(|x|) \text{ and } 1 \in I_Y \text{ and } Y^{\mathrm{c}} = 0 \tag{2.9}$$

$$\Lambda(x) = O(|x|^p) \text{ and } p \in I_Y \cap (0, 1] \text{ and } Y^{\mathrm{d}} = Y^{\mathrm{c}} = 0. \tag{2.10}$$

Each O or o here (and everywhere in this paper) denotes an $x \to 0$ relation.

We have written $\Delta Y_s := Y_s - Y_{s-}$ (which is nonzero for countably many s, by right-continuity and left-limits of Y). Let brackets $[\cdot]$ denote quadratic variation, and let $\mathrm{TV}(\cdot)$ denote the total variation of a finite variation process.

Note that $Y - Y^{\mathrm{c}}$ ("the jumps") will have finite variation if and only if $1 \in I_Y$; moreover Y will have finite variation if and only if $1 \in I_Y$ and $Y^{\mathrm{c}} = 0$.

Definition 2.2 (G-variation)**.** *For $G \in \mathbb{V}(Y)$, define the G-variation of Y to be*

$$V_t^{Y,G} := \alpha_G \operatorname{TV}(Y^{\mathrm{d}})_t + \beta_G(Y_t - Y_0) + \gamma_G[Y^{\mathrm{c}}]_t + \sum_{0<s\leq t}(G(\Delta Y_s) - \beta_G \Delta Y_s), \tag{2.11}$$

where we define $\alpha_G \operatorname{TV}(Y^{\mathrm{d}}) := 0$ in the case that $\alpha_G = 0$. (This convention simplifies notation, by avoiding a separate definition for the infinite-variation jump case).

If the jumps have finite variation, then (2.11) may be rewritten as

$$V_t^{Y,G} := \alpha_G \operatorname{TV}(Y^{\mathrm{d}})_t + \beta_G(Y_t^{\mathrm{c}} + Y_t^{\mathrm{d}}) + \gamma_G[Y^{\mathrm{c}}]_t + \sum_{0<s\leq t} G(\Delta Y_s), \tag{2.12}$$

which separates the continuous part (first three terms) and jump part (last term) of $V_t^{Y,G}$. Our definition (2.11), more broadly, makes sense even if the jumps have infinite variation.

The following convergence result, a direct corollary of Jacod ([13] Theorem 2.2, which guided the formulations of our Definitions 2.1 and 2.2) serves two purposes for us. First, it gives the rationale for why we chose to define G-variation by (2.11): namely, that G-variation is the continuous-sampling limit of "discrete G-variation," meaning the sum of G applied to the sampled increments of Y. Second, it shows that, even if the decomposition of G into the form (2.4) is not unique, the definition (2.11) is invariant to the decomposition, and depends only on G and Y.

Proposition 2.3. *For any $G \in \mathbb{V}(Y)$, and any sequence $\Delta_n \to 0$, let*

$$V^{Y,G}(n)_t := \sum_{j=1}^{\lfloor t/\Delta_n \rfloor} G(Y_{j\Delta_n} - Y_{(j-1)\Delta_n}) \tag{2.13}$$

For any *decomposition of the form (2.4), the $V^{Y,G}$ defined by (2.11) satisfies*

$$V^{Y,G}(n) \longrightarrow V^{Y,G} \tag{2.14}$$

in probability in the Skorokhod sense, as $n \to \infty$.

As a corollary, $V^{Y,G}$ does not depend on the decomposition of G.

Proof. Let $G_0(x) := \alpha_G|x|$ and $G_1(x) := \beta_G x$ and $G_2(x) := \gamma_G x^2$. Let

$$A_t := \alpha_G \operatorname{TV}(Y^{\mathrm{d}})_t + \sum_{0<s\leq t} G_0(\Delta Y_s) \tag{2.15}$$

$$B_t := \beta_G(Y_t - Y_0) \tag{2.16}$$

$$C_t := \gamma_G[Y^{\mathrm{c}}]_t + \sum_{0<s\leq t} G_2(\Delta Y_s) \tag{2.17}$$

$$D_t := \sum_{0<s\leq t} \Lambda_G(\Delta Y_s) \tag{2.18}$$

As $n \to \infty$, we have the Skorohod limits in probability

$$V^{Y,G_0}(n) \longrightarrow A \tag{2.19}$$

$$V^{Y,G_2}(n) \longrightarrow C \tag{2.20}$$

$$V^{Y,\Lambda_G}(n) \longrightarrow D \tag{2.21}$$

by parts (c), (b), and (a) respectively, of Jacod [13] Theorem 2.2, whose proof of (b) is indeed valid for the case $r = 2 \in I_Y$.

Moreover $V^{Y,G_1}(n)_t = \beta_G(Y_{\Delta_n \lfloor t/\Delta_n \rfloor} - Y_0)$ implies that we have the pathwise Skorokhod limit

$$V^{Y,G_1}(n) \longrightarrow B. \tag{2.22}$$

Summing (2.19) to (2.22) produces (2.14). Finally the decomposition of G does not affect the left-hand side of (2.14), hence it does not affect the right-hand side. □

With a view toward the hedging applications of Section 8, we need a "tower" property, that the G_2-variation of the G_1-variation of Y is the $(G_2 \circ G_1)$-variation of Y, where $\circ$ denotes composition.

Proposition 2.4 (Tower property). *Let $G_1 \in \mathbb{V}(Y)$ and $G_2 \in \mathbb{V}(V^{Y,G_1})$. Then*

$$V^{V^{Y,G_1},G_2} = V^{Y,G_2 \circ G_1}. \tag{2.23}$$

Proof. Let $\alpha_{G_1}, \beta_{G_1}, \gamma_{G_1}, \Lambda_{G_1}$ be an arbitrary decomposition consistent with Definition 2.1. Then

$$V_t^{Y,G_1} = \alpha_{G_1} \operatorname{TV}(Y^{\mathrm{d}})_t + \beta_{G_1}(Y_t - Y_0) + \gamma_{G_1}[Y^{\mathrm{c}}]_t + \sum_{0<s\le t} (G_1(\Delta Y_s) - \beta_{G_1}\Delta Y_s). \tag{2.24}$$

It suffices to prove (2.23) for four cases: $G_2 = x$, $G_2 = |x|$, $G_2 = x^2$, and $G_2 \in \mathbb{J}(V^{Y,G_1})$. The result for general $G_2 \in \mathbb{V}(V^{Y,G_1})$ follows by linearity.

If $G_2(x) = x$, then

$$V_t^{V^{Y,G_1},G_2} = V_t^{Y,G_1} = V_t^{Y,G_2 \circ G_1}. \tag{2.25}$$

If $G_2 \in \mathbb{J}(V^{Y,G_1})$, then

$$V_t^{V^{Y,G_1},G_2} = \sum_{0<s\le t} G_2(G_1(\Delta Y_s)) = V_t^{Y,G_2 \circ G_1}. \tag{2.26}$$

If $G_2(x) = x^2$, then either Y has finite variation, which implies by (2.12) that

$$V_t^{V^{Y,G_1},G_2} = \sum_{0<s\le t} (G_1(\Delta Y_s))^2 = V_t^{Y,G_1^2}, \tag{2.27}$$

or else $\alpha_{G_1} = 0$, which implies

$$V_t^{V^{Y,G_1},G_2} = [V^{Y,G_1}]_t \tag{2.28}$$

$$= \beta_{G_1}^2 [Y]_t + \sum_{0<s\le t} (G_1(\Delta Y_s) - \beta_{G_1}\Delta Y_s)^2 + 2 \sum_{0<s\le t} \beta_{G_1}\Delta Y_s (G_1(\Delta Y_s) - \beta_{G_1}\Delta Y_s) \tag{2.29}$$

$$= \beta_{G_1}^2 [Y^{\mathrm{c}}]_t + \sum_{0<s\le t} (G_1(\Delta Y_s))^2 \tag{2.30}$$

$$= V_t^{Y,G_2 \circ G_1}, \tag{2.31}$$

where (2.29) is because $[A+B]=[A]+[B]+2[A,B]$ for any semimartingales A,B.

If $G_2(x)=|x|$ then either $\alpha_{G_1}=\beta_{G_1}=0$ or Y has finite variation. In the former subcase,

$$V_t^{V^{Y,G_1},G_2}=\mathrm{TV}(\gamma_{G_1}[Y^{\mathrm{c}}])_t+\sum_{0<s\le t}|G_1(\Delta Y_s)|=V_t^{Y,G_2\circ G_1}, \tag{2.32}$$

as claimed; here the last step is because $|G_1(x)|-|\gamma_{G_1}||x|=O(G_1(x)-\gamma_{G_1}|x|)$ implies that $|G_1(x)|=|\gamma_{G_1}x|+O(\Lambda_{G_1}(x))$. Otherwise, in the finite variation subcase, let

$$\alpha_{|G_1|}:=(|\alpha_{G_1}+\beta_{G_1}|+|\alpha_{G_1}-\beta_{G_1}|)/2 \tag{2.33}$$

$$\beta_{|G_1|}:=(|\alpha_{G_1}+\beta_{G_1}|-|\alpha_{G_1}-\beta_{G_1}|)/2. \tag{2.34}$$

Because $\mathbb{J}(Y)$ contains $\Lambda_{G_1}(x)=G_1(x)-(\alpha_{G_1}|x|+\beta_{G_1}x)-\gamma_{G_1}x^2$ and all $O(x^2)$ functions, it therefore contains also

$$\Lambda_{|G_1|}(x):=|G_1(x)|-|\alpha_{G_1}|x|+\beta_{G_1}x|-\gamma_{G_1}x^2=|G_1(x)|-\alpha_{|G_1|}|x|+\beta_{|G_1|}x-\gamma_{G_1}x^2. \tag{2.35}$$

Thus $|G_1|$ has the following decomposition, consistent with Definition 2.1:

$$|G_1(x)|=\alpha_{|G_1|}|x|+\beta_{|G_1|}x+\gamma_{G_1}x^2+\Lambda_{|G_1|}(x). \tag{2.36}$$

Now take the Jordan decomposition of $\mathrm{TV}(Y^{\mathrm{d}})$ into increasing processes $\mathrm{NV}(Y^{\mathrm{d}})$ and $\mathrm{PV}(Y^{\mathrm{d}})$ such that $\mathrm{TV}(Y^{\mathrm{d}})=\mathrm{PV}(Y^{\mathrm{d}})-\mathrm{NV}(Y^{\mathrm{d}})$ and $Y^{\mathrm{d}}=\mathrm{PV}(Y^{\mathrm{d}})+\mathrm{NV}(Y^{\mathrm{d}})$. Then

$$V_t^{V^{Y,G_1},G_2}=\mathrm{TV}\left(\alpha_{G_1}\,\mathrm{TV}(Y^{\mathrm{d}})+\beta_{G_1}Y^{\mathrm{d}}\right)_t+\sum_{0<s\le t}|G_1(\Delta Y_s)| \tag{2.37}$$

$$=\mathrm{TV}\left((\alpha_{G_1}+\beta_{G_1})\,\mathrm{PV}(Y^{\mathrm{d}})+(\alpha_{G_1}-\beta_{G_1})\,\mathrm{NV}(Y^{\mathrm{d}})\right)_t+\sum_{0<s\le t}|G_1(\Delta Y_s)| \tag{2.38}$$

$$=|\alpha_{G_1}+\beta_{G_1}|\,\mathrm{PV}(Y^{\mathrm{d}})_t+|\alpha_{G_1}-\beta_{G_1}|\,\mathrm{NV}(Y^{\mathrm{d}})_t+\sum_{0<s\le t}|G_1(\Delta Y_s)| \tag{2.39}$$

$$=\alpha_{|G_1|}\,\mathrm{TV}(Y^{\mathrm{d}})_t+\beta_{|G_1|}Y_t^{\mathrm{d}}+\sum_{0<s\le t}|G_1(\Delta Y_s)| \tag{2.40}$$

$$=V_t^{Y,|G_1|}, \tag{2.41}$$

where the last step uses (2.12), (2.36), and $Y^{\mathrm{c}}=0$. This completes the $G_2(x)=|x|$ case. □

Finally, with a view toward applications where Y is a log-returns process, we define *dual* or *share-weighted* G-variation as follows.

Definition 2.5 (Dual G-variation and $\tilde{\mathbb{V}}$)**.** *If Y has finite variation, let $\tilde{\mathbb{V}}(Y):=\mathbb{V}(Y)$. Otherwise, let $\tilde{\mathbb{V}}(Y):=\{G\in\mathbb{V}(Y): G$ has a Definition-2.1 decomposition such that $\alpha_G=\beta_G=0\}$.*

In any case, for $G\in\tilde{\mathbb{V}}(Y)$, define the dual G-variation *or* share-weighted G-variation *of Y by*

$$\tilde{V}_t^{Y,G}:=\int_0^t e^{Y_s}\mathrm{d}V_s^{Y,G} \tag{2.42}$$

where the integral is defined pathwise in the Riemann-Stieltjes sense.

The Riemann-Stieltjes integral exists pathwise because the integrator is of finite variation and the integrand is right-continuous with left limits.

3 Variation swaps on time-changed Lévy processes

In this section we price contracts paying the G-variation of time-changed Lévy processes, in terms of log contracts and the *G-multiplier*, which depends only on the driving Lévy process.

3.1 The G-multiplier

For a Lévy process X satisfying integrability conditions, let us define a family $\mathbb{W}(X)$ of functions G. Whereas the family $\mathbb{V}$ comprised G for which G-variation is finite, the family $\mathbb{W}$ comprises G for which the G-variation is moreover integrable.

Definition 3.1 (The family $\mathbb{W}$). *Let X be a nondeterministic Lévy process with $\mathbb{P}$-Lévy measure ν, such that $\int_{|x|>1} e^x \mathrm{d}\nu(x) < \infty$ and $\int_{|x|>1} |x| \mathrm{d}\nu(x) < \infty$. Let*

$$\mathbb{W}(X, \mathbb{P}) = \{G \in \mathbb{V}(X, \mathbb{P}) : \int |G(x) - \beta_G x| \mathrm{d}\nu(x) < \infty\}. \tag{3.1}$$

In the case $1 \in I_X$ (jumps have finite variation), condition (3.1) *is equivalent to*

$$\mathbb{W}(X, \mathbb{P}) = \{G \in \mathbb{V}(X, \mathbb{P}) : \int |G(x)| \mathrm{d}\nu(x) < \infty\}. \tag{3.2}$$

We may write simply $G \in \mathbb{W}(X)$ when the measure is understood.

For any given (X, G), note that whether $G \in \mathbb{W}(X)$ does not depend on the choice of β_G; either a unique β_G is consistent with Definition 2.1, so there is no choice; or else $1 \in I_X$, hence $\int |x| \mathrm{d}\nu(x) < \infty$, hence condition (3.1) reduces to condition (3.2) on G alone

For X such that $\mathbb{E}e^{X_1} < \infty$, define the drift-adjusted process

$$X'_u := X_u - u \log \mathbb{E}e^{X_1}. \tag{3.3}$$

Proposition 3.2 (G-multiplier). *For $G \in \mathbb{W}(X, \mathbb{P})$, define the G-multiplier by*

$$Q^{X,G} := Q^{X,G,\mathbb{P}} := \frac{\mathbb{E}V_1^{X',G}}{-\mathbb{E}X'_1} = \frac{\mathbb{E}V_1^{X',G}}{\log \mathbb{E}e^{X_1} - \mathbb{E}X_1} \tag{3.4}$$

where $\mathbb{E}$ denotes $\mathbb{P}$-expectation. Then, for any $\alpha_G, \beta_G, \gamma_G$ consistent with Definition 2.1, we have

$$Q^{X,G} = \frac{\alpha_G \left| \sigma^2/2 + \int (e^x - 1) \mathrm{d}\nu(x) \right| + \gamma_G \sigma^2 + \int (G(x) - \beta_G x) \mathrm{d}\nu(x)}{\sigma^2/2 + \int (e^x - 1 - x) \mathrm{d}\nu(x)} - \beta_G, \tag{3.5}$$

where (A, σ^2, ν) denotes the generating $\mathbb{P}$-triplet of X.

Proof. Sato [18] Theorem 25.17 and Example 25.12 imply that

$$\log \mathbb{E}e^{X_1} - \mathbb{E}X_1 = \sigma^2/2 + \int (e^x - 1 - x\mathbf{1}_{|x| \le 1}) \mathrm{d}\nu(x) + A - (A + \int_{|x| \ge 1} x \mathrm{d}\nu(x)) \tag{3.6}$$

(and that the integrals are finite). This gives the denominator of (3.5), which is positive by convexity of exp and nonconstancy of X_1.

From Definition 2.2, the numerator is the sum of four terms. The first term is because either $\alpha_G = 0$ or $X_1'^{\mathrm{d}} = \sigma^2/2 + \int (e^x - 1)\mathrm{d}\nu(x)$. The second term is because $X_1^{\mathrm{c}} = \sigma^2$. To obtain the third term, apply Sato [18], Propositions 19.2 and 19.5, to the restriction of ν to $\{x : 1/m < |x| < m\}$ for each $m > 0$, together with dominated convergence as $m \to \infty$. The fourth term is $\beta_G \mathbb{E} X_1'$, which is divided by $-\mathbb{E}X_1'$ to obtain $-\beta_G$. $\square$

3.2 Time changed Lévy processes

We build upon the time-changed Lévy framework in [4], so this subsection's assumptions are quoted from there, and are in force throughout the remainder of this paper.

We begin with a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_u\}_{u \geq 0}, \mathbb{P})$ satisfying the usual conditions.

Fix a time horizon $T > 0$.

Let the interest rate be a deterministic right-continuous left-limits process r with $\int_0^T |r_s| \mathrm{d}s < \infty$. Let

$$\bar{r}_t := \int_0^t r_s \mathrm{d}s. \tag{3.7}$$

Let F denote a positive underlying T-expiry forward or futures price process, and let

$$Y_t := \log(F_t/F_0) \tag{3.8}$$

denote the log return on F. Let

$$F_t^* := F_t e^{\bar{r}_t - \bar{r}_T}. \tag{3.9}$$

denote the associated underlying spot price, and

$$Y_t^* := \log(F_t^*/F_0^*) = Y_t + \bar{r}_t \tag{3.10}$$

denote the log return on F^*. Assume that

$$Y_t = X'_{\tau_t} \tag{3.11}$$

where X is a Lévy process such that $\mathbb{E}e^{X_1} < \infty$, and where the time change

$$\{\tau_t : t \in [0, T]\} \tag{3.12}$$

is a continuous increasing family of finite stopping times. We do *not* assume independence of X and τ.

Financially, we regard X, indexed by "business" time, as a "driving" or "background" Lévy process, which induces the drift-adjusted process X' such that $e^{X'}$ is a martingale. We regard τ as an unspecified stochastic clock that maps calendar time t to business time τ_t. The resulting $\{\mathcal{F}_{\tau_t}\}$-adapted process Y can exhibit stochastic volatility, stochastic jump-intensity, volatility clustering, and "leverage" effects, the latter via skewed jump distributions, or via correlation of X and τ.

3.3 Pricing variation swaps in terms of log contracts

Define the T-expiry *log contract* to pay at time T

$$-Y_T, \tag{3.13}$$

where the sign convention was chosen to make log contracts have nonnegative value. Log contract values are, in principle, observable given T-expiry call and put prices of all strikes.

For $G \in \mathbb{V}(Y)$, define a (zero-fixed-leg) *G-variation swap* on F to pay at time T

$$V_T^{Y,G}. \tag{3.14}$$

Assume that $\mathbb{P}$ is a martingale measure for log contracts and G-variation swaps; thus the T-expiry log contract and G-variation swap have respective time-0 values $e^{-\bar{r}_T}\mathbb{E}(-Y_T)$ and $e^{-\bar{r}_T}\mathbb{E}V_T^{Y,G}$, if finite.

Proposition 3.3 (G-variation swap valuation)**.** *If $G \in \mathbb{W}(X, \mathbb{P})$ and $\mathbb{E}\tau_T < \infty$ then*

$$\mathbb{E}V_T^{Y,G} = Q^{X,G}\mathbb{E}(-Y_T), \tag{3.15}$$

where $\mathbb{E}$ denotes $\mathbb{P}$-expectation. The multiplier $Q^{X,G}$ does not depend on the time-change.

Proof. The definition of $Q^{X,G}$ implies that the Lévy process

$$V_u^{X',G} + Q^{X,G}X'_u \tag{3.16}$$

is a martingale. Because $\mathbb{E}\tau_T < \infty$, we have, by Wald's first equation in continuous time [11],

$$\mathbb{E}\big(V_{\tau_T}^{X',G} + Q^{X,G}X'_{\tau_T}\big) = 0. \tag{3.17}$$

Moreover $\mathbb{E}V_{\tau_T}^{X',G} < \infty$, again by Wald's first equation, so

$$\mathbb{E}V_{\tau_T}^{X',G} = Q^{X,G}\mathbb{E}(-X'_{\tau_T}). \tag{3.18}$$

It remains to show that $V_T^{Y,G} = V_{\tau_T}^{X',G}$. By linearity, we need only prove this for four cases of G. Case 1: If $G(x) = x$, then clearly $Y_T = X'_{\tau_T}$. Case 2: If $G(x) = x^2$, then Jacod [12] Theorem 10.17 implies $[Y]_T = [X']_{\tau_T}$, by continuity of τ. Case 3: If $G(x) = |x|$, then pathwise

$$\begin{aligned} \mathrm{TV}(Y)_T &= \sup\{\textstyle\sum |Y_{t_{i+1}} - Y_{t_i}| : \{t_i\} \text{ is a partition of } [0,T]\} \\ &= \sup\{\textstyle\sum |X'_{u_{i+1}} - X'_{i_i}| : \{u_i\} \text{ is a partition of } [0,\tau_T]\} = \mathrm{TV}(X')_{\tau_T}, \end{aligned} \tag{3.19}$$

where the equality of the suprema holds because each partition of $[0,T]$ induces (via $t \mapsto \tau_t$) a partition of $[0,\tau_T]$ having the same sum, and vice versa (via the right-continuous inverse τ^{-1}, which exists by continuity of τ). Case 4: If $G \in \mathbb{J}(X)$, then $\sum_{0<t\leq T} G(\Delta Y_t) = \sum_{0<u\leq\tau_T} G(\Delta X_u)$ because $\Delta Y_t = \Delta X_{\tau_t}$ by continuity of τ. $\square$

Thus the G-variation swap value $e^{-\bar{r}_T}\mathbb{E}V_T^{Y,G}$ equals $Q^{X,G}$ times the log contract value $e^{-\bar{r}_T}\mathbb{E}(-Y_T)$. Equivalently, restated in terms of forward-settled payments, the G-variation swap's fair forward price $\mathbb{E}V_T^{Y,G}$ equals $Q^{X,G}$ times the log contract's forward price $\mathbb{E}(-Y_T)$.

The *multiplier* $Q^{X,G}$ depends only on G and the characteristics of the background Lévy process X. It does *not* depend on the time-change.

Likewise, for the spot underlying, the G-variation swap on F^* can be defined to pay $V_T^{Y^*,G}$. However, if G has a Definition-2.1-compliant decomposition in which $\alpha_G = \beta_G = 0$, then no distinction exists between G-variation swaps on futures and spot, because $Y^* - Y = \bar{r}$ has finite variation and no jumps, which implies $V^{Y,G} = V^{Y^*,G}$. We have established the following.

Corollary 3.4 (G-variation swaps, on spot underlying)**.** *If $G \in \mathbb{W}(X,\mathbb{P})$ and $\alpha_G = \beta_G = 0$ then*

$$\mathbb{E}V_T^{Y^*,G} = Q^{X,G}\mathbb{E}(-Y_T), \tag{3.20}$$

provided that $\mathbb{E}\tau_T < \infty$.

4 Financial examples of G-variation swaps

We list some financial examples of G-variation swaps. Each of these variation contracts admits pricing in terms of log contracts, subject to the conditions of Proposition 3.3.

$$\text{Logarithmic } \textit{variance swap}: \quad G(x) := x^2. \tag{4.1}$$

$$\text{Simple-return } \textit{variance swap}: \quad G(x) := (e^x - 1)^2. \tag{4.2}$$

$$\text{Logarithmic } \textit{pth moment swap}: \quad G(x) := x^p. \tag{4.3}$$

$$\text{Logarithmic } \textit{absolute pth moment swap}: \quad G(x) := |x|^p. \tag{4.4}$$

Simple-return and *absolute simple-return* moment swaps are defined by replacing x with $e^x - 1$ in (4.3) and (4.4) respectively. A *capped-jump* version of a G-variation contract replaces G with

$$\bar{G}(x) := G(\min(\max(x,a),b)), \tag{4.5}$$

for some $-\infty \leq a < b \leq \infty$. A *capped-$G$* version of a G-variation contract replaces G with

$$\bar{G}(x) := \min(G(x), M) \tag{4.6}$$

for some constant M.

Remark 4.1. Choosing $G(x) := |x|$ gives a "total variation" swap, and $G(x) := \min(x^2, M)$ gives a "capped-quadratic" variation swap. These specifications, with sub-quadratic tails, are particularly relevant in the wake of market events of 2008-09, as traders have sought contracts which allow some form of volatility to be sold, without admitting quadratic exposure to the risk of sharp movements.

Examples (4.1), (4.3), (4.4), and some cases of (4.5), belong to the (logarithmic) *semi-moment* family, defined by

$$G(x) := |x|^p(\mathbf{1}_{x>0}U + \mathbf{1}_{x<0}D), \tag{4.7}$$

where U and D can be chosen in $\{-1, 0, 1\}$, in accordance with the desired payoff.

In particular, $U = D = 1$ gives the absolute pth moment swap, $(U, D) = (1, (-1)^p)$ for integer p gives the pth moment swap, and with $p = 2$, the choices $(U, D) = (0, 1)$ and $(U, D) = (1, 0)$ give *down semivariance* and *up semivariance* respectively. Unlike corridor variance, the semivariance applies an up/down filter to each fluctuation, not to the cumulative return.

5 Dual G-variation

In addition to G-variation contracts, we shall moreover price contracts paying *dual* G-variation, also described as *share-weighted* G-variation, in the sense of Definition 2.5:

$$\tilde{V}_t^{Y,G} = \int_0^t \frac{F_s}{F_0}\mathrm{d}V_s^{Y,G} = \int_0^t \frac{F_s}{F_0}\mathrm{d}U_s + \sum_{0<s\leq t}\frac{F_s}{F_0}G(\Delta Y_s), \tag{5.1}$$

where

$$U := \alpha_G\,\mathrm{TV}(Y^{\mathrm{d}}) + \beta_G Y^{\mathrm{d}} + \gamma_G[Y^{\mathrm{c}}], \tag{5.2}$$

with our notational convention that the first term and/or second term is 0 if $\alpha_G = 0$ or $\beta_G = 0$ respectively. The $\tilde{V}_t^{Y,G}$ may be regarded as a payment of G-variation "in shares".

Specifically, for time-changed Lévy processes, we price the dual G-variation contract in terms of "$F \log F$" contracts, which by definition pay

$$Y_T e^{Y_T} = \frac{F_T}{F_0}\log\frac{F_T}{F_0}, \tag{5.3}$$

and which have prices *observable*, in principle, given prices of T-expiry calls and puts at all strikes. According to Propositions 5.1 and 5.2, the dual G-variation contract is worth a multiple of the $F \log F$ contract, where the multiplier depends only on G and the driving Lévy process, not on the time change. The assumptions of subsection 3.2 remain in effect.

Proposition 5.1 (Dual G-multiplier)**.** *Define measure $\tilde{\mathbb{P}}$ by*

$$\frac{\mathrm{d}\tilde{\mathbb{P}}_u}{\mathrm{d}\mathbb{P}_u} = \exp X'_u \tag{5.4}$$

where $\tilde{\mathbb{P}}_u$ and $\mathbb{P}_u$ denote the restrictions of $\tilde{\mathbb{P}}$ and $\mathbb{P}$ to $\mathcal{F}_u$. Let $\tilde{X} := -X$ and $\tilde{G}(x) := G(-x)$.

For $G \in \tilde{\mathbb{W}}(X, \mathbb{P}) := \{G \in \tilde{\mathbb{V}}(X, \mathbb{P}) : \tilde{G} \in \mathbb{W}(\tilde{X}, \tilde{\mathbb{P}})\}$, define the dual G-multiplier *of X by*

$$\tilde{Q}^{X,G,\mathbb{P}} := Q^{\tilde{X},\tilde{G},\tilde{\mathbb{P}}}. \tag{5.5}$$

Then, for any choice of $\alpha_G, \beta_G, \gamma_G$ consistent with Definition 2.1,

$$\tilde{Q}^{X,G,\mathbb{P}} = \frac{\alpha_G \left| \sigma^2/2 + \int (1-e^x)\mathrm{d}\nu(x) \right| + \gamma_G \sigma^2 + \int e^x (G(x) - \beta_G x)\mathrm{d}\nu(x)}{\sigma^2/2 + \int (1 - e^x + xe^x)\mathrm{d}\nu(x)} + \beta_G, \tag{5.6}$$

where (A, σ^2, ν) denotes the generating $\mathbb{P}$-triplet of X.

Proof. By Sato [18], X has generating $\tilde{\mathbb{P}}$-triplet $(\tilde{A}, \sigma^2, \tilde{\nu})$, where $\mathrm{d}\tilde{\nu}(x)/\mathrm{d}\nu(x) = e^x$.

By Proposition 3.2 we have

$$Q^{\tilde{X},\tilde{G},\tilde{\mathbb{P}}} = \frac{\alpha_G \left| \sigma^2/2 + \int (e^{-x} - 1)\mathrm{d}\tilde{\nu}(x) \right| + \gamma_G \sigma^2 + \int \tilde{G}(-x) - (-\beta_G)(-x)\mathrm{d}\tilde{\nu}(x)}{\sigma^2/2 + \int (e^{-x} - 1 + x)\mathrm{d}\tilde{\nu}(x)} - (-\beta_G) \tag{5.7}$$

which simplifies to (5.6). □

Proposition 5.2 (Dual G-variation swap valuation). *If $G \in \tilde{\mathbb{W}}(X, \mathbb{P})$ and $\mathbb{E}F_T^2 < \infty$ and $\mathbb{E}\tau_T^2 < \infty$ and $\tilde{\mathbb{E}}\tau_T < \infty$, then*

$$\mathbb{E}\tilde{V}_T^{Y,G} = \tilde{Q}^{X,G}\mathbb{E}(e^{Y_T} Y_T). \tag{5.8}$$

The multiplier $\tilde{Q}^{X,G}$ does not depend on the time-change.

Proof. Let $\tilde{\mathbb{E}}$ denote $\tilde{\mathbb{P}}$-expectation. We have

$$\tilde{X}'_u := \tilde{X}_u - u \log \tilde{\mathbb{E}} e^{\tilde{X}_1} = -X_u - u(-\log \mathbb{E} e^{X_1}) = -X'_u \tag{5.9}$$

hence

$$\tilde{Y}_t := \tilde{X}'_{\tau_t} = -X'_{\tau_t} = -Y_t. \tag{5.10}$$

Moreover, for all $s < T$ we have

$$\frac{\mathrm{d}\tilde{\mathbb{P}}_{\tau_s}}{\mathrm{d}\mathbb{P}_{\tau_s}} = \exp X'_{\tau_s} = \frac{F_s}{F_0} \tag{5.11}$$

by the $\mathbb{P}$-a.s. and $\tilde{\mathbb{P}}$-a.s. finiteness of τ_T (see, for instance, [14] Appendix B).

Let Γ denote either $|G|$ or G, and let $\tilde{\Gamma}(x) := \Gamma(-x)$. Then

$$\mathbb{E} \sum_{0<s\leq T} \frac{F_s}{F_0}\Gamma(\Delta Y_s) = \sum_{0<s\leq T} \mathbb{E}\frac{F_s}{F_0}\Gamma(\Delta Y_s) = \sum_{0<s\leq T} \tilde{\mathbb{E}}\Gamma(\Delta Y_s) = \tilde{\mathbb{E}} \sum_{0<s\leq T} \tilde{\Gamma}(\Delta \tilde{Y}_s). \tag{5.12}$$

For $\Gamma := |G|$, the Fubini step is justified by $\Gamma \geq 0$; moreover, the right-hand-side of (5.12) is finite because $\tilde{G} \in \mathbb{W}(\tilde{X}, \tilde{\mathbb{P}})$. This in turn justifies usage of Fubini for $\Gamma := G$.

Recall the U definition (5.2). For any sequence of partitions $0 = t_{n,0} < t_{n,1} < \ldots < t_{n,n} = T$ such that $\max_k\{t_{n,k} - t_{n,k-1}\} \to 0$ as $n \to \infty$, we have

$$\begin{aligned}
\mathbb{E}\int_0^T \frac{F_t}{F_0}\mathrm{d}U_t &= \mathbb{E} \lim_{n\to\infty} \sum_{k=1}^n \frac{F_{t_{n,k}}}{F_0}(U_{t_{n,k}} - U_{t_{n,k-1}}) && (5.13)\\
&= \lim_{n\to\infty} \sum_{k=1}^n \mathbb{E}\frac{F_{t_{n,k}}}{F_0}(U_{t_{n,k}} - U_{t_{n,k-1}}) && (5.14)\\
&= \lim_{n\to\infty} \sum_{k=1}^n \tilde{\mathbb{E}}(U_{t_{n,k}} - U_{t_{n,k-1}}) = \tilde{\mathbb{E}}U_T, && (5.15)
\end{aligned}$$

where (5.13) is by continuity of the finite variation integrator and right-continuity-left-limits of the integrand, and the interchange in (5.14) is by dominated convergence: for each n the Riemann sum is bounded by $|U_T| \sup_{t \leq T} F_t/F_0$, which is L^1 because each factor is L^2: the first because $\mathbb{E}\tau_T^2 < \infty$, and the second by Doob's inequality.

By summing (5.15) and (5.12) with $\Gamma := G$, then applying Proposition 3.3, then (5.10), we have

$$\mathbb{E}\tilde{V}_T^{Y,G} = \tilde{\mathbb{E}}V_T^{\tilde{Y},\tilde{G}} = \tilde{Q}^{X,G}\tilde{\mathbb{E}}(-\tilde{Y}_T) = \tilde{Q}^{X,G}\mathbb{E}(Y_T e^{Y_T}) \tag{5.16}$$

as claimed. □

6 Financial examples of dual G-variation swaps

We give some financial examples of dual G-variation swaps. Each of these variation contracts admits pricing in terms of $F \log F$ contracts, subject to the conditions of Proposition 5.2.

Example 6.1. The canonical example of a dual G-variation swap is the *gamma swap*, which takes $G(x) = x^2$ and therefore pays

$$\int_0^T \frac{F_s}{F_0} \mathrm{d}[Y]_s, \tag{6.1}$$

according to (5.1). Gamma swaps allow investors to acquire variance exposures proportional to the underlying level, which may be of practical importance for several reasons. First, the investor may be bullish on Y. Second, the investor may have the view that the market's downward volatility skew is too steep, making down-variance expensive relative to up-variance. Third, the investor may be seeking to hedge vega exposure that grows as Y increases, such as what arises in dispersion trading of a basket's volatility against its components' volatilities. Fourth, the investor may wish to trade single-stock variance without the caps often embedded in variance swaps to protect the seller from crash risk; in a gamma swap, the weighting inherently dampens the downside variance, without imposing a cap.

The standard theory of gamma swaps assumes continuous paths and finds that a gamma swap has the value of two $F \log F$ contracts. Proposition 5.2 extends the standard theory in two ways: first, by allowing time-changed Lévy processes with *jumps*; and second, by pricing share-weighted G-variation for not just quadratic but *general* G, motivated by investment objectives analogous to the gamma swap motivations outlined in Example 6.1.

Example 6.2 (Share-weighted moment swaps). Generalizing the gamma swap, a share-weighted moment swap pays

$$\int_0^T \frac{F_s}{F_0} \mathrm{d}V_s^{Y,G}, \tag{6.2}$$

where $G(x) = x^p$ or $G(x) = |x|^p$ or $G(x) = (e^x - 1)^p$ or $G(x) = |e^x - 1|^p$, producing pth-moment swaps on logarithmic, absolute logarithmic, simple, and absolute simple returns respectively.

Example 6.3 (Pre-jump share-weighted G-variation swaps). Let us modify the share-weighted G-variation payoff, by applying *pre-jump* share weights to define the payoff

$$\int_0^T \frac{F_{s-}}{F_0} \mathrm{d}V_s^{Y,G}, \tag{6.3}$$

which is tractable within our framework, because

$$\int_0^T \frac{F_{s-}}{F_0} \mathrm{d}V_s^{Y,G} = \int_0^T \frac{F_{s-}}{F_0} \mathrm{d}U_s + \sum_{0<s\leq t} \frac{F_{s-}}{F_0} G(\Delta Y_s) \tag{6.4}$$

$$= \int_0^T \frac{F_s}{F_0} \mathrm{d}U_s + \sum_{0<s\leq t} \frac{F_s}{F_0} e^{-\Delta Y_s} G(\Delta Y_s), \tag{6.5}$$

where the integrals are pathwise Riemann-Stieltjes.

The pre-jump share-weighted G-variation, therefore, still admits pricing in terms of $F \log F$ contracts, via the $\tilde{Q}$ multiplier, but with respect to the function $e^{-x}G(x)$ instead of $G(x)$.

7 Multiplier calculations

In the following examples of driving Lévy processes X, we will not need to specify the "drift" component of X, because passing to X' via (3.3) resets the drift anyway, to make $e^{X'}$ a martingale.

Each example's scope includes a *family* of log return processes $Y_t = X'_{\tau_t}$, because the time change τ is general and unspecified. Without modeling the stochastic clock τ, Proposition 3.3 prices the G-variation swap payoff $V_T^{Y,G}$ in terms of log contracts, and Proposition 5.2 prices the share-weighted G-variation swap payoff $\tilde{V}_T^{Y,G}$ in terms of $F \log F$ contracts, via the multiplier or dual-multiplier that depends only on X and G, not on the clock dynamics.

We shall solve for the G-multipliers and dual G-multipliers of the following examples of Lévy processes: Brownian motion, a two-possible-jump-size process, and the (extended) CGMY, including Variance Gamma.

7.1 Example: Brownian motion

Let X be Brownian motion, and let $G(x) = x^2 + o(x^2)$. This includes $G(x) = (e^x - 1)^2$ and $G(x) = x^2$. By Propositions 3.2 and 5.1,

$$Q^{X,G} = 2 \tag{7.1}$$

$$\tilde{Q}^{X,G} = 2. \tag{7.2}$$

For both simple and logarithmic returns, then, a log contract multiplier of 2 prices variance swaps, and an $F \log F$ multiplier of 2 prices gamma swaps – on *all* positive continuous local martingales, because their log return dynamics are all induced by time changes of drift-adjusted Brownian motion, by the following corollary to Dambis/Dubins-Schwarz [7, 9].

Proposition 7.1. *Let S be a positive continuous local martingale relative to a filtration $\{\mathcal{G}_t\}_{t\geq 0}$. If $\mathbb{E}[\log S]_T < \infty$ and $[\log S]_\infty = \infty$, then there exist a filtration $\mathbb{F} := \{\mathcal{F}_u\}_{u\geq 0}$, an $\mathbb{F}$-Brownian motion W, and a continuous $\mathbb{F}$-time change τ with $\mathbb{E}\tau_T < \infty$, such that $\log(S_t/S_0) = W_{\tau_t} - \tau_t/2$.*

Proof. See [4]. □

Including and extending the known valuations of variance and gamma swaps on continuous underliers, Propositions 3.3 and 5.2 allow general time-changes of *general* Lévy processes X with jumps, and allow *general* specifications of G-variation, not necessarily quadratic.

7.2 Example: Two jump sizes

Let X have Brownian component zero, and Lévy measure

$$\lambda_1\delta_{c_1} + \lambda_2\delta_{c_2}, \tag{7.3}$$

where δ_c denotes a point mass at c, and $c_2 < 0 < c_1$; thus up-jumps have magnitude c_1 and down-jumps have magnitude $|c_2|$. By Propositions 3.2 and 5.1, for arbitrary G,

$$Q^{X,G} = \frac{\alpha_G|\mu| + \beta_G\mu + \lambda_1 G(c_1) + \lambda_2 G(c_2)}{\lambda_1(e^{c_1} - 1 - c_1) + \lambda_2(e^{c_2} - 1 - c_2)} \tag{7.4}$$

$$\tilde{Q}^{X,G} = \frac{\alpha_G|\mu| + \beta_G\mu + \lambda_1 e^{c_1} G(c_1) + \lambda_2 e^{c_2} G(c_2)}{\lambda_1(1 - e^{c_1} + c_1 e^{c_1}) + \lambda_2(1 - e^{c_2} + c_2 e^{c_2})} \tag{7.5}$$

where

$$\mu := \lambda_1(1 - e^{c_1}) + \lambda_2(1 - e^{c_2}).$$

This prices G-variation and dual G-variation swaps via Propositions 3.3 and 5.2.

Moreover, in the case of piecewise constant paths $\mu = 0$, our valuations are enforceable by perfect replication strategies. Let

$$q^{X,G} := \frac{c_2 G(c_1) - c_1 G(c_2)}{c_1(1 - e^{c_2}) + c_2(e^{c_1} - 1)} \tag{7.6}$$

$$\tilde{q}^{X,G} := \frac{c_1 G(c_2) - c_2 G(c_1)}{c_1(e^{-c_2} - 1) + c_2(1 - e^{-c_1})}. \tag{7.7}$$

Proposition 7.2 shows that holding $Q^{X,G}$ log contracts statically, together with $q^{X,G} Z_t/F_{t-}$ futures dynamically, replicates the G-variation payoff $V_T^{Y,G}$; and that holding $\tilde{Q}^{X,G}$ log contracts statically, together with $(\tilde{q}^{X,G} + Y_{t-})Z_t/F_0$ futures dynamically, replicates the share-weighted G-variation payoff $\tilde{V}_T^{Y,G}$, where $Z_t := e^{\bar{r}_t - \bar{r}_T}$.

Proposition 7.2 (Replication)**.** *Let X have zero Brownian part, $\mu = 0$, and Lévy measure (7.3). Then*

$$V_T^{Y,G} = Q^{X,G}\log(F_0/F_T) + \int_0^T \frac{q^{X,G}}{F_{t-}}\mathrm{d}F_t \tag{7.8}$$

and

$$\tilde{V}_T^{Y,G} = \tilde{Q}^{X,G}\frac{F_T}{F_0}\log\frac{F_T}{F_0} + \int_0^T \frac{\tilde{q}^{X,G} + Y_{t-}}{F_0}\mathrm{d}F_t. \tag{7.9}$$

Proof. We have

$$X'_u = c_1 N_u^1 + c_2 N_u^2, \tag{7.10}$$

where N^1 and N^2 are independent Poisson processes. Reindexing by calendar time,

$$Y_t = c_1\tilde{N}_t^1 + c_2\tilde{N}_t^2, \tag{7.11}$$

where $\tilde{N}_t^j := N_{\tau_t}^j$ for $j = 1, 2$. By Itô's rule, the futures price $F_t = F_0\exp(Y_t)$ satisfies

$$\mathrm{d}F_t = (e^{c_1} - 1)F_{t-}\mathrm{d}\tilde{N}_t^1 + (e^{c_2} - 1)F_{t-}\mathrm{d}\tilde{N}_t^2 \tag{7.12}$$

and

$$\mathrm{d}(Y_t e^{Y_t}) = (c_1 e^{c_1} + Y_{t-}(e^{c_1} - 1))(F_{t-}/F_0)\mathrm{d}\tilde{N}_t^1 + (c_2 e^{c_2} + Y_{t-}(e^{c_2} - 1))(F_{t-}/F_0)\mathrm{d}\tilde{N}_t^2, \tag{7.13}$$

Combining (7.11) and (7.12),

$$-Q^{X,G}\mathrm{d}\log F_t + \frac{q^{X,G}}{F_{t-}}\mathrm{d}F_t = \sum_{j=1,2}(q^{X,G}e^{c_j} - q^{X,G} - Q^{X,G}c_j)\mathrm{d}\tilde{N}_t^j = G(c_1)\mathrm{d}\tilde{N}_t^1 + G(c_2)\mathrm{d}\tilde{N}_t^2. \tag{7.14}$$

which implies (7.8). Combining (7.12) and (7.13),

$$\tilde{Q}^{X,G}\mathrm{d}(Y_t e^{Y_t}) - \frac{\tilde{q}^{X,G} + Y_{t-}}{F_0}\mathrm{d}F_t = \sum_{j=1,2}(\tilde{q}^{X,G}(1 - e^{c_j}) + \tilde{Q}^{X,G}c_j e^{c_j})\frac{F_{t-}}{F_0}\mathrm{d}\tilde{N}_t^j \tag{7.15}$$

$$= G(c_1)\frac{F_t}{F_0}\mathrm{d}\tilde{N}_t^1 + G(c_2)\frac{F_t}{F_0}\mathrm{d}\tilde{N}_t^2. \tag{7.16}$$

which implies (7.9). □

7.3 Example: Generalized CGMY

Let X have no Brownian component. Let X have the *generalized CGMY* Lévy density

$$\nu(x) = \frac{C_\mathrm{d}}{|x|^{1+Y_\mathrm{d}}}e^{-M_\mathrm{d}|x|}\mathbf{1}_{x<0} + \frac{C_\mathrm{u}}{|x|^{1+Y_\mathrm{u}}}e^{-M_\mathrm{u}|x|}\mathbf{1}_{x>0}, \tag{7.17}$$

where $C_\mathrm{u}, C_\mathrm{d} > 0$, and $M_\mathrm{d} > 0$, $M_\mathrm{u} > 1$, and $Y_\mathrm{u}, Y_\mathrm{d} < 2$. In (7.18) and (7.19), the top line in each bracket is for the case $\{Y_\mathrm{u}, Y_\mathrm{d}\} \cap \{0, 1\} = \emptyset$, while the bottom line is for the case $Y_\mathrm{u} = Y_\mathrm{d} = 0$, which includes the Variance Gamma (VG) model.

According to Proposition 3.2, the $Q^{X,G}$ has denominator

$$\mathrm{Denom}(C_\mathrm{d}, C_\mathrm{u}, M_\mathrm{d}, M_\mathrm{u}, Y_\mathrm{d}, Y_\mathrm{u}) := \tag{7.18}$$
$$\begin{cases} C_\mathrm{d}\Gamma(-Y_\mathrm{d})[(M_\mathrm{d}+1)^{Y_\mathrm{d}} - M_\mathrm{d}^{Y_\mathrm{d}} - Y_\mathrm{d}M_\mathrm{d}^{Y_\mathrm{d}-1}] + C_\mathrm{u}\Gamma(-Y_\mathrm{u})[(M_\mathrm{u}-1)^{Y_\mathrm{u}} - M_\mathrm{u}^{Y_\mathrm{u}} + Y_\mathrm{u}M_\mathrm{u}^{Y_\mathrm{u}-1}], \\ \text{or} \\ C_\mathrm{d}(1/M_\mathrm{d} - \log(1 + 1/M_\mathrm{d})) - C_\mathrm{u}(1/M_\mathrm{u} + \log(1 - 1/M_\mathrm{u})) \end{cases}$$

In the case that G is simple-return variance, for $M_\mathrm{u} > 2$, the $Q^{X,G}$ has numerator

$$\mathrm{Numer}^G(C_\mathrm{d}, C_\mathrm{u}, M_\mathrm{d}, M_\mathrm{u}, Y_\mathrm{d}, Y_\mathrm{u}) := \tag{7.19}$$

$$\begin{cases} C_\mathrm{d}\Gamma(-Y_\mathrm{d})(M_\mathrm{d}^{Y_\mathrm{d}} - 2(M_\mathrm{d}+1)^{Y_\mathrm{d}} + (M_\mathrm{d}+2)^{Y_\mathrm{d}}) + C_\mathrm{u}\Gamma(-Y_\mathrm{u})((M_\mathrm{u}-2)^{Y_\mathrm{u}} - 2(M_\mathrm{u}-1)^{Y_\mathrm{u}} + M_\mathrm{u}^{Y_\mathrm{u}}) \\ \text{or} \\ -C_\mathrm{d}(\log M_\mathrm{d} - 2\log(M_\mathrm{d}+1) + \log(M_\mathrm{d}+2)) - C_\mathrm{u}(\log(M_\mathrm{u}-2) - 2\log(M_\mathrm{u}-1) + \log M_\mathrm{u}) \end{cases}$$

In the case that G belongs to the semi-moment family (4.7), if $p > \max(1, Y_\mathrm{u}, Y_\mathrm{d})$ (which can be relaxed if $U = 0$ or $D = 0$), then the $Q^{X,G}$ has numerator

$$\mathrm{Numer}^G(C_\mathrm{d}, C_\mathrm{u}, M_\mathrm{d}, M_\mathrm{u}, Y_\mathrm{d}, Y_\mathrm{u}) := C_\mathrm{u}\Gamma(p - Y_\mathrm{u})M^{Y_\mathrm{u}-p}U + C_\mathrm{d}\Gamma(p - Y_\mathrm{d})G^{Y_\mathrm{d}-p}D. \tag{7.20}$$

In any case, the multiplier equals $\mathrm{Numer}^G / \mathrm{Denom}$, and the *dual* multiplier can be expressed by evaluating the Numer^G and Denom functions at adjusted arguments. Explicitly,

$$Q^{X,G} = \frac{\mathrm{Numer}^G(C_\mathrm{d}, C_\mathrm{u}, M_\mathrm{d}, M_\mathrm{u}, Y_\mathrm{d}, Y_\mathrm{u})}{\mathrm{Denom}(C_\mathrm{d}, C_\mathrm{u}, M_\mathrm{d}, M_\mathrm{u}, Y_\mathrm{d}, Y_\mathrm{u})}, \qquad \tilde{Q}^{X,G} = \frac{\mathrm{Numer}^G(C_\mathrm{d}, C_\mathrm{u}, M_\mathrm{d}+1, M_\mathrm{u}-1, Y_\mathrm{d}, Y_\mathrm{u})}{\mathrm{Denom}(C_\mathrm{u}, C_\mathrm{d}, M_\mathrm{u}-1, M_\mathrm{d}+1, Y_\mathrm{u}, Y_\mathrm{d})} \tag{7.21}$$

by Propositions 3.2 and 5.1.

7.4 Skewness effects on valuations of quadratic-type payoffs

By (7.1,7.2), for G-variation contracts on continuous processes, where $G(x) = x^2 + o(x^2)$, the multiplier and the dual multiplier both equal 2. Introducing jumps may make the multiplier larger or smaller than 2, depending on the jump distribution and on G.

Proposition 7.3. *Let $G(x) = x^2 + o(x^2)$.*

Under the Proposition 3.3 hypotheses, let $\mathrm{JP}^{G,\nu} := \int (G(x) + 2x + 2 - 2e^x)\mathrm{d}\nu(x)$. *Then*

$$\mathbb{E}V_T^{Y,G} - 2\mathbb{E}(-Y_T) = (\mathbb{E}\tau_T)\,\mathrm{JP}^{G,\nu} \tag{7.22}$$

$$\mathrm{sgn}(Q^{X,G} - 2) = \mathrm{sgn}(\mathrm{JP}^{G,\nu}) \tag{7.23}$$

Under the Proposition 5.2 hypotheses, let $\tilde{\mathrm{JP}}^{G,\nu} := \int (e^x G(x) - 2xe^x + 2e^x - 2)\mathrm{d}\nu(x)$. *Then*

$$\mathbb{E}\tilde{V}_T^{Y,G} - 2\mathbb{E}(e^{Y_T}Y_T) = (\tilde{\mathbb{E}}\tau_T)\,\tilde{\mathrm{JP}}^{G,\nu} \tag{7.24}$$

$$\mathrm{sgn}(\tilde{Q}^{X,G} - 2) = \mathrm{sgn}(\tilde{\mathrm{JP}}^{G,\nu}). \tag{7.25}$$

Proof. The numerator in (3.5), minus twice the denominator, equals $\mathrm{JP}^{G,\nu}$. This justifies (7.23) and also the last step in

$$\mathbb{E}V_T^{Y,G} - 2\mathbb{E}(-Y_T) = (Q^{X,G} - 2)\mathbb{E}(-Y_T) = (Q^{X,G} - 2)\mathbb{E}\tau_T\mathbb{E}(-\tilde{X}_1') = (\mathbb{E}\tau_T)\,\mathrm{JP}^{G,\nu},$$

where the middle step is by Wald's identity. Likewise, the numerator in (5.6), minus twice the denominator, equals $\tilde{\mathrm{JP}}^{G,\nu}$. This justifies (7.25) and also the last step in

$$\mathbb{E}\tilde{V}_T^{Y,G} - 2\mathbb{E}(e^{Y_T}Y_T) = (\tilde{Q}^{X,G} - 2)\mathbb{E}(e^{Y_T}Y_T) = (\tilde{Q}^{X,G} - 2)\tilde{\mathbb{E}}\tau_T\tilde{\mathbb{E}}(-X_1') = (\tilde{\mathbb{E}}\tau_T)\,\tilde{\mathrm{JP}}^{G,\nu},$$

where the middle step is by Wald's identity. $\square$

Table 1: Jump premia for various contracts G and, in the case of pre-jump weights, $G^* := e^{-x}G$

G	F-weighted?	Jump Premium	...in terms of skewness
x^2	No	$\mathrm{JP}^{G,\nu} = \int(-e^x + x^2/2 + x + 1)\mathrm{d}\nu(x)$	$= -\frac{1}{3}\,\mathrm{SK}_1(\nu)$
		$= \int(-x^3/3 - x^4/12 + O(x^5))\mathrm{d}\nu(x)$	
x^2	Yes, post	$\tilde{\mathrm{JP}}^{G,\nu} = \int(e^x x^2/2 - xe^x + e^x - 1)\mathrm{d}\nu(x)$	$= +\frac{1}{3}\,\mathrm{SK}_2(\nu)$
		$= \int(x^3/3 + x^4/4 + O(x^5))\mathrm{d}\nu(x)$	
x^2	Yes, pre	$\tilde{\mathrm{JP}}^{G^*,\nu} = \int(x^2/2 - xe^x + e^x - 1)\mathrm{d}\nu(x)$	$= -\frac{2}{3}\,\mathrm{SK}_3(\nu)$
		$= \int(-2x^3/3 - x^4/4 + O(x^5))\mathrm{d}\nu(x)$	
$(e^x-1)^2$	No	$\mathrm{JP}^{G,\nu} = \int(-e^x + (e^x-1)^2/2 + x + 1)\mathrm{d}\nu(x)$	$= +\frac{2}{3}\,\mathrm{SK}_4(\nu)$
		$= \int(2x^3/3 + x^4/2 + O(x^5))\mathrm{d}\nu(x)$	
$(e^x-1)^2$	Yes, post	$\tilde{\mathrm{JP}}^{G,\nu} = \int(e^x(e^x-1)^2/2 - xe^x + e^x - 1)\mathrm{d}\nu(x)$	$= +\frac{4}{3}\,\mathrm{SK}_5(\nu)$
		$= \int(4x^3/3 + 11x^4/6 + O(x^5))\mathrm{d}\nu(x)$	
$(e^x-1)^2$	Yes, pre	$\tilde{\mathrm{JP}}^{G^*,\nu} = \int((e^x-1)^2/2 - xe^x + e^x - 1)\mathrm{d}\nu(x)$	$= +\frac{1}{3}\,\mathrm{SK}_6(\nu)$
		$= \int(x^3/3 + x^4/3 + O(x^5))\mathrm{d}\nu(x)$	

The left-hand side of (7.22) is the G-variation contract's value, minus its value in the absence of jump risk. The equality (7.22), therefore, explains why we describe JP as the *jump premium.* Likewise, (7.24) explains why we regard $\tilde{\mathrm{JP}}$ as the share-weighted jump premium. By (7.23,7.25), a positive (negative) jump premium is equivalent to a multiplier greater (smaller) than 2.

For quadratic-type G, the jump premium is a constant multiple of the Lévy measure's *skewness.*

Definition 7.4. *Let* SK *be a real-valued function defined on a set of Lévy measures. We say that* SK *is a* (noncentral) generalized skewness *if there exists a Borel function $\psi(x) = x^3 + O(x^4)$ such that, for all Lévy measures ν with $\int|\psi|\mathrm{d}\nu < \infty$, the function* SK *is defined and* $\mathrm{SK}(\nu) = \int \psi \mathrm{d}\nu$.

Proposition 7.5. *Under the Proposition 7.3 hypotheses, if $G(x) = x^2 + c_G x^3 + O(x^4)$ for some constant c_G, then there exists a generalized skewness* SK *and a generalized skewness* $\tilde{\mathrm{SK}}$ *such that*

$$\mathrm{JP}^{G,\nu} = (c_G - 1/3)\,\mathrm{SK}(\nu) \quad \textit{and} \quad \tilde{\mathrm{JP}}^{G,\nu} = (c_G + 1/3)\tilde{\mathrm{SK}}(\nu).$$

Proof. Apply Taylor's theorem to each e^x in the integrands in Proposition 7.3 □

Table 1 illustrates Proposition 7.3 by expressing jump premia for various G contracts in terms of the Lévy measure's skewness. The subscripts on SK emphasize that each SK is a distinct member of the generalized skewness family in Definition 7.4. With logarithmic returns, the variance swap and pre-jump gamma swap are "short" skewness, while the post-jump gamma swap is "long" skewness. With simple returns, the variance swap and both gamma swaps are all long skewness.

7.5 Multipliers of empirically calibrated processes

Carr-Geman-Madan-Yor [2] calibrate various time-changed Lévy processes to four cross-sections of S&P 500 options data from March, June, September, and December 2000, producing the parameter estimates in Table 2. In Table 3, we compute the multipliers and dual multipliers associated with those parameter estimates.

The driving Lévy process X has CGMY, VG, or NIG dynamics. In each case, the time-change is by a CIR stochastic clock. We do not report the estimated parameters of the CIR clock, because the multiplier depends only on X.

In the "Variance", "Simple variance", and "Third moment" columns, we have respectively $G(x) = x^2$, $G(x) = (e^x - 1)^2$, and $G(x) = x^3$. For each G we compute, via Propositions 3.2 and 5.1, the unweighted, post-F-weighted, and pre-F-weighted multipliers, by which we mean, respectively, $Q^{X,G}$, $\tilde{Q}^{X,G}$, and $\tilde{Q}^{X,G^*}$, where $G^*(x) := e^{-x}G(x)$ includes the pre-jump weighting adjustment.

The results are consistent with negatively skewed jumps: The third-moment multipliers are negative; and (as expected in light of Table 1), the variance and post-F-weighted variance multipliers are greater than 2, while the pre-F-weighted variance and all simple-return variance multipliers are smaller than 2.

For most of the models in Table 3, using the no-jump multiplier 2.00 in the presence of jumps would underprice variance swaps by 5-10%, and overprice gamma swaps by 5-10%, and underprice simple-return variance swaps by 8-20%.

Table 2: Calibration of time-changed Lévy processes, by Carr-Geman-Madan-Yor [2], using four cross-sections of S&P 500 options data.

Lévy driver	Data	Lévy parameters
CGMY	Mar	$C_\mathrm{d}/C_\mathrm{u} = 0.2883, M_\mathrm{d} = 0.697, M_\mathrm{u} = 22.0, Y_\mathrm{d} = 1.45, Y_\mathrm{u} = -3.65$
CGMY	Jun	$C_\mathrm{d}/C_\mathrm{u} = 0.0526, M_\mathrm{d} = 0.423, M_\mathrm{u} = 24.6, Y_\mathrm{d} = 1.67, Y_\mathrm{u} = -4.51$
CGMY	Sep	$C_\mathrm{d}/C_\mathrm{u} = 0.0676, M_\mathrm{d} = 1.64, M_\mathrm{u} = 16.9, Y_\mathrm{d} = 1.54, Y_\mathrm{u} = -2.90$
CGMY	Dec	$C_\mathrm{d}/C_\mathrm{u} = 0.0855, M_\mathrm{d} = 3.68, M_\mathrm{u} = 52.9, Y_\mathrm{d} = 1.22, Y_\mathrm{u} = -2.12$
VG	Mar	$M_\mathrm{d} = 7.33, M_\mathrm{u} = 32.4$
VG	Jun	$M_\mathrm{d} = 11.0, M_\mathrm{u} = 30.1$
VG	Sep	$M_\mathrm{d} = 12.4, M_\mathrm{u} = 33.6$
VG	Dec	$M_\mathrm{d} = 11.7, M_\mathrm{u} = 42.7$
NIG	Mar	$\alpha = 96.4, \beta = -92.0$
NIG	Jun	$\alpha = 69.7, \beta = -62.1$
NIG	Sep	$\alpha = 99.8, \beta = -91.1$
NIG	Dec	$\alpha = 274.8, \beta = -265.4$

Table 3: Multipliers $Q^{X,G}$ and $\tilde{Q}^{X,G}$ computed from the calibrated Lévy parameters in Table 2

Lévy driver		Variance			Simple variance			Third moment		
			F-weighted			F-weighted			F-weighted	
			Post	Pre		Post	Pre		Post	Pre
CGMY	Mar	2.43	1.75	2.85	1.53	1.37	1.80	-1.92	-0.57	-2.25
CGMY	Jun	2.37	1.81	2.70	1.62	1.53	1.85	-1.85	-0.42	-2.11
CGMY	Sep	2.17	1.87	2.33	1.76	1.62	1.89	-0.61	-0.33	-0.65
CGMY	Dec	2.13	1.88	2.27	1.78	1.63	1.89	-0.45	-0.31	-0.48
VG	Mar	2.17	1.85	2.35	1.72	1.52	1.86	-0.56	-0.41	-0.60
VG	Jun	2.10	1.91	2.20	1.83	1.69	1.92	-0.32	-0.25	-0.34
VG	Sep	2.09	1.92	2.18	1.84	1.72	1.92	-0.28	-0.23	-0.30
VG	Dec	2.10	1.91	2.21	1.82	1.67	1.91	-0.33	-0.27	-0.34
NIG	Mar	2.21	1.82	2.45	1.66	1.43	1.84	-0.73	-0.49	-0.81
NIG	Jun	2.12	1.89	2.25	1.79	1.63	1.90	-0.39	-0.31	-0.42
NIG	Sep	2.11	1.90	2.22	1.81	1.66	1.91	-0.35	-0.28	-0.36
NIG	Dec	2.10	1.90	2.21	1.82	1.67	1.91	-0.33	-0.27	-0.35

Table 4: Hedges minimizing expected ϱ-variation of tracking error Π (H-variation less G-variation)

Contract G		Optimal coefficients of H_0, H_1, H_2					
		Risk measured by $\varrho(x) = (x \wedge 0)^2$			Risk measured by $\varrho(x) = x^2$		
		$\hat{a}_0$	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_0$	$\hat{a}_1$	$\hat{a}_2$
$\frac{1}{100}\|x\|$	Mar	0.58	0.58	-0.18	0.24	0.25	-0.09
$\frac{1}{100}\|x\|$	Jun	1.18	1.18	-0.43	0.68	0.69	-0.28
$\frac{1}{100}\|x\|$	Sep	1.43	1.43	-0.54	0.84	0.84	-0.35
$\frac{1}{100}\|x\|$	Dec	1.30	1.30	-0.48	0.66	0.66	-0.27
$(x \wedge 0)^2$	Mar	-1.47	-1.46	1.62	-0.93	-0.91	1.34
$(x \wedge 0)^2$	Jun	-4.97	-4.96	3.24	-3.20	-3.16	2.31
$(x \wedge 0)^2$	Sep	-5.47	-5.45	3.49	-3.51	-3.48	2.47
$(x \wedge 0)^2$	Dec	-2.95	-2.94	2.33	-1.97	-1.96	1.81
x^3	Mar	6.70	6.71	-3.35	7.79	7.80	-3.85
x^3	Jun	6.43	6.43	-3.22	7.12	7.12	-3.54
x^3	Sep	6.38	6.38	-3.19	7.00	7.00	-3.49
x^3	Dec	6.43	6.43	-3.21	7.10	7.10	-3.53

8 Hedging G-variation using H-variation

Suppose that we hedge a G-variation payoff, by using an H-variation payoff, assuming that the latter payoff is exactly available.

Example 8.1. Consider the following cases of H-variation: $H = H_0$, $H = H_1$, and $H = H_2$, where

$$H_0(x) := e^x - 1 \tag{8.1}$$

$$H_1(x) := -x \tag{8.2}$$

$$H_2(x) := x^2. \tag{8.3}$$

The H_0-variation payoff is exactly attainable, by collecting the profit/loss from dynamically holding $1/F_{t-}$ futures. This follows from applying Itô's rule:

$$\int_0^T \frac{1}{F_{t-}} \mathrm{d}F_t = \frac{1}{2}[Y^{\mathrm{c}}]_T + \log(F_T/F_0) + \sum_{0<t\leq T} \left(e^{\Delta Y_t} - 1 - \Delta Y\right) \tag{8.4}$$

which equals V_T^{Y,H_0} because decomposing $e^x - 1 = x + x^2/2 + (e^x - 1 - x - x^2/2)$ satisfies Def 2.1.

The H_1-variation payoff is exactly available, by statically holding 1 log contract.

The H_2-variation payoff is exactly available, if a static holding of 1 variance swap is available. In that case, linear combinations $H \in \operatorname{span}\{H_0, H_1, H_2\} := \{a_0 H_0 + a_1 H_1 + a_2 H_2 : a_0, a_1, a_2 \in \mathbb{R}\}$ are then also exactly attainable.

In markets where other types of variation swaps are liquid (or exactly synthesizable), they become available as hedging instruments, and expand the set of exactly attainable H.

Let us hedge a short position in G-variation by going long H-variation. The resulting profit/loss process (in the realized sense, not a mark-to-market sense), or "hedging error" or "tracking error" is defined by

$$\Pi_t := V_t^{Y,H} - V_t^{Y,G}. \tag{8.5}$$

We will apply to Π some measures of hedge risk: variance and ϱ-variation, to be defined below.

Two possible modifications of the hedging error definition (8.5) do not add any complication. First, one could modify the definition by adding to it an initial profit/loss, such as the difference between the time-0 prices of the H-variation and G-variation contracts; this modification would not affect our analysis, because all of our hedge risk measures will be invariant to translation of Π by a constant. Second, one could consider hedging a *long* position in G-variation by going *short* H-variation; if ϱ is asymmetric, then this modification *would* affect the measurement of risk by ρ-variation, but in a way that falls within the scope of our results, which can simply be applied to $-G$ and $-H$.

In parallel to our conclusions about valuation, some of our conclusions about hedging (8.8,8.20) will express each hedge risk measure as a multiplier times the log contract value, where the multiplier does not depend on the time change. This allows us to solve, at least numerically, for hedges H which minimize the risk measure.

8.1 Hedge risk, measured by terminal variance

First consider measuring the risk of the tracking error Π by using terminal variance $\mathrm{Var}(\Pi_T)$.

Lemma 8.1. *For $k = 1, 2$, let $G_k \in \mathbb{V}(X)$ and let β_{G_k} be the coefficient of x in an arbitrary decomposition of $G_k(x)$ consistent with Definition 2.1. Then G_1G_2 has decomposition*

$$G_1G_2(x) = \beta_{G_1}\beta_{G_2}x^2 + \Lambda(x)$$

for some $\Lambda \in \mathbb{J}(X)$.

Moreover, $G_1G_2 \in \mathbb{W}(X)$ if and only if $\int |G_1G_2|\mathrm{d}\nu < \infty$, where ν is the Lévy measure of X.

Proof. If X has infinite variation, then $G_k(x) = \beta_{G_k}x + o(x)$ for $k = 1, 2$, by Definition 2.1. So

$$G_1(x)G_2(x) - \beta_{G_1}\beta_{G_2}x^2 = o(x^2) \in \mathbb{J}(X). \tag{8.6}$$

If X has finite variation, then $\mathbb{J}(X)$ contains the functions $G_1(x)$ and x^2, so

$$G_1(x)G_2(x) - \beta_{G_1}\beta_{G_2}x^2 = O(G_1(x) + x^2) \in \mathbb{J}(X). \tag{8.7}$$

as claimed. The "if and only if" conclusion follows, because (8.6) reduces (3.1) into (3.2). $\square$

Proposition 8.2 (Hedging G-variation using H-variation. Risk measured by terminal variance)**.** *Let $G, H \in \mathbb{W}(X)$. Let $R := H - G$. Assume that $R^2 \in \mathbb{W}(X)$ and $\mathbb{E}\tau_T < \infty$.*

(a) If the hedge is initially cost-neutral (also known as "dollar-neutral" to traders of USD-denominated assets), meaning that $\mathbb{E}V_T^{Y,G} = \mathbb{E}V_T^{Y,H}$ or equivalently $Q^{X,G} = Q^{X,H}$, then

$$\mathrm{Var}(\Pi_T) = Q^{X,R^2}\mathbb{E}(-Y_T). \tag{8.8}$$

So the hedging error's variance is again a multiple of the log contract value, where the multiplier, given by (8.10), depends only on $H - G$ and the Lévy driver, not the time change.

(b) If X and the clock τ are independent, then (without assuming cost-neutrality),

$$\mathrm{Var}(\Pi_T) = Q^{X,R^2}\mathbb{E}(-Y_T) + (Q^{X,R})^2(\mathbb{E}Y_T^2 - (\mathbb{E}Y_T)^2 - Q^{X,\mathrm{sq}}\mathbb{E}(-Y_T)). \tag{8.9}$$

So the hedging error's variance is a function of the log contract value $-\mathbb{E}Y_T$, the log-squared contract value $\mathbb{E}Y_T^2$, and $H - G$, and the Lévy driver X – regardless of the time change.

In both cases we have

$$Q^{X,R^2} = \frac{\beta_R^2\sigma^2 + \int R^2(x)\mathrm{d}\nu(x)}{\sigma^2/2 + \int(e^x - 1 - x)\mathrm{d}\nu(x)}. \tag{8.10}$$

In (8.9), sq *denotes the function $x \mapsto x^2$; thus $Q^{X,\mathrm{sq}} = (\sigma^2 + \int x^2\mathrm{d}\nu(x))/(\sigma^2/2 + \int(e^x - 1 - x)\mathrm{d}\nu(x))$.*

Proof. Definitions 2.2 and 3.1 imply that $R \in \mathbb{W}(X)$ and $\Pi := V^{Y,H} - V^{Y,G} = V^{Y,R}$.

Let $m = \mathbb{E}V_1^{X',R}$. By the Lévy, strong Markov, and martingale properties of $V^{X',R} - mt$, along with finiteness of $\mathbb{E}\tau_T$ and Wald's first identity, we have that $V_t^{Y,R} - m\tau_t$ is a martingale.

Moreover, by the same argument as (2.30), the quadratic variation $[V^{X',R}]_1$ equals a constant plus $\sum_{0<s\leq 1} R(\Delta X_s)^2$. Therefore $\mathbb{E}[V^{X',R}]_1 < \infty$, by $R^2 \in \mathbb{W}(X)$ and Lemma 8.1. Equivalently, $\operatorname{Var} V_1^{X',R} < \infty$ by ([4] Proposition 2.1). Hence $\mathbb{E}(V_t^{Y,R} - m\tau_t)^2 < \infty$ by ([11] Theorem 3i). Indeed

$$\mathbb{E}(V_T^{Y,R} - m\tau_T)^2 = \mathbb{E}[V^{Y,R} - m\tau.]_T = \mathbb{E}[V^{Y,R}]_T \tag{8.11}$$

$$= \mathbb{E}V_T^{Y,R^2} \tag{8.12}$$

$$= Q^{X,R^2}\mathbb{E}(-Y_T) \tag{8.13}$$

where (8.11) is by, for instance, Protter [17] Corollary II.27.3 together with the continuity and finite variation of $\tau.$; (8.12) is by the tower property (Proposition 2.4); and (8.13) is by Proposition 3.3.

In the case (a), $m = 0$, so we are done. In the case (b),

$$\mathbb{E}(V_T^{Y,R})^2 = \mathbb{E}(V_T^{Y,R} - m\tau_T)^2 + 2m\mathbb{E}(\tau_T V_T^{Y,R}) - m^2\mathbb{E}\tau_T^2 \tag{8.14}$$

$$= \mathbb{E}(V_T^{Y,R} - m\tau_T)^2 + m^2\mathbb{E}\tau_T^2 \tag{8.15}$$

$$= \mathbb{E}(V_T^{Y,R} - m\tau_T)^2 + (Q^{X,R})^2(\mathbb{E}Y_T^2 - Q^{X,\mathrm{sq}}\mathbb{E}(-Y_T)) \tag{8.16}$$

where (8.15) is by conditioning on τ and using the independence of X and τ; and (8.16) is true, again using independence, because

$$Q^{X,\mathrm{sq}}\mathbb{E}(-Y_T) = \mathbb{E}[Y]_T = \mathbb{E}(Y_T - \tau_T\mathbb{E}X_1')^2 \tag{8.17}$$

$$= \mathbb{E}Y_T^2 - \mathbb{E}(\tau_T\mathbb{E}X_1')^2 = \mathbb{E}Y_T^2 - m^2\mathbb{E}\tau_T^2/(Q^{X,R})^2. \tag{8.18}$$

Subtracting $(\mathbb{E}V_T^{Y,R})^2 = (Q^{X,R}\mathbb{E}Y_T)^2$ from (8.16) gives

$$\operatorname{Var}(V_T^{Y,R}) = Q^{X,R^2}\mathbb{E}(-Y_T) + (Q^{X,R})^2(\mathbb{E}Y_T^2 - (\mathbb{E}Y_T)^2 - Q^{X,\mathrm{sq}}\mathbb{E}(-Y_T)), \tag{8.19}$$

which verifies (8.9). Finally, (8.10) comes from Proposition 3.2 and Lemma 8.1. □

8.2 Hedge risk, measured by expected ϱ-variation

Whereas Proposition 8.2 measured hedge risk by the variance of the random variable Π_T, this section measures hedge risk by $\mathbb{E}V_T^{\Pi,\varrho}$, the expected ϱ-variation of the *process* Π, where ϱ is chosen in accordance with some notion of risk.

Example 8.2. The choice $\varrho(x) = x^2$ measures risk according to the quadratic variation of the hedged portfolio – a path dependent quantity, in contrast to the previous section's terminal variance, which depended only on Π_T.

Depending on the choice of ϱ, this ϱ-variation measure of hedge risk has the flexibility to treat losses asymmetrically from gains – a feature desirable from a risk-management standpoint.

Example 8.3. The "realized semivariance" $\varrho(x) = (x \wedge 0)^2$ penalizes losses quadratically without penalizing gains.

Here we assume neither the cost-neutrality nor the independence condition of Proposition 8.2.

Proposition 8.3 (Hedging G-variation using H-variation. Risk measured by expected ϱ-variation)**.** *Let $R := H - G$. If $\varrho \in \mathbb{W}(V^{X',R})$ then*

$$\mathbb{E}V_T^{\Pi,\varrho} = Q^{X,\varrho\circ R}\mathbb{E}(-Y_T). \tag{8.20}$$

So the hedging error's expected ϱ-variation is again a multiple of the log contract value, where the multiplier depends only on ϱ and $H - G$ and the Lévy driver, not the time change.

Proof. Combine our "tower" proposition for G-variation and our log-contract conclusion, to obtain

$$\mathbb{E}V_T^{\Pi,\varrho} = \mathbb{E}V_T^{V^{Y,R},\varrho} = \mathbb{E}V_T^{Y,\varrho\circ R} = Q^{X,\varrho\circ R}\mathbb{E}(-Y_T) \tag{8.21}$$

by Propositions 2.4 and 3.3. □

In Propositions 8.2 and 8.3, our setting is, in two directions, more general than Crosby's [6], because we hedge contracts on G-variation, not just quadratic variation, and we measure risk by expected ϱ-variation (including asymmetric ϱ), not just variance. (In a different direction, our setting is less general because our Lévy driver is one-dimensional, but this gives us explicit formulas (8.8,8.9,8.20) expressing the risk metrics directly in terms of log contract values.)

8.3 Optimal hedges of G-variation using H-variation: quadratic ρ

Given a family of exactly attainable H-variation payoffs, if we can find H which minimizes $Q^{X,\varrho\circ R}$, then it minimizes $\mathbb{E}V_T^{\Pi,\varrho}$. Such minimal H then hedges G-variation *optimally*, with respect to ϱ-variation of the tracking error Π.

Solving for H is tractable, and in some cases straightforward, because $Q^{X,\varrho\circ R} = Q^{X,\varrho\circ(H-G)}$ depends on H in a way that is easily computable and in some cases expressible explicitly.

Indeed, for the quadratic case $\varrho(x) = x^2$, Proposition 8.5 will show that H is optimized by solving an explicit linear system of equations that does not depend on the time change.

Proposition 8.4. *Let $\mathbb{W}^2(X) := \{G \in \mathbb{V}(X) : G^2 \in \mathbb{W}(X)\}$. The operation*

$$\langle G_1|G_2\rangle := Q^{X,G_1G_2}, \tag{8.22}$$

is a well-defined mapping $\mathbb{W}^2(X) \times \mathbb{W}^2(X) \to \mathbb{R}$. Under this operation $\mathbb{W}^2(X)$ is a real inner product space.

Proof. If $G_1, G_2 \in \mathbb{W}^2(X)$, then $\int G_k^2 \mathrm{d}\nu < \infty$ for $k = 1, 2$ by Lemma 8.1, so $\int |G_1G_2|\mathrm{d}\nu < \infty$ by Cauchy-Schwarz, hence $G_1G_2 \in \mathbb{W}(X)$, again by Lemma 8.1. The operation is therefore well-defined. The remaining inner product properties are easily verified. □

Proposition 8.5. *Let $G \in \mathbb{W}^2(X)$, and let $H_1, \ldots, H_M \in \mathbb{W}^2(X)$.*

Let $\mathbb{H} := \operatorname{span}\{H_1, \ldots, H_M\} := \{a_1 H_1 + \cdots + a_M H_M : a_m \in \mathbb{R}\}$. If $\hat{a}_1, \ldots, \hat{a}_M$ solve the linear system

$$\sum_{m=1}^{M} \hat{a}_m Q^{X, H_m H_n} = Q^{X, G H_n}, \qquad n = 1, \ldots, M, \tag{8.23}$$

then the expected quadratic variation of the hedging error $\Pi := V^{Y,H} - V^{Y,G}$ is minimized, among all $H \in \mathbb{H}$, by $H = \hat{a}_1 H_1 + \cdots + \hat{a}_M H_M$.

Proof. By standard results on best approximation in subspaces of inner product spaces, any solution of the normal equations (8.23) minimizes $Q^{X,(H-G)^2}$. □

8.4 Optimal hedges: numerical examples including asymmetric ϱ

Although quadratic variation of the tracking error is easily minimized via Proposition 8.5, the quadratic criterion has a drawback, in that $\varrho(x) = x^2$ penalizes gains and losses symmetrically.

In this section we allow general ϱ, and hence a wider range of ways to measure the variability of the tracking error. For general ϱ, without Proposition 8.5, the optimization will be numerical. The problem, of finding $H \in \operatorname{span}\{H_1, \ldots, H_M\}$ to minimize $\mathbb{E} V^{\Pi,\varrho}$, takes the equivalent form

$$\min\{Q^{X, \varrho\circ(a_1 H_1 + \cdots + a_M H_M - G)} : (a_1, \ldots, a_M) \in \mathbb{R}^M\}, \tag{8.24}$$

For a one-sided risk measure such as semivariance, the unconstrained optimization problem (8.24) will not be well-posed, because Q can always be improved, for instance, by increasing the coefficient of the x^2 component of H (or equivalently, by adding variance swaps to the hedge portfolio). It is therefore appropriate to introduce a budget constraint, such as $\mathbb{E} V_T^{Y,H} \le \mathbb{E} V_T^{Y,G}$, or equivalently

$$\sum_{m=1}^{M} a_m Q^{X, H_m} \le Q^{X,G}, \tag{8.25}$$

an explicit linear constraint that again does not depend on the time-change.

In Table 4 we solve numerically the minimization problem (8.24) subject to constraint (8.25). We take time-changed VG processes (with the same parameters as in Table 2), which have finite variation, so that total variation contracts are well-defined. We examine three different contracts: total variation $G(x) = |x|$ (times notional 1/100, chosen for rough comparability with variance contracts), semivariance $G(x) = (x \wedge 0)^2$, and third moment $G(x) = x^3$. For hedging purposes, we assume the availability of H-variation payoffs, where $H \in \{H_0, H_1, H_2\}$ as defined above. For the risk criterion, we take two cases: semivariance $\varrho(x) = (x \wedge 0)^2$ and quadratic variation $\varrho(x) = x^2$.

The Table 4 results have the following intuitive interpretation. Both optimization criteria found hedges that turn out to be nearly "delta-neutral" in the sense that $a_0 - a_1$ nearly vanishes. The two criteria disagree, however, on the sizes of the positions; in hedging the total variation and semivariance contracts, where $G \ge 0$ everywhere (which makes plausible the heuristic that optimally $H \geq 0$ also), optimizing for semi-quadratic ϱ results in larger absolute hedge position sizes than optimizing for quadratic ϱ. Intuitively, scaling up a positive H may possibly worsen a two-sided risk measure, whereas it improves a one-sided measure that does not penalize profits. This same phenomenon occurs in hedging the $G(x) = x^3$ contract, which has mixed sign, but the $x < 0$ region is more influential, because we have a left-skewed jump distribution; within this region, we have $G \leq 0$ (which makes plausible the heuristic that in this region, optimally $H \leq 0$ also), therefore scaling *down* the absolute size of H improves the one-sided risk measure.

9 Conclusion

Under arbitrary exponential Lévy dynamics, stochastically time-changed by an arbitrary continuous clock having arbitrary correlation with the Lévy driver X, we prove that a multiple Q of the log contract prices the G-variation swap, and that a multiple $\tilde{Q}$ of the $F \log F$ contract prices the share-weighted G-variation swap, under integrability conditions. The *multiplier* Q and *dual multiplier* $\tilde{Q}$ depend only on G and the characteristics of X, not on the time change. Hence our results allow stochastic volatility and jump arrival rates, while avoiding the model risk of misspecifying or miscalibrating those rates.

We calculate explicitly the multipliers for various examples of Lévy drivers. We recover the standard no-jump valuation formula as a special case, because all positive continuous martingales are time-changes of driftless geometric Brownian motion, which has multiplier 2 for quadratic-type G. We then find the multipliers for general G under jump dynamics, including time-changes of CGMY, VG, and two-jump-size processes; in the latter case, our valuations admit enforcement via hedging strategies which perfectly replicate G-variation (or share-weighted G-variation) by holding log contracts (or $F \log F$ contracts) statically and trading the underlier dynamically.

We show that the direction in which the multiplier differs from 2 is determined (positively or negatively, depending on G) by the Lévy measure's skewness. Given some examples of empirically-calibrated Lévy parameters, our computations of the associated multipliers show jump premia of signs consistent with negatively-skewed jump risk. In most of the examples at hand, using the no-jump multiplier 2.00 in the presence of jumps would underprice variance swaps by 5-10%, and overprice gamma swaps by 5-10%, and underprice simple-return variance swaps by 8-20%.

We hedge G-variation, for general X, using a family of strategies indexed by functions H. The riskiness of each strategy's hedging error can be measured by a family of risk statistics, indexed by functions ϱ. The resulting risk measurement, we prove, equals the $\varrho \circ (H - G)$-variation. Some key statistics of hedging errors, therefore, have expected values again given by explicit multiples of the log contract value. We then solve for strategies to minimize various measures of hedge risk.

This paper extends the existing literature in several directions. Extending the known gamma swap solution for continuous underlying paths, we introduce empirically relevant *jumps*, together with an arbitrary time-change, which can generate stochastic jump intensity and can reproduce options-implied volatility skews at both long and short expiries. Extending the analysis of unweighted variance in [4], we solve for valuations of *share-weighted* variance, motivated by the popular features of gamma swaps. Extending both the gamma swap and Carr-Lee-Wu [4] analyses, which were based on quadratic variation, we price the (unweighted and share-weighted) *G-variation*, which includes not just quadratic variation, but also other variability measures relevant in finance; for instance, contracts on total variance and capped-jump variance allow traders (wary of extreme event risk in the aftermath of recent financial crises) to sell volatility while avoiding tail risk exposure that blows up quadratically. Finally, extending the applications of our tools from pricing to *hedging*, we show that finding a risk-minimizing hedge of G-variation reduces to a tractable matter of minimizing an explicit multiplier.

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References

[1] Mark Broadie and Ashish Jain. Pricing and hedging volatility derivatives. *Journal of Derivatives*, 15(3):7–24, 2008.

[2] Peter Carr, Hélyette Geman, Dilip Madan, and Marc Yor. Stochastic volatility for Lévy processes. *Mathematical Finance*, 13(3):345–382, 2003.

[3] Peter Carr, Hélyette Geman, Dilip Madan, and Marc Yor. Pricing options on realized variance. *Finance and Stochastics*, 9(4):453–475, 2005.

[4] Peter Carr, Roger Lee, and Liuren Wu. Variance swaps on time-changed Lévy processes. *Finance and Stochastics*, Forthcoming.

[5] Peter Carr and Dilip Madan. Towards a theory of volatility trading. In R. Jarrow, editor, *Volatility*, pages 417–427. Risk Publications, 1998.

[6] John Crosby. Optimal hedging of variance derivatives. University of Glasgow, 2010.

[7] K. E. Dambis. On the decomposition of continuous submartingales. *Theory of Probability and Its Applications*, 10(3):401–410, 1965.

[8] Emanuel Derman, Kresimir Demeterfi, Michael Kamal, and Joseph Zou. A guide to volatility and variance swaps. *Journal of Derivatives*, 6(4):9–32, 1999.

[9] Lester E. Dubins and Gideon Schwarz. On continuous martingales. *Proceedings of the National Academy of Sciences of the United States of America*, 53(5):913–916, 1965.

[10] Bruno Dupire. Arbitrage pricing with stochastic volatility. Société Générale, 1992.

[11] W. J. Hall. On Wald's equations in continuous time. *J Applied Probability*, 7:59–68, 1970.

[12] Jean Jacod. *Calcul Stochastique et Problèmes de Martingales*. Lecture Notes in Mathematics. Springer-Verlag, 1979.

[13] Jean Jacod. Asymptotic properties of realized power variations and related functionals of semimartingales. *Stochastic Processes and Their Applications*, 118:517–559, 2008.

[14] Uwe Küchler and Michael Sørensen. *Exponential Families of Stochastic Processes*. Springer-Verlag, 1997.

[15] Nicolas Mougeot. Variance swaps and beyond. BNP Paribas, 2005.

[16] Anthony Neuberger. Volatility trading. London Business School working paper, 1990.

[17] Philip Protter. *Stochastic Integration and Differential Equations*. Springer, 2nd edition, 2004.

[18] Ken-Iti Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, 1999.