CALCULUS 153: MIDTERM 1 SOLUTIONS

Please answer all questions in a blue book that's provided to you (even the true/false). Don't forget to write your name. There are two sides to this exam.

Problem 1 (16 points). Determine the least upper bound and greatest lower bound of the following sets, or state that they do not exist. You do not need to justify your answer (4 points each).

(1)
$$(\pi, \infty),$$

(2)
$$\{x : |x - 2| < 1\},$$

(3)
$$\left\{\sin(\frac{\pi}{2n}) : n = 1, 2, \ldots\right\},$$

(4)
$$\{.9, .99, .999, \ldots\}.$$

Solution

(1) The lub does not exist and the glb is π.
(2) The lub is 3 and the glb is 1.
(3) The lub is 1 and the glb is 0.
(4) The lub is 1 and the glb is .9.

Problem 2 (21 points). For each of the following sequences, determine whether the sequence converges or diverges. If it converges, find its limit. Show your work. (7 points each).

(1)
$$a_n = (2n)^{1/n};$$

(2)
$$b_n = \cos\left(\pi + \frac{\ln n}{n}\right).$$

(3)
$$c_n = \frac{3^n}{n!}\cos(\ln n).$$

Solution

(1) Note that
$$\lim_{n\to\infty}(2n)^{1/n} = \lim_{n\to\infty} 2^{1/n} \cdot \lim_{n\to\infty} n^{1/n} = 1 \cdot 1 = 1.$$
We have used two important limits here ($x^{1/n} \to 1$ for $x > 0$ and $n^{1/n} \to 1$).

(2) Since cos is continuous,

$$\lim_{n\to\infty} \cos\left(\pi + \frac{\ln}{n}\right) = \cos\left(\pi + \lim_{n\to\infty} \frac{\ln n}{n}\right) = \cos(\pi + 0) = -1.$$

Here we have used the important limit $\ln n/n \to 0$.

(3) Since $-1 \leq \cos(\ln n) \leq 1$, it follows that

$$-\frac{3^n}{n!} \leq \frac{3^n}{n!}\cos(\ln n) \leq \frac{3^n}{n!}.$$

Since $3^n/n! \to 0$ (important limit), it follows by the squeeze theorem, that

$$\lim_{n\to\infty} \frac{3^n}{n!}\cos(\ln n) = 0.$$

Problem 3 (21 points). Find the following limits. Show your work (7 points each).

(1)

$$\lim_{x\to 0} \frac{\sin(2x) + x^2 - 2x}{e^x - 1 - x},$$

(2)

$$\lim_{x\to\infty} \cos\left(\frac{1}{x}\right)^x,$$

(3)

$$\lim_{x\to\infty} \ln(x+1) - \ln x.$$

Solution

(1) This is an indeterminate form $0/0$. We apply L'Hôpital's Rule twice to obtain

$$\lim_{x\to 0} \frac{\sin(2x) + x^2 - 2x}{e^x - 1 - x} = \lim_{x\to 0} \frac{2\cos(2x) + 2x - 2}{e^x - 1} = \lim_{x\to 0} \frac{-4\sin(2x) + 2}{e^x} = 2.$$

(2) This is an indeterminate form 1^∞. Note that

$$\lim_{x\to\infty} \ln\left(\cos\left(\frac{1}{x}\right)^x\right) = \lim_{x\to\infty} x\ln\left(\cos\left(\frac{1}{x}\right)\right) = \lim_{x\to\infty} \frac{\ln\left(\cos\left(\frac{1}{x}\right)\right)}{1/x}.$$

The last limit is an indeterminate form $0/0$, so we may apply L'Hôpital's Rule to obtain

$$\lim_{x\to\infty} \frac{\ln\left(\cos\left(\frac{1}{x}\right)\right)}{1/x} = \lim_{x\to\infty} \frac{(1/\cos(1/x))(-\sin(1/x))(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} \frac{-\sin(1/x)}{\cos(1/x)} = 0.$$

Therefore,

$$\lim_{x\to\infty} \cos\left(\frac{1}{x}\right)^x = e^0 = 1.$$

(3) Note that

$$\lim_{x\to\infty} [\ln(x+1) - \ln(x)] = \lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right) = \ln\left(\lim_{x\to\infty} 1 + \frac{1}{x}\right) = \ln(1) = 0.$$

Note that we used the fact that ln is continuous to move the limit inside the natural log.

Problem 4 (8 points). State the $\epsilon - K$ definition of $\lim_{n\to\infty} a_n = L$.

Problem 5 (14 points). Prove that if $a_n \to L$ and $b_n \to M$ then $a_n + b_n \to L + M$.

Problem 6 (20 points). Please answer true or false (you do not need to justify your answer).

(1) If M is an upper bound for S, and $M \in S$, then M is the least upper bound for S.
(2) The least upper bound of a set S must be contained in S.
(3) The least upper bound of a set of irrational numbers must be irrational.
(4) A sequence that is bounded and increasing must converge.
(5) If $a_n + b_n$ converges then both a_n and b_n must converge.

Solution T,F,F,T,F.