CALCULUS 153: SAMPLE MIDTERM 1 SOLUTIONS

Problem 1 (16 points)**.** *Determine the least upper bound and greatest lower bound of the following sets, or state that they do not exist. You do not need to justify your answer (4 points each).*

(1) $(-\infty, 3] \cup [4, 7)$,
(2) $\{x : 2 < \ln(x+1) < 4\}$,
(3) $\{a_n\}$ *where* $a_n = 2^{1/n}$,
(4) $\{x \in \mathbb{Q} : e < x < \pi\}$.

Solution

(1) The lub is 7, the glb does not exist.
(2) The lub is $e^4 - 1$, the glb is $e^2 - 1$.
(3) The lub is 2, the glb is 1 (note that the sequence is decreasing and converges to 1).
(4) The lub is π and the glb is e. Note that for any real number a there are rational numbers arbitrarily close to a; therefore, specifying that x must be a rational won't affect the lub and glb.

Problem 2 (20 points)**.** *For each of the following sequences, determine whether the sequence converges or diverges. If it converges, find its limit. Show your work. (5 points each).*

(1)
$$a_n = \frac{2^n + (-1)^n}{n!};$$

(2)
$$b_n = \left(1 + \frac{x}{n}\right)^{2n};$$

(3)
$$c_n = \frac{\sin(n^2 + \ln n) n^{1/n}}{n+1}.$$

(4)
$$d_n = x^{\frac{n}{n+1}} \text{ where } x > 0.$$

Solution

(1)
$$\lim_{n\to\infty} \frac{2^n + (-1)^n}{n!} = \lim_{n\to\infty} \frac{2^n}{n!} + \lim_{n\to\infty} \frac{(-1)^n}{n!}.$$
The first limit is an important limit; it is equal to 0. To find the second limit, notice that $-1/n! \le (-1)^n/n! \le 1/n!$. Therefore, we can use the squeeze theorem and the fact that $1/n! \to 0$ to conclude the second limit is 0. Thus,
$$\lim_{n\to\infty} \frac{2^n + (-1)^n}{n!} = 0.$$

(2)

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{2n} = \lim_{n\to\infty}\left[\left(1+\frac{x}{n}\right)^{n}\right]^2 = \left[\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}\right]^2 = (e^x)^2 = e^{2x}.$$

(3) We'll use the squeeze theorem. Note that $-1 \le \sin(n^2+\ln n) \le 1$ and therefore,

$$-\frac{n^{1/n}}{n+1} \le \frac{\sin(n^2+\ln n)n^{1/n}}{n+1} \le \frac{n^{1/n}}{n+1}.$$

Furthermore, since $n^{1/n}\to 1$ (important limit),

$$\lim_{n\to\infty}\frac{n^{1/n}}{n+1} = 0.$$

Therefore, by the squeeze theorem,

$$\lim_{n\to\infty}\frac{\sin(n^2+\ln n)n^{1/n}}{n+1} = 0.$$

(4) Note that

$$\frac{n}{n+1} = \frac{n+1-1}{n+1} = 1-\frac{1}{n+1}.$$

Therefore,

$$\lim_{n\to\infty} x^{\frac{n}{n+1}} = \lim_{n\to\infty} x^{1-\frac{1}{n+1}} = x\cdot\lim_{n\to\infty}\frac{1}{x^{\frac{1}{n+1}}}.$$

However, $x^{1/n}\to 1$ (important limit) and therefore $x^{1/(n+1)}\to 1$. Thus,

$$\lim_{n\to\infty} x^{\frac{n}{n+1}} = x.$$

Problem 3 (20 points)**.** *Find the following limits. Show your work (5 points each).*

(1)

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x},$$

(2)

$$\lim_{x\to 0^+} x^{\sin x},$$

(3)

$$\lim_{x\to 1}\frac{1}{x-1}-\frac{x}{\ln x},$$

(4)

$$\lim_{x\to 0}\frac{1+x-e^x}{x(e^x-1)}.$$

Solution

(1) This is an indeterminate form 0/0, so we can apply L'Hôpital's rule directly:

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x} = \lim_{x\to 0}\frac{e^x+e^{-x}}{\cos x} = \frac{1+1}{1} = 2.$$

(2) This is an indeterminate form 0^0. Note that $\ln(x^{\sin x}) = \sin x \ln x$, and
$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/(\sin x)},$$
which is indeterminate ∞/∞. We apply L'Hôpital's rule twice to obtain
$$\lim_{x\to 0^+} \frac{\ln x}{1/(\sin x)} = \lim_{x\to 0^+} \frac{1/x}{-\cos x/(\sin^2 x)} = \lim_{x\to 0^+} \frac{\sin^2 x}{-x\cos x} = \lim_{x\to 0^+} \frac{2\sin x \cos x}{-\cos x + x \sin x} = 0.$$
Therefore,
$$\lim_{x\to 0^+} x^{\sin x} = e^0 = 1.$$

(3) This is an indeterminate form $\infty - \infty$. Note that
$$\lim_{x\to 1} \frac{1}{x-1} - \frac{x}{\ln x} = \lim_{x\to 1} \frac{\ln x - x(x-1)}{(x-1)\ln x}$$
which is indeterminate $0/0$. We apply L'Hôpital's rule to obtain
$$\lim_{x\to 1} \frac{\ln x - x(x-1)}{(x-1)\ln x} = \lim_{x\to 1} \frac{1/x - 2x + 1}{\ln x + (x-1)/x}.$$
This is again indeterminate $0/0$, so we apply L'Hôpital's rule again to obtain
$$\lim_{x\to 1} \frac{1/x - 2x + 1}{\ln x + (x-1)/x} = \lim_{x\to 1} \frac{-1/x^2 - 2}{1/x + 1/x^2} = \frac{-3}{2}.$$

(4) This is an indeterminate form $0/0$ and so we apply L'Hôpital's rule twice to get
$$\lim_{x\to 0} \frac{1 + x - e^x}{x(e^x - 1)} = \lim_{x\to 0} \frac{1 - e^x}{xe^x + e^x - 1} = \lim_{x\to 0} \frac{-e^x}{xe^x + 2e^x} = \frac{-1}{2}.$$

Problem 4 (10 points)**.** *State the ϵ – K definition of* $\lim_{n\to\infty} a_n = L$.

Solution In your notes and in the book. Please state it carefully! A random jumble of ϵ's, K's, "there exists", "for all" and $|a_n - L| < \epsilon$ won't cut it.

Problem 5 (14 points)**.** *Prove that if T is a subset of S, then* $\sup T \leq \sup S$.

Solution This is #29 from 11.1. The solution is written up in the "Solutions to selected homework problems".

Problem 6 (20 points)**.** *For each of the following give an example, or state that no example exists (you do not need to justify your answer).*

(1) *A set S whose least upper bound is not an element of S.*
(2) *A set of negative numbers whose least upper bound is strictly positive.*
(3) *A set of rational numbers whose least upper bound is irrational.*
(4) *A set of irrational numbers whose least upper bound is rational.*
(5) *A sequence that's bounded but divergent.*

Solution

(1) $(0, 1)$.
(2) This is not possible (homework problem #32 from 11.1).
(3) $\{3, 3.1, 3.14, 3.141, \ldots\}$.
(4) $\{-\pi, -\pi/2, -\pi/3, \ldots\}$.

(5) $0, 1, 0, 1, 0, \ldots$.