CALCULUS 153: SOLUTIONS TO SELECTED HOMEWORK PROBLEMS

11.1 #27 Suppose that M is any upper bound for the set S. Then by definition, $s \leq M$ for all $s \in S$. But $b \in S$, and therefore $b \leq M$. Thus, b is an upper bound for S that is less than or equal to all other upper bounds. By defnition, $b = \sup S$.

11.1 #29 (a) Let M be an upper bound for S. Then, by definition, $s \leq M$ for all $s \in S$. But since T is a subset of S, $t \leq M$ for all $t \in T$. Therefore, M is also an upper bound for T and therefore T is bounded.

(b) We first show that $\sup T \leq \sup S$. We showed in part (a) that every upper bound of S is an upper bound for T. In particular, $\sup S$ is an upper bound for T. By definition of $\sup T$, this implies that $\sup T \leq \sup S$. The fact that $\inf T \leq \sup T$ is immediate. The fact that $\inf S \leq \inf T$ is proved in the same way as we proved that $\sup T \leq \sup S$.

11.1 #32 (a) Let S be a set of negative numbers. Then 0 is an upper bound for S. In particular $0 \geq \sup S$. Therefore, the least upper bound of a set of negative numbers cannot be strictly positive.

(b) is proved in a similar fashion.

11.3 #52 Since b_n is bounded, there exists $M > 0$ such that for all n,

$$-M \leq b_n \leq M.$$

Therefore, if $a_n \geq 0$,

$$-a_n M \leq a_n b_n \leq a_n M,$$

and if $a_n \leq 0$,

$$a_n M \leq a_n b_n \leq -a_n M.$$

In either case, $a_n b_n$ is between $-a_n M$ and $a_n M$. Since $a_n \to 0$, so does $-a_n M$ and $a_n M$. Therefore, by the squeeze theorem, $a_n b_n \to 0$.

11.3 # 53 We prove this by contradiction. Suppose not, so that $M < L$. Then there exists $\epsilon > 0$ such that $M < L - \epsilon$ (you can take $\epsilon = (L - M)/2$). Since $a_n \to L$, there exists K so that for $n \geq K$, $|a_n - L| < \epsilon$. This implies that for $n \geq K$,

$$-\epsilon < a_n - L < \epsilon,$$

so that

$$a_n > L - \epsilon > M.$$

This contradicts the fact that $a_n \leq M$ for all n.

8.2 # 68 Use IBP and let $u = (\ln x)^n$ and $dv = dx$.

8.2 # 78 a) Consider

$$\int_a^b f''(x)(x-b)\,dx.$$

Letting $u = (x-b)$ and $dv = f''(x)\,dx$, we obtain

$$\int_a^b f''(x)(x-b)\,dx = [(x-b)f'(x)]_a^b - \int_a^b f'(x)\,dx = -f'(a)(a-b) - (f(b)-f(a)).$$

Rearranging everything gives the desired result.

8.3 # 53 a) a) Letting $u = \sin^{n-1} x$ and $dv = \sin x\,dx$, one gets that

$$\begin{aligned}\int \sin^n x\,dx &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx \\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx \\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx.\end{aligned}$$

Rearranging gives the desired answer.

12.3 # 38 All $p > 1$. Use the fact that $\ln k < k^\alpha$ for k large and all $\alpha > 0$.

12.3 # 49 a) We apply the limit comparison test:

$$\lim_{k\to\infty} \frac{a_k^2}{a_k} = \lim_{k\to\infty} a_k.$$

However, this last limit is 0, since $\sum a_k$ converges. Therefore, by the limit comparison test, $\sum a_k^2$ converges since $\sum a_k$ does.

b) If $a_k = 1/k^2$ then both $\sum a_k^2$ and $\sum a_k$ converge. If $a_k = 1/k$ then $\sum a_k^2$ converges, while $\sum a_k$ diverges.

12.8 # 44

$$\sum \frac{x^k}{r^k}.$$