CALCULUS 153: TESTS FOR CONVERGENCE OF SERIES

There are five essential components of positive series: $\ln k$, k^p where p is a fixed positive power, r^k where r is a fixed positive number, $k!$ and k^k. The terms in most series we consider are made up of sums products and quotients of the above. In order to get a good intuition for whether a series converges or diverges, and what test to use to show it, it's important to remember how fast they grow with respect to each other

$$\begin{aligned}
\lim_{k\to\infty} \frac{\ln k}{k^p} &= 0 \quad \text{for any } p > 0; \\
\lim_{k\to\infty} \frac{k^p}{r^k} &= 0 \quad \text{for any } p > 0,\ r > 1\ ; \\
\lim_{k\to\infty} \frac{r^k}{k!} &= 0 \quad \text{for any } r > 1. \\
\lim_{k\to\infty} \frac{k!}{k^k} &= 0
\end{aligned}$$

Therefore, $\ln k$ grows more slowly than k^p which grows more slowly than r^k which grows more slowly than $k!$ which grows more slowly than k^k. What is more useful for series however is the equivalent statement: $1/k^k$ goes to zero more quickly than $1/(k!)$ goes to zero more quickly than $1/r^k$ which goes to zero more quickly than $1/k^p$ which goes to zero more quickly than $1/\ln k$. In order for a positive series to converge, the terms must go to zero quickly enough so that the sum doesn't get too large.

It is also important to have examples of series which converge or diverge. There are two classes of examples to remember: geometric series and p-series. Recall that a geometric series is a series of the form

$$\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + r^3 + r^4 \ldots,$$

and that this series converges if and only if $-1 < r < 1$ (and the sum is equal to $1/(1-r)$). A p-series is a series of the form

$$\sum_{k=0}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots.$$

This series converges if and only if $p > 1$. Note that when $p = 1$, the series is called the harmonic series and diverges.

The tests that you need to know are the kth Term Test for Divergence (12.2.5)), the Integral Test (12.3.2), the Ordinary Comparison Test (12.3.6), the Limit Comparison Test (12.3.7), the Root Test (12.4.1) and the Ratio Test (12.4.2).

You should always apply the kth Term Test for Divergence first. In other words, given a series $\sum a_k$, check whether or not $\lim_{k\to\infty} a_k = 0$. If it doesn't, then you can immediately conclude that it diverges. If it does, then you can't conclude anything yet, and you need to apply another test.

If your series contains $k!$, try the Ratio Test; it will probably work. If it contains k^k or r^k try either the Root Test or the Ratio Test. If it contains neither r^k, $k!$ or k^k, the Root and Ratio Test will most likely be inconclusive.

Next, I would try the integral test. Write down the corresponding integral. By now, you should have a feel for which functions are easy to integrate and which aren't. If you can evaluate the integral, then do so. If it converges, your series converges; if it diverges, then your series diverges.

If the above tests don't work, then you will need to use either of the comparison tests. Note that the Limit Comparison Test is the more powerful of the two. You can often just use that. In order to use it, you want to compare your series to either a geometric series or to a p-series. Usually it ends up being the latter. To do so it's important to remember that $\ln k$ grows more slowly than k^p. Note that if your series involves the ratio of two polynomials, for example

$$\sum_{k=1}^{\infty} \frac{k^3 - k^2 + 19}{k^4 + 4k^2 + k + 2},$$

then compare it to the ratio of the largest power in the numerator and the largest power in the denominator. In this case that would be $k^3/k^4 = 1/k$. Therefore, we would use the limit comparison test to compare it to $\sum 1/k$ and conclude it diverges since the latter diverges.