CALCULUS 131: MIDTERM 1 SOLUTIONS

Please answer all questions in a blue book provided to you. Calculators are not allowed. The exam is worth 50 points with 5 possible bonus points.

Problem 1 (10 points). Please answer true or false. You do not need to justify your answer.

(a) If $Q \Rightarrow P$ is true, then $P \Rightarrow Q$ is true.
(b) The sum of two irrational numbers is always irrational.
(c) If $a < b$ then $a + c < b + c$ for all real numbers c.
(d) If f is odd, then $f(0) = 0$.
(e) $\sqrt{x^2} = x$ for all x.

Solution (a) is false. For example, "if n is a natural number, then $n \geq 0$" is true. But "if $n \geq 0$ then n is a natural number" is false.

(b) is false. You had this problem in your homework. For instance, $\sqrt{2}$ and $-\sqrt{2}$ are irrational, but $-\sqrt{2} + \sqrt{2} = 0$ is rational.

(c) is true. I wrote this down explicitly in class when we started talking about solving inequalities.

(d) is true. There are two ways to see it. Either you remember that the graph of an odd function is symmetric with respect to the origin, and therefore must go through the origin. Or, notice that by the definition of odd function, $f(-0) = -f(0)$. But $-0 = 0$, so $f(0) = -f(0)$. The only number that's equal to its opposite is zero. So $f(0) = 0$.

(e) is false. You showed in your homework that $\sqrt{x^2} = |x|$. For instance, $\sqrt{(-2)^2} = \sqrt{4} = 2$.

Problem 2 (8 points). Solve the inequality $(x - 3)(x + 1) < 0$.

Solution This is a quadratic inequality, and the first step is to bring everything to one side and factor. But this is already done. The next step is to split $\mathbb{R}$ up into $(-\infty, -1)$, $(-1, 3)$ and $(3, \infty)$ and figure out the sign of $(x - 3)(x + 1)$ on each interval.

On $(-\infty, -1)$, $(x - 3)$ is negative, $(x + 1)$ is negative and so $(x - 3)(x + 1)$ is positive. On $(-1, 3)$, $(x + 1)$ is positive, $(x - 3)$ is negative and so $(x + 1)(x - 3)$ is negative. On $(3, \infty)$, $(x + 1)$ is positive, $(x - 3)$ is positive and so $(x + 1)(x - 3)$ is positive. Therefore, our solution is $(-1, 3)$ (we don't include the endpoints, because we have a strict inequality $<$).

Problem 3 (10 points).

(a) What is the equation of the circle with center $(-3, 2)$ and radius 4? (5 points)
(b) Find the equation of the line that passes through the points $(-2, 11)$ and $(1, 5)$. Is $(2, 3)$ on the line? (5 points)

Solution

(a) The equation of the circle with center (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2.$$

Therefore, the equation of the circle with center $(-3, 2)$ and radius 4 is

$$(x+3)^2 + (y-2)^2 = 16.$$

(b) The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-11}{1-(-2)} = \frac{-6}{3} = -2.$$

Therefore, our equation is of the form $y = -2x + b$. To find b, we just plug in one of our points:

$$5 = -2(1) + b \Rightarrow b = 7.$$

Therefore, our equation is $y = -2x + 7$. $(2, 3)$ is on the line, because $(2, 3)$ satisfies the equation of the line:

$$-2(2) + 7 = 3.$$

Problem 4 (22 points).

(a) Solve the inequality $|x-2| > 1$. (6 points)

Let $f(x) = \frac{1}{\sqrt{x-1}}$ and $g(x) = |x-2|$.

(b) Find the natural domains of f and g. (4 points)
(c) Find the domain of $f+g$ and find a formula for $(f+g)(x)$. (4 points)
(d) Find the domain of $f \circ g$ and the formula for $(f \circ g)(x)$. (*Hint: Use (a).*) (4 points)
(e) Find the domain of $g \circ g$ and the formula for $(g \circ g)(x)$. (4 points)

Solution

(a)

$$\begin{aligned} |x-2| > 1 \quad &\Longleftrightarrow \quad x-2 > 1 \quad \text{or} \quad x-2 < -1 \\ &\Longleftrightarrow \quad x > 3 \quad \text{or} \quad x < 1. \end{aligned}$$

Therefore, the solution set is $(-\infty, 1) \cup (3, \infty)$.

(b) In order for $f(x)$ to be well-defined, we need to ensure that we don't take the square root of a negative number, and also that the denominator is nonzero. We can combine these conditions into the condition $x - 1 > 0$, i.e. $x > 1$. Therefore, the natural domain of f is $(1, \infty)$. We can take the absolute value of any real number. Therefore, the natural domain of g is $\mathbb{R}$.

(c) The domain of $f+g$ is the intersection of the domain of f and the domain of g. Therefore, the domain of $f+g$ is $(1, \infty)$. The formula for $(f+g)(x)$ is given by

$$(f+g)(x) = f(x) + g(x) = \frac{1}{\sqrt{x-1}} + |x-2|.$$

(d) The formula for $(f \circ g)(x)$ is given by

$$(f \circ g)(x) = f(g(x)) = f(|x-2|) = \frac{1}{\sqrt{|x-2|-1}}.$$

Therefore, for $f \circ g$ to be defined, we need $|x-2|-1>0$. But we solved this inequality in (a)! The set of x that satisfy that is $(-\infty,1)\cup(3,\infty)$, and that's the domain of $f \circ g$.

(e) The domain of g is all real numbers, so the domain of $g \circ g$ will also be $\mathbb{R}$.

$$(g \circ g)(x) = g(g(x)) = g(|x-2|) = ||x-2|-2|\,,$$

which can't be simplified further!

Problem 5 (5 bonus points). Prove that the product of a nonzero rational number and an irrational number is irrational.

Solution We will prove this by contradiction. Let a be any irrational number, and b be any nonzero rational number. Suppose to the contrary, that ab is rational. Then by definition of rational numbers, we can write $b = \frac{p}{q}$ for some integers p and q ($p \neq 0$), and also $ab = \frac{m}{n}$ for some integers m and n. Therefore,

$$a(\frac{p}{q}) = \frac{m}{n} \Rightarrow a = \frac{mq}{np}.$$

So we've written a as the ration of two integers (mq and np), which implies that a is rational. This is a contradiction, since a was assumed to be an irrational number.