

CALCULUS 131: SAMPLE MIDTERM 1 SOLUTIONS

Problem 1 (10 points). Please answer true or false. You do not need to justify your answer.

- (1) The contrapositive of $P \Rightarrow Q$ is $Q \Rightarrow P$.
- (2) The product of two irrational numbers is always irrational.
- (3) The sum of two odd functions is odd.
- (4) For every f and g , $f \circ g = g \circ f$.
- (5) The function $f(x) = |x| + 2$ is even.

Solution (1) is false; the contrapositive of $P \Rightarrow Q$ is $\sim Q \Rightarrow \sim P$. $Q \Rightarrow P$ is the converse. (2) is false; $\sqrt{2}$ is irrational, but $\sqrt{2}$ times itself is 2 which is rational. (3) is true. If f and g are odd, then by definition, $f(-x) = -f(x)$ and $g(-x) = -g(x)$ for all x . Then,

$$(f+g)(-x) = f(-x) + g(-x) = (-f(x)) + (-g(x)) = -(f(x) + g(x)) = -(f+g)(x)$$

and so $f+g$ is odd. (4) is false. I said this in class, and it won't be true for pretty much any example of f and g you come up with. (5) is true.

$$f(-x) = |-x| + 2 = |x| + 2 = f(x),$$

since $|-x| = |x|$ for all x .

Problem 2 (8 points). Solve the inequality $(x-1)(x+4) \leq 0$ and write its solution in interval notation.

Solution This is a quadratic inequality, and the first step is to bring everything to one side and factor. But this is already done. The next step is to split \mathbb{R} up into $(-\infty, -4)$, $(-4, 1)$ and $(1, \infty)$ and figure out the sign of $(x-1)(x+4)$ on each interval. It's best to draw a table as I did in class, but it will take me too long to type it up.

On $(-\infty, -4)$, $(x-1)$ is negative, $(x+4)$ is negative and so $(x-1)(x+4)$ is positive. On $(-4, 1)$, $(x-1)$ is negative, $(x+4)$ is positive and so $(x-1)(x+4)$ is negative. On $(1, \infty)$, $(x-1)$ is positive, $(x+4)$ is positive and so $(x-1)(x+4)$ is positive. Therefore, our solution is $[-4, 1]$ (we include the endpoints, because the inequality is \leq and therefore 0 is in the solution set).

Problem 3 (8 points). Find the center and radius of the circle whose equation is $x^2 - 2x + y^2 + 4y = 4$.

Solution To solve these problems we need to complete squares!

$$\begin{aligned} x^2 - 2x + y^2 + 4y &= 4 \\ \iff x^2 - 2x + 1 + y^2 + 4y &= 5 \\ \iff (x-1)^2 + y^2 + 4y &= 5 \\ \iff (x-1)^2 + y^2 + 4y + 4 &= 9 \\ \iff (x-1)^2 + (y+2)^2 &= 9. \end{aligned}$$

By the formula given in class, it follows that this equation describes the circle with center $(1, -2)$ and radius $\sqrt{9} = 3$.

Problem 4 (14 points). Let $f(x) = \sqrt{x} + 2$, and $g(x) = \frac{1}{x-1}$.

- (1) Find the natural domains of f and g . (6 points)
- (2) Find the domain of $f + g$ and find a formula for $(f + g)(x)$. (4 points)
- (3) Find the domain of $f \circ g$ and a formula for $(f \circ g)(x)$. (4 points)

Solution (1) For \sqrt{x} to be defined, we need $x \geq 0$. Therefore, the natural domain of f is $[0, \infty)$. For g to be defined, its denominator can't be zero. Therefore, we need $x - 1 \neq 0$, or $x \neq 1$. Thus, the natural domain of g is $\mathbb{R} \setminus \{1\}$.

- (2) The domain of $f + g$ is the intersection of the domain of f and the domain of g . In this case it's $[0, \infty) \setminus \{1\}$. The formula for $f + g$ is simply

$$(f + g)(x) = f(x) + g(x) = \sqrt{x} + \frac{1}{x-1} + 2.$$

- (3) Let's find the formula for $(f \circ g)(x)$ first.

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x-1}\right) = \frac{1}{\sqrt{x-1}} + 2.$$

In order for this to be defined, we need $x - 1 > 0$ (we need $x - 1 \geq 0$ in order to take a square root, but we also need $x - 1 \neq 0$ so we don't divide by zero). Therefore, the domain of $f \circ g$ is $(1, \infty)$.

Problem 5 (10 points). Let $f(x) = \frac{x+2}{x-1}$.

- (1) What is the natural domain of f ? (4 points)
- (2) Show that the natural domain of $f \circ f$ is the same as that of f , and that $(f \circ f)(x) = x$ for all x in this domain. (6 points)

Solution (1) The only problem we can have is for the denominator to be zero. This happens when $x = 1$. Therefore, the natural domain of f is $\mathbb{R} \setminus \{1\}$.

- (2) In order for $(f \circ f)(x)$ to be defined, x must be in the domain of f and $f(x)$ must also be in the domain of f . For the former to occur, we must have $x \neq 1$. For the latter to occur, we must have $f(x) \neq 1$, i.e.

$$\begin{aligned} \frac{x+2}{x-1} &\neq 1 \\ x+2 &\neq x-1 \\ 2 &\neq -1. \end{aligned}$$

Therefore, it is always true that $f(x) \neq 1$. So the domain of $f \circ f$ is the same as that of f , i.e. $\mathbb{R} \setminus \{1\}$.

To find the formula for $(f \circ f)(x)$, we write

$$\begin{aligned}(f \circ f)(x) &= f\left(\frac{x+2}{x-1}\right) \\ &= \frac{\frac{x+2}{x-1} + 2}{\frac{x+2}{x-1} - 1} \\ &= \frac{\frac{(x+2) + (2x-2)}{x-1}}{\frac{(x+2) - (x-1)}{x-1}} \\ &= \frac{\frac{3x}{x-1}}{\frac{3}{x-1}} \\ &= \frac{3x}{3} = x.\end{aligned}$$

Notice that for the penultimate step (where we cancel the $1/(x-1)$ terms) to make sense we need $x \neq 1$. Therefore $(f \circ f)(x) = x$ only when $x \neq 1$. When $x = 1$, the function is not defined.

Instead of finding the natural domain of $f \circ f$ as we did before, we could have found the formula above noting that it was valid iff $x \neq 1$. We would have then concluded that the natural domain is $\mathbb{R} \setminus \{1\}$. Either method is fine.