CALCULUS 131: SAMPLE MIDTERM 2 SOLUTIONS

Problem 1 (10 points). Please answer true or false. You do not need to justify your answer.

(1) If $\lim_{x\to c} f(x) = L$ then $\lim_{x\to c} |f(x)| = |L|$.
(2) $\lim_{h\to 0} f(x+h) = f(x)$ for any function f.
(3) If $f(x)+g(x)$ is continuous at c then both $f(x)$ and $g(x)$ are continuous at c.
(4) $|x|$ is differentiable at all points.
(5) The function
$$f(x) = \begin{cases} x-3 & \text{if } x \le -1; \\ x^2-5 & \text{if } x > -1. \end{cases}$$
is continuous at all points.

Solution (1) is true; I stated this fact when I said that $|x|$ is continuous. (2) is false; $\lim_{h\to 0} f(x+h) = f(x)$ is only true if f is continuous at x. (3) is false; this is very similar to Problem B from assignment 5. Let
$$f(x) = \begin{cases} -1 & \text{if } x \le 0; \\ 1 & \text{if } x > 0. \end{cases},$$
$$g(x) = \begin{cases} 1 & \text{if } x \le 0; \\ -1 & \text{if } x > 0. \end{cases}$$
Then $f+g=0$ which is continuous, but neither f nor g is continuous at 0. (4) is false. $|x|$ is not differentiable at 0, as shown in class. (5) is true. The only place where f could be discontinuous is at -1. However, $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x-3 = -4$, $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x^2-5 = -4$, and $f(-1) = -4$. So $\lim_{x\to -1} f(x) = f(-1)$ and so f is continuous at -1.

Problem 2 (10 points).

(1) State the ϵ-δ definition of $\lim_{x\to c} f(x) = L$. (3 points)
(2) Using the definition, show that $\lim_{x\to -2} -2x+1 = 5$. (7 points)

Solution The ϵ-δ definition is in your notes and in the book. To prove that $\lim_{x\to -2} -2x+1 = 5$, we first do a preliminary analysis, then a formal proof.

Preliminary Analysis: We need to find a $\delta > 0$ so that whenever $0 < |x+2| < \delta$ then $|(-2x+1)-5| < \epsilon$. However,
$$|(-2x+1)-5| = |-2x-4| = |-2(x+2)| = 2\,|x+2|\,.$$
Therefore, $|(-2x+1)-5| < \epsilon \iff |x+2| < \epsilon/2$. So let $\delta = \epsilon/2$.

Formal Proof: Let $\epsilon > 0$ be given. Let $\delta = \epsilon/2$. Then if $|x+2| < \delta$,
$$|(-2x+1)-5| = |-2x-4| = |-2(x+2)| = 2\,|x+2| < 2\delta = \epsilon.$$

Problem 3 (15 points). Compute the following; please show your work.

(1)
$$\lim_{x\to\infty} \frac{x^7 - x^4 + 2x + 1}{3x^7 - \pi},$$

(2)
$$D_x((x^2+1)^{1/3}(x+1)^2),$$

(3)
$$D_x((x+\sqrt{x})^7 + x^{-2/7}).$$

Solution

(1)
$$\begin{aligned} \lim_{x\to\infty} \frac{x^7 - x^4 + 2x + 1}{3x^7 - \pi} &= \lim_{x\to\infty} \frac{(x^7 - x^4 + 2x + 1)x^{-7}}{(3x^7 - \pi)x^{-7}} \\ &= \lim_{x\to\infty} \frac{1 - x^{-3} + 2x^{-6} + x^{-7}}{3 - \pi x^{-7}}. \end{aligned}$$

However, as $x \to \infty$, x^{-3}, x^{-6} and x^{-7} tend to 0. Therefore, applying the MLT, we get that
$$\lim_{x\to\infty} \frac{1 - x^{-3} + 2x^{-6} + x^{-7}}{3 - \pi x^{-7}} = \frac{1+0+0}{3-0} = \frac{1}{3}.$$

(2)
$$\begin{aligned} D_x((x^2+1)^{1/3}(x+1)^2) &= D_x((x^2+1)^{1/3})(x+1)^2 + (x^2+1)^{1/3} D_x((x+1)^2) \\ &= [\frac{1}{3}(x^2+1)^{-2/3} D_x(x^2+1)](x+1)^2 \\ &+ (x^2+1)^{1/3}[2(x+1)D_x(x+1)] \\ &= \frac{1}{3}(x^2+1)^{-2/3}(2x)(x+1)^2 + (x^2+1)^{1/3}(2(x+1)). \end{aligned}$$

Don't bother simplifying further.

(3)
$$\begin{aligned} D_x((x+\sqrt{x})^7 + x^{-2/7}) &= D_x((x+\sqrt{x})^7) + D_x(x^{-2/7}) \\ &= 7(x+\sqrt{x})^6 D_x(x+\sqrt{x}) - \frac{2}{7}x^{-9/7} \\ &= 7(x+\sqrt{x})^6(1 + \frac{1}{2\sqrt{x}}) - \frac{2}{7}x^{-9/7}. \end{aligned}$$

Problem 4 (8 points).

(1) Define what it means for f to be continuous at c. (3 points)
(2) Prove that if f and g are continuous at c, then $f + g$ is continuous at c. (5 points)

Solution Both of these are in the notes and in the book.

Problem 5 (7 points). Suppose that the distance d in feet covered by a car in t seconds is given by $d(t) = 5t^2 + 10t$ (not very realistic). At what time t is the car moving at 90 feet per second?

Solution The velocity of the car is given by $d'(t) = 10t + 10$. To find the time t when it's moving at 90 feet per second, we solve $d'(t) = 90$, i.e. $10t + 10 = 90$ which gives $t = 8$.