CALCULUS 133: BONUS PROBLEMS

Please hand these in to me before the final exam. All questions are worth 10 points and count towards your cumulative homework score.

Problem 1. *Show that for any* $-1 \leq x \leq 1$, $\sin^{-1}(x) + \cos^{-1}(x) = \pi/2$.

Problem 2. *Prove that*

$$\lim_{x\to\infty} \ln(x) = \infty.$$

Problem 3. *Give an example of a function* f *defined on* $\mathbb{R}$ *such that*

(1) f *is continuous on* $\mathbb{R}$*;*
(2) $\int_0^\infty f(x)\,dx$ *converges;*
(3) $\lim_{x\to\infty} f(x) \neq 0$.

Hint: think about the integral as an area. Thinking about series could also be useful here.

Problem 4. *Find the limit of the sequence*

$$\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\}.$$

Hint: write down a recursion formula. Use the monotone sequence theorem to show the limit exists, then use the recursion formula to find what the limit is equal to. This is similar to a problem that was done in class.

Problem 5. *Prove that if* $a_k \geq 0$ *and* $\sum_{k=1}^\infty a_k$ *converges, then so does* $\sum_{k=1}^\infty a_k^2$.

Problem 6. *However, show that the above is no longer true if we remove the hypothesis* $a_k \geq 0$. *In other words, give an example of a series* $\sum_{k=1}^\infty a_k$ *that converges and such that* $\sum_{k=1}^\infty a_k^2$ *diverges (the* a_k *can be negative here).*

Problem 7. *Consider the series*

$$\sum_{k=1}^\infty \frac{k}{2^k}.$$

By the ratio test one can show that this converges (you do not need to show this) to a sum S. *The goal of this question is to determine what* S *is.*

(1) *Manipulate the sum to show that*

$$(1 - \frac{1}{2})S = \sum_{k=1}^\infty 2^{-k}.$$

(2) *Use the above to show that* $S = 2$.

Problem 8. *Use both the result and the method from the above problem to find*

$$\sum_{k=1}^\infty \frac{k^2}{2^k}.$$

Problem 9. *Let*

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We stated in class that $f^{(n)}(0) = 0$ *for all* n*. Prove this (note that you will need to use the limit definition of the derivative).*