CALCULUS 133: FINAL REVIEW SHEET

The purpose of this review sheet is to give an overview of what we did this quarter, and to remind you of what you are supposed to know for the final. It's not enough to just study from this sheet! Go over your notes, the book, the homework and the two midterms.

1. Limits

Computing limits has come up in various situations this quarter: when we studied L'Hôpital's rule, when we computed improper integrals, when we found limits of sequences, and when we applied convergence tests. Here is a list of specific limits you should know:

$$\begin{aligned}
\lim_{x\to\infty} e^x &= \infty, \\
\lim_{x\to\infty} \ln(x) &= \infty, \\
\lim_{x\to 0^+} \ln(x) &= -\infty, \\
\lim_{x\to\infty} x^p &= \infty \quad p>0, \\
\lim_{x\to 0^+} x^p &= \infty \quad p<0, \\
\lim_{k\to\infty} r^k &= \infty \quad r>1, \\
\lim_{k\to\infty} r^k &= 0 \quad -1<r<1, \\
\lim_{k\to\infty} k! &= \infty.
\end{aligned}$$

It is also important to know how fast the above grow with respect to each other:

$$\begin{aligned}
\lim_{k\to\infty} \frac{\ln k}{k^p} &= 0 \quad \text{for any } p>0; \\
\lim_{k\to\infty} \frac{k^p}{r^k} &= 0 \quad \text{for any } p>0,\ r>1\ ; \\
\lim_{k\to\infty} \frac{r^k}{k!} &= 0 \quad \text{for any } r>1.
\end{aligned}$$

Therefore, $\ln k$ grows more slowly than k^p which grows more slowly than r^k which grows more slowly than $k!$. What is more useful for series however is the equivalent statement: $1/(k!)$ goes to zero more quickly than $1/r^k$ which goes to zero more quickly than $1/k^p$ which goes to zero more quickly than $1/\ln k$.

2. Integrals

When you're given an integral to compute, you should first ask yourself if it is a run of the mill integral, or if it's an improper integral. In the latter case, you need to write this integral as a limit; you evaluate the integral first, and then take the limit. To evaluate these integrals, you should remember the various techniques we saw in Chapter 9: integration by substitution, integration by parts, trigonometric integrals and rationalizing substitutions. These are reviewed in a lot of detail in the handout called "Techniques of integration" on the website.

As an example, consider the following three integrals:

(1)
$$\int_2^\infty \frac{2x}{x^2-3}\,dx,$$

(2)
$$\int_1^2 \frac{x}{\sqrt{x-1}}\,dx,$$

(3)
$$\int_0^\pi x\sin x\,dx.$$

To compute (1), notice that the function is being integrated over an infinite interval. Therefore, by definition,

$$\int_2^\infty \frac{2x}{x^2-3}\,dx = \lim_{b\to\infty}\int_2^b \frac{2x}{x^2-3}\,dx.$$

To evaluate the integral on the right, simply use the substitution $u = x^2 - 3$. After evaluating the integral and taking the limit, you should get $+\infty$ as your answer.

To compute (2), notice that the function "blows up" (tends to infinity) at $x = 1$. Therefore, this is an improper integral and so by definition,

$$\int_1^2 \frac{x}{\sqrt{x-1}}\,dx = \lim_{a\to 1^+}\int_a^2 \frac{x}{\sqrt{x-1}}\,dx.$$

To evaluate the integral on the right, use the rationalizing substitution technique, i.e. let $u = \sqrt{x-1}$. After evaluating the integral and taking the limit, you should get $8/3$ as your answer.

To compute (3), notice that this integral is not improper. Therefore, we need only apply integration by parts. You should get π as your answer.

3. L'Hôpital's Rule

Please see the corresponding handout.

4. Sequences

This is covered in section 9.1. Make sure you understand the intuitive definition of the limit of a sequence. You should also know the theorems stated in the book and in your notes for determining the limit of a sequence. Note that if $a_n = f(n)$ for some function f, then

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x).$$

You can then use any methods that you know for finding limits at infinity of functions, in particular L'Hôpital's Rule. Also pay attention to the squeeze theorem. The squeeze theorem is often useful when you have sequences involving sin, cos and $(-1)^n$. Note that $-1 \le \sin x \le 1$, $-1 \le \cos x \le 1$ and $-1 \le (-1)^k \le 1$.

5. Series

The main focus of the final will be on series. Here is a summary of what we've done. A series is essentially an infinite sum of real numbers, and is written as $\sum_{n=1}^{\infty} a_n$. You should know some specific examples:

(1) The geometric series

$$\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \dots$$

converges for $-1 < r < 1$ and diverges otherwise. If $-1 < r < 1$, then the sum is equal to $1/(1-r)$.

(2) The p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots$$

converges for $p > 1$ and diverges otherwise. When $p = 1$ this is the harmonic series. Thus, the harmonic series diverges. However, the alternating harmonic series,

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

converges (to $\ln 2$).

We then saw a few tests for determining whether a series converges or diverges. Please see the handout on "Tests for Convergence of Series" for more information. Note that the Integral Test, the Limit and Ordinary Comparison Tests and the Ratio Test are only valid for positive series. For an alternating series one can use the Alternating Series Test (Thm A page 475); one can also use the Absolute Ratio Test (Thm C page 477) for any series.

The last topic we covered is power series. A power series in $x - a$ (or centered at a) is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots.$$

Here, x is a variable, and therefore whether or not the power series converges depends on x. It is important to figure out for what x the power series converges; this is the convergence set of the power series. You should know Theorem A page 481 which tells us that convergence sets are intervals centered at a. The "radius" of this interval is called the radius of convergence. Given a power series, you should also be able to find its convergence set; to do so, we always use the Absolute Ratio Test (see the examples done in class, in the book and assignment 9).

On its interval of convergence, a power series is a function. You should know how to apply Theorem A page 484, which tells us that we can integrate and differentiate this function, simply by integrating or differentiating the power series term by term. This allowed us to derive useful power series representations of functions. You should know 1, 2, 4, 5 and 6 on page 495, and how to use these known series to find the power series representations of related functions (see for example the problems in assignment 9 from section 9.7, and also problems in the book and class).

The final question about power series that we asked is: given a function, does it have a power series representation in $x - a$? Theorem A page 489 tells us that if it does, it must be given by its Taylor series:

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots.$$

The question is whether the Taylor series actually does represent the function. The answer, unfortunately, is maybe. It depends on the function. Theorem C page 491 (Taylor's Theorem) gives us a criterion for determining this, however I won't require you to know it. What you do need to know how to do is, given a specific function, how to find the first few terms of its Taylor series. One can always do this by computing derivatives and using the formula above. However, sometimes one can be clever and use the Maclaurin series 1,2,4,5,6 from page 495 to derive Maclaurin (Taylor) series of related functions as in section 9.7. I did a few examples where I called this method the "clever" method.