CALCULUS 133: INDETERMINATE FORMS AND L'HÔPITAL'S RULE

We saw four different indeterminate forms: $0/0$, ∞/∞, $0 \cdot \infty$ and $\infty - \infty$. There are other indeterminate forms, but these are the ones we deal with in this class. Therefore, if your limit does not involve one of these forms, it is not indeterminate, and you can find the limit immediately.

When doing problems involving infinite limits, it is important to remember the following (the easiest way to remember this is to remember the graphs of these functions):

$$\lim_{x\to\infty} e^x = \infty,$$
$$\lim_{x\to\infty} \ln(x) = \infty,$$
$$\lim_{x\to 0^+} \ln(x) = -\infty,$$
$$\lim_{x\to\infty} x^r = \infty \quad r > 0,$$
$$\lim_{x\to 0^+} x^r = \infty \quad r < 0.$$

It is also important to think about what you are doing, and use common sense, and not to go through these problems blindly.

We have two L'Hôpital's Rules. They allow us to deal with the indeterminate forms $0/0$ and ∞/∞:

Theorem 1 (L'Hôpital's Rule for the form $0/0$)**.** *If* $\lim_{x\to u} f(x) = 0$ *and* $\lim_{x\to u} g(x) = 0$ *then*

$$\lim_{x\to u} \frac{f(x)}{g(x)} = \lim_{x\to u} \frac{f'(x)}{g'(x)}.$$

Here u can denote $\pm\infty$, a, a^+ or a^-.

Theorem 2 (L'Hôpital's Rule for the form ∞/∞)**.** *If* $\lim_{x\to u} f(x) = \pm\infty$ *and* $\lim_{x\to u} g(x) = \pm\infty$ *then*

$$\lim_{x\to u} \frac{f(x)}{g(x)} = \lim_{x\to u} \frac{f'(x)}{g'(x)}.$$

Here u can denote $\pm\infty$, a, a^+ or a^-.

Please see the book, the class notes, and homework problems for examples applying L'Hôpital's Rule to the indeterminate forms $0/0$ and ∞/∞.

There is no L'Hôpital's rule for the forms $0 \cdot \infty$ and $\infty - \infty$. To deal with the $0 \cdot \infty$ form, rewrite it as one of the forms above.

Example

$$\lim_{x\to 0^+} \sin(x)\ln(x)$$

is of the form $0 \cdot \infty$. We can rewrite

$$\lim_{x\to 0^+} \sin(x)\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{\sin x}},$$

which has the form ∞/∞. Therefore,

$$\begin{aligned}\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{\sin x}} &= \lim_{x\to 0^+} \frac{1/x}{-\cos(x)/\sin^2(x)} \\ &= \lim_{x\to 0^+} -\frac{\sin^2(x)}{x\cos(x)}.\end{aligned}$$

This last limit is again of the form $0/0$ so we apply L'Hôpital's Rule again.

$$\lim_{x\to 0^+} -\frac{\sin^2(x)}{x\cos(x)} = \lim_{x\to 0^+} -\frac{2\sin x\cos x}{\cos x - x\sin x} = \frac{0}{1-0} = 0.$$

The form $\infty - \infty$ is harder. If possible, try to put it in the form $0/0$ or ∞/∞ and apply L'Hôpital's Rule. Sometimes that won't work and you have to be clever.

Example

$$\lim_{x\to 0^+} \frac{1}{x\sin(x)} - \frac{1}{x},$$

is of the from $\infty - \infty$. This limit can be rewritten as

$$\lim_{x\to 0^+} \frac{1}{x\sin(x)} - \frac{1}{x} = \lim_{x\to 0^+} \frac{1-\sin(x)}{x\sin(x)}.$$

Notice that the limit of the numerator is 1, while the denominator tends to 0 (and is positive). Therefore,

$$\lim_{x\to 0^+} \frac{1-\sin(x)}{x\sin(x)} = \infty.$$

We never had to use L'Hôpital's rule at all!