CALCULUS 133: TECHNIQUES OF INTEGRATION

The purpose of these techniques is the following: you are given the problem of finding an antiderivative of a complicated function, and these techniques allow you to reduce it to finding an antiderivative of a simpler function. Eventually you reduce the problem to finding an antiderivative of a "standard function", for instance sin or cos or e^x. A list of standard functions, for which we know the antiderivatives is given on page 383. I expect you to know 1,2,3,5,6,11, 13 and 14.

1. Integration by substitution

This technique involves introducing a new variable u for x, which will hopefully simplify the integral. We let $u = f(x)$ be some function of x and then $du = f'(x)\,dx$. Then substitute in u and du in your integral. You have to be clever to make sure all the x's cancel. Be careful when you integrate **definite** integrals to change the bounds.

Example To find $\int \frac{e^x}{1+e^x}\,dx$, let $u = 1 + e^x$ so that $du = e^x\,dx$. Then

$$\int \frac{e^x}{1+e^x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln(1+e^x) + C.$$

Example To find $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx$, let $u = \sin x$ so that $du = \cos x\,dx$. Then when $x = 0$, $u = 0$ and when $x = \pi/2$, $u = 1$. Therefore,

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx &= \int_0^1 \frac{1}{1+u^2}\,du \\ &= \left[\tan^{-1}(u)\right]_0^1 \\ &= \tan^{-1}(1) - \tan^{-1}(0) = \frac{\sqrt{2}}{2} - 0 = \frac{\sqrt{2}}{2}.\end{aligned}$$

Example To find $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$, let $u = \cos x$, then $du = -\sin x\,dx$. Therefore,

$$\int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C = -\ln|\cos x| + C.$$

2. Trigonometric Integrals

To compute $\int \sin^n x\,dx$, $\int \cos^n x\,dx$ or $\int \sin^m x \cos^n x\,dx$, where m and n are positive integers, the following ideas are handy. If $\sin^n$ (or $\cos^n$) appears where n is odd, then simply let $u = \cos x$ ($u = \sin x$). Then use the fact that $\sin^2 x = 1 - \cos^2 x$ ($\cos^2 x = 1 - \sin^2 x$) to write everything in terms of u.

Example To find $\int \sin^3 x\,dx$, let $u = \cos x$ so that $du = -\sin x$. Then

$$\begin{aligned}\int \sin^3 x\,dx &= \int (1-\cos^2 x)\sin x\,dx\\ &= -\int (1-u^2)\,du\\ &= \frac{u^3}{3} - u + C\\ &= \frac{\cos^3 x}{3} - \cos x + C.\end{aligned}$$

Example To find $\int \cos^3 x \sin^2 x\,dx$, let $u = \sin x$ so that $du = \cos x\,dx$. Then

$$\begin{aligned}\int \cos^3 x\sin^2 x\,dx &= \int \cos^2 x\sin^2 x\cos x\,dx\\ &= \int (1-\sin^2 x)\sin^2 x\cos x\,dx\\ &= \int (1-u^2)u^2\,du\\ &= \int u^2 - u^4\,du\\ &= \frac{u^3}{3} - \frac{u^5}{5} + C\\ &= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.\end{aligned}$$

If only even powers appear, use the identities $\sin^2 x = (1-\cos 2x)/2$ and $\cos^2 x = (1+\cos 2x)/2$ to put your integral into an easier form.

Example

$$\begin{aligned}\int \sin^2 x\cos^2 x\,dx &= \int \frac{1}{4}(1-\cos 2x)(1+\cos 2x)\,dx\\ &= \frac{1}{4}\int 1-\cos^2(2x)\,dx\\ &= \frac{x}{4} - \frac{1}{4}\int \frac{1}{2}(1+\cos 4x)\,dx\\ &= \frac{x}{4} - \frac{x}{8} - \frac{1}{8}\int \cos 4x\,dx\\ &= \frac{x}{8} - \frac{1}{8}\int \cos 4x\,dx.\end{aligned}$$

Now let $u = 4x$ so that $du = 4\,dx$ to get

$$\begin{aligned}\int \sin^2 x\cos^2 x\,dx &= \frac{x}{8} - \frac{1}{32}\int \cos u\,du\\ &= \frac{x}{8} - \frac{1}{32}\sin u + C\\ &= \frac{x}{8} - \frac{\sin 4x}{32} + C.\end{aligned}$$

Phew!

Finally to compute integrals of the form $\int(\sin mx)(\sin nx)\,dx$, $\int(\cos mx)(\cos nx)\,dx$ and $\int(\sin mx)(\cos nx)\,dx$ just use the identities at the bottom of page 395.

Example

$$\int(\sin 3x)(\sin x)\,dx = \int -\frac{1}{2}(\cos 4x - \cos 2x)\,dx = \frac{1}{2}\int \cos 2x\,dx - \frac{1}{2}\int \cos 4x\,dx.$$

After u substituting in both integrals you should get

$$\int(\sin 3x)(\sin x)\,dx = \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + C.$$

3. Rationalizing substitutions

This technique allows you to get rid of n-th roots in your integral. The resulting integral will hopefully be easier than what you started out with. This technique tells you that if $(ax+b)^{1/n}$ appears in your integral, let $u=(ax+b)^{1/n}$. If $\sqrt{a^2-x^2}$ appears let $x = a\sin t$ so that $t=\sin^{-1}(x/a)$; note that t is your new variable!

Sometimes you can just do a usual u-substitution though! Don't forget about your previous techniques just because you learn new ones. For instance to find $\int 2x\sqrt{1-x^2}\,dx$, just let $u = 1-x^2$.

Example To find $\int \frac{1}{x+x^{1/3}}\,dx$, let $u = x^{1/3}$ so that $u^3 = x$. Then $3u^2\,du = dx$ so that

$$\int \frac{1}{x+x^{1/3}}\,dx = \int \frac{3u^2}{u^3+u}\,du = 3\int \frac{u}{u^2+1}\,du.$$

Now let $v = u^2+1$ so that $dv = 2u\,du$. Then

$$3\int \frac{u}{u^2+1}\,du = \frac{3}{2}\int \frac{1}{v}\,dv = \frac{3}{2}\ln|v| + C = \frac{3}{2}\ln(u^2+1)+C = \frac{3}{2}\ln(x^{2/3}+1)+C.$$

Example To find $\int \frac{x}{\sqrt{1-x^2}}\,dx$, we let $x=\sin t$ so that $dx = \cos t\,dt$ and $\sqrt{1-x^2} = \cos t$. Then we get

$$\begin{aligned}\int \frac{x}{\sqrt{1-x^2}}\,dx &= \int \frac{\sin t}{\cos t}\cos t\,dt \\ &= \int \sin t\,dt \\ &= -\cos t + C \\ &= -\cos(\sin^{-1}(x)) + C \\ &= -\sqrt{1-x^2} + C.\end{aligned}$$

Notice that we could have also solved this using u-substitution.

4. Integration by Parts

The formula for integration by parts is

$$\int u\,dv = uv - \int v\,du.$$

Usually you should try other methods first and if they don't work, try integration by parts, although with more practice you will recognize situations where you should use integration by parts almost immediately.

To use integration by parts you must write your $f(x)\,dx$ in the form $u\,dv$. This means you must choose u and v. A useful way to do this that will work almost every time is to remember LIATE (Log, Inverse trig, Algebra, Trig and Exponential). If ln occurs in your function, you let $u = \ln x$ and let the rest be dv. If not then if an inverse trig function occurs, you let u be that and the rest dv, if not and an algebra term (like x or x^2) occurs then you let u be that and the rest be dv etc.

Example To find $\int x \cos x \, dx$ we notice that by the LIATE method above, we should let $u = x$ so that $du = dx$, and $dv = \cos x \, dx$ so that $v = \sin x$. Then we plug into the formula above to get

$$\int x \cos x \, dx = x \sin x - \int \cos x \, dx = x \sin x - \sin x + C.$$

To evaluate definite integrals, just add your bounds:

Example

$$\int_0^{\pi/2} x \cos x \, dx = [x \sin x]_0^{\pi/2} - \int_0^{\pi/2} \cos x \, dx = \frac{\pi}{2} - [\sin x]_0^{\pi/2} = \pi/2 - 1.$$

Sometimes you have to integrate by parts twice:

Example To find $\int x^2 e^x \, dx$, let $u = x^2$ so that $du = 2x \, dx$ and $dv = e^x \, dx$ so that $v = e^x$. Then

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

Now to deal with the second integral, let $u = x$ so that $du = dx$, and $dv = e^x \, dx$ so that $v = e^x$. Then we get

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx = x^2 e^x - 2 \left(x e^x - \int e^x \, dx \right) = x^2 e^x - 2x e^x + 2e^x + C.$$

Finally, you may integrate by parts twice and end up with what you started with. This does not necessarily mean that you have failed (although sometimes it does mean that).

Example To find $\int \sin x e^x \, dx$ let $u = \sin x$ so that $du = \cos x \, dx$ and $dv = e^x \, dx$ so that $v = e^x$. Then integration by parts gives

$$\int \sin x e^x \, dx = e^x \sin x - \int \cos x e^x \, dx.$$

Let's integrate by parts again, letting $u = \cos x$ so that $du = -\sin x \, dx$ and $dv = e^x \, dx$ so that $v = e^x$ in the second integral. Then

$$\begin{aligned} \int \sin x e^x \, dx &= e^x \sin x - \int \cos x e^x \, dx \\ &= e^x \sin x - \left(e^x \cos x - \int (-\sin x) e^x \, dx \right) \\ &= e^x \sin x - e^x \cos x - \int \sin x e^x \, dx. \end{aligned}$$

Now simply bring the $\int \sin x e^x \, dx$ to the left side to get that

$$\int \sin x e^x \, dx = \frac{1}{2} \left(e^x \sin x - e^x \cos x \right) + C.$$

5. Miscellaneous techniques

Sometimes you will need to manipulate your function algebraically to put it in a form that is more convenient for integrating. One of the more common ways of doing this is to complete the square:

Example

$$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{(x+1)^2+1}\,dx.$$

Let $u = x+1$ so that $du = dx$. Then

$$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{u^2+1}\,du = \tan^{-1}(u) + C = \tan^{-1}(x+1) + C.$$