CALCULUS 133: MIDTERM 2 SOLUTIONS

Problem 1 (12 points). Evaluate each integral, or show that it diverges (6 points each):

(1)

$$\int_2^\infty xe^{-x^2}\,dx.$$

(2)

$$\int_1^2 \frac{1}{\sqrt{x-1}}\,dx.$$

Solution

(1)

$$\int_2^\infty xe^{-x^2}\,dx = \lim_{b\to\infty}\int_2^b xe^{-x^2}\,dx.$$

Let $u=-x^2,\ du=-2x\,dx$; when $x=2$, $u=-4$ and when $x=b$, $u=-b^2$. Then

$$\begin{aligned}\lim_{b\to\infty}\int_2^b xe^{-x^2}\,dx &= \lim_{b\to\infty} -\frac{1}{2}\int_{-4}^{-b^2} e^u\,du \\ &= \lim_{b\to\infty} -\frac{1}{2}\left[e^u\right]_{-4}^{-b^2} \\ &= \lim_{b\to\infty} -\frac{1}{2e^{b^2}}+\frac{e^{-4}}{2}=\frac{e^{-4}}{2}.\end{aligned}$$

(2)

$$\int_1^2 \frac{1}{\sqrt{x-1}}\,dx = \lim_{a\to 1^+}\int_a^2 \frac{1}{\sqrt{x-1}}\,dx.$$

Let $u=x-1$ so that $du=dx$. When $x=a$, $u=a-1$ and when $x=2$, $u=1$. Then

$$\begin{aligned}\lim_{a\to 1^+}\int_a^2 \frac{1}{\sqrt{x-1}}\,dx &= \lim_{a\to 1^+}\int_{a-1}^1 u^{-1/2}\,dx \\ &= \lim_{a\to 1^+} 2\left[\sqrt{u}\right]_{a-1}^1 \\ &= \lim_{a\to 1^+} 2-2\sqrt{a-1}=2.\end{aligned}$$

Problem 2 (9 points). For each of the following sequences, determine whether the sequence converges or diverges. If it converges, find its limit (3 points each).

(1)

$$a_n=\frac{n+1}{n^3+4};$$

(2)
$$b_n = \frac{\ln n}{2^n};$$

(3)
$$c_n = \frac{\sin n}{n^3}$$

(Hint: use the Squeeze Theorem).

Solution (1) By L'Hôpital's Rule,
$$\lim_{n\to\infty} \frac{n+1}{n^3+4} = \lim_{n\to\infty} \frac{1}{3n^2} = 0.$$

(2) $\ln n$ grows more slowly than r^n for any $r > 1$. Therefore,
$$\lim_{n\to\infty} \frac{\ln n}{2^n} = 0.$$

(3) $-1 \leq \sin n \leq 1$ and therefore,
$$\frac{-1}{n^3} \leq \frac{\sin n}{n^3} \leq \frac{1}{n^3}.$$
Since $1/n^3$ and $-1/n^3$ converge to 0, by the Squeeze Theorem, so does c_n.

Problem 3 (8 points).

(1) Use the integral test to show that
$$\sum_{k=1}^{\infty} \frac{\ln k}{k}$$
diverges (5 points).

(2) Now prove it using the ordinary comparison test (3 points).

Solution

(1)
$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x}\,dx.$$
Let $u = \ln x,\ du = (1/x)\,dx$. When $x = 1$, $u = 0$ and when $x = b$, $u = \ln b$. Therefore,
$$\begin{aligned} \lim_{b\to\infty} \int_1^b \frac{\ln x}{x}\,dx &= \lim_{b\to\infty} \int_0^{\ln b} u\,du \\ &= \lim_{b\to\infty} \left[\frac{u^2}{2}\right]_0^{\ln b} \\ &= \lim_{b\to\infty} \frac{(\ln b)^2}{2} = \infty. \end{aligned}$$
Therefore, by the integral test, the series diverges (since the corresponding integral diverges).

(2) Since $\ln k \geq 1$ for all $k \geq 3$, it follows that

$$\frac{\ln k}{k} \geq \frac{1}{k}$$

for all $k \geq 3$. Therefore, since the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

diverges, by the ordinary comparison test, so does

$$\sum_{k=1}^{\infty} \frac{\ln k}{k}.$$

Problem 4 (16 points). Using the test or method of your choice, find whether the following series converge or diverge. Please show the details; do not simply write down the answer (4 points each).

(1)

$$\sum_{k=1}^{\infty} \frac{k}{1+k^3},$$

(2)

$$\sum_{k=1}^{\infty} \frac{3^k}{k!},$$

(3)

$$\sum_{k=2}^{\infty} \frac{k^2}{k^2+k-3},$$

(4)

$$\sum_{k=2}^{\infty} \frac{\ln k}{k^3}.$$

Solution

(1) We will use the limit comparison test to compare this series to $\sum(1/k^2)$ which converges (p-series for $p = 2$).

$$\lim_{k\to\infty} \frac{k/(1+k^3)}{1/k^2} = \lim_{k\to\infty} \frac{k^3}{1+k^3} = 1.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{k}{1+k^3}$$

converges.

(2) We use the ratio test. Let $a_k = 3^k/k!$. Then,

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{3^{k+1}/(k+1)!}{3^k/k!} = \lim_{k\to\infty} \frac{3^{k+1}k!}{3^k(k+1)!} = \lim_{k\to\infty} \frac{3}{k+1} = 0.$$

Since the limit above is strictly less than 1, by the ratio test, the series converges.

(3) Note that
$$\lim_{k\to\infty} \frac{k^2}{k^2+k-3} = 1 \neq 0.$$
Therefore, by the k-th term test for divergence, the series diverges.

(4) One could solve this by the integral test, but the limit comparison test is shorter. We will compare the series to $\sum 1/k^2$ which converges:
$$\lim_{k\to\infty} \frac{(\ln k)/k^3}{1/k^2} = \lim_{k\to\infty} \frac{\ln k}{k} = 0.$$
Therefore, since $\sum 1/k^2$ converges, by the limit comparison test, so does $\sum \ln k/k^3$.

Problem 5 (5 points). Write $.2424242424\ldots$ as a fraction. Show your work.

Solution
$$\begin{aligned} .2424242424\ldots &= .24 + .0024 + .000024 + \ldots \\ &= .24(1 + 1/100 + 1/100^2 + \ldots) \\ &= .24(\frac{1}{1-1/100}) \\ &= .24 \cdot \frac{100}{99} = \frac{8}{33}. \end{aligned}$$

Problem 6 (Bonus problem worth an addition 5 points). Consider the series
$$\sum_{k=1}^{\infty} \frac{k}{2^k}.$$
By the ratio test one can show that this converges (you do not need to show this) to a sum S. The goal of this question is to determine what S is.

(1) Manipulate the sum to show that
$$(1-\frac{1}{2})S = \sum_{k=1}^{\infty} 2^{-k}.$$

(2) Use the above to show that $S = 2$.

Solution Actually, this is now assigned as a homework bonus problem.