CALCULUS 133: SAMPLE FINAL SOLUTIONS

Problem 1 (25 points)**.** *Evaluate each of the following integrals, or show that it diverges (5 points each):*

(1)
$$\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx,$$

(2)
$$\int_1^2 \frac{x+1}{\sqrt{x-1}}\,dx,$$

(3)
$$\int_0^{\pi/2} \sin^3 x\,dx,$$

(4)
$$\int_1^\infty \frac{x}{1+x^2}\,dx,$$

(5)
$$\int_0^1 \ln x\,dx.$$

Hint: Use integration by parts and then L'Hôpital's rule.

Solution

(1) Notice that $1/\sqrt{1-x^2}$ blows up when $x=-1$ or $x=1$. However, neither of those points is in the interval we're integrating over. Therefore, this is not an improper integral.

$$\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = \left[\sin^{-1}(x)\right]_0^{1/2} = \sin^{-1}(1/2) - \sin^{-1}(0) = \pi/6.$$

(2) In this case, our function blows up at 1 and 1 is in the interval we are integrating over. Therefore,

$$\int_1^2 \frac{x+1}{\sqrt{x-1}}\,dx \quad = \quad \lim_{a\to 1^+} \int_a^2 \frac{x+1}{\sqrt{x-1}}\,dx$$

To evaluate the integral, we do a rationalizing substitution. Let $u = \sqrt{x-1}$, so that $x = u^2+1$ and $dx = 2u\,du$. When $x=a$, $u=\sqrt{a-1}$ and when $x = 2$, $u = 1$. Then,

$$\begin{aligned}
\lim_{a\to 1^+}\int_a^2 \frac{x+1}{\sqrt{x-1}}\,dx &= \lim_{a\to 1^+}\int_{\sqrt{a-1}}^1 \frac{u^2+2}{u}(2u\,du)\\
&= \lim_{a\to 1^+}\int_{\sqrt{a-1}}^1 (2u^2+4)\,du\\
&= \lim_{a\to 1^+}\left[\frac{2}{3}u^3+4u\right]_{\sqrt{a-1}}^1\\
&= \lim_{a\to 1^+}\frac{2}{3}+4-\frac{2}{3}(a-1)^{3/2}-4\sqrt{a-1} = \frac{14}{3}.
\end{aligned}$$

(3)

$$\begin{aligned}
\int_0^{\pi/2}\sin^3(x)\,dx &= \int_0^{\pi/2}\sin^2(x)\sin(x)\,dx\\
&= \int_0^{\pi/2}(1-\cos^2(x))\sin x\,dx.
\end{aligned}$$

Now let $u = \cos x$ so that $du = -\sin x\,dx$. When $x = 0$, $u = 1$ and when $x = \pi/2$, $u = 0$. Thus,

$$\begin{aligned}
\int_1^0 (1-u^2)(-\,du) &= \int_0^1 (1-u^2)\,du\\
&= \left[u-\frac{u^3}{3}\right]_0^1 = 2/3.
\end{aligned}$$

(4)

$$\int_1^\infty \frac{x}{1+x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{x}{1+x^2}\,dx.$$

Let $u = 1+x^2$ so that $du = 2x\,dx$. When $x = 1$, $u = 2$; when $x = b$, $u = b^2+1$. Therefore,

$$\begin{aligned}
\lim_{b\to\infty}\int_1^b \frac{x}{1+x^2}\,dx &= \lim_{b\to\infty}\int_2^{1+b^2}\frac{1}{u}(\frac{1}{2}\,du)\\
&= \lim_{b\to\infty}\frac{1}{2}\left[\ln u\right]_2^{b^2+1}\\
&= \frac{1}{2}\lim_{b\to\infty}\ln(b^2+1)-\ln 2 = \infty.
\end{aligned}$$

(5) Note that $\ln x$ blows up at 0 ($\lim_{x\to 0^+}\ln x = -\infty$). So,

$$\int_0^1 \ln x\,dx = \lim_{a\to 0^+}\int_a^1 \ln x\,dx.$$

To evaluate the integral, we use integration by parts: let $u = \ln x$, $du = (1/x)\,dx$ and $dv = dx$ so that $v = x$. Then,

$$\begin{aligned}
\lim_{a\to 0^+}\int_a^1 \ln x\,dx &= \lim_{a\to 0^+}\left[x\ln x\right]_a^1 - \int_a^1 dx\\
&= \lim_{a\to 0^+} a\ln a - 1 + a.
\end{aligned}$$

However, by L'Hôpital's rule,

$$\lim_{a\to 0^+} a\ln a = \lim_{a\to 0^+} \frac{\ln a}{1/a} = \lim_{a\to 0^+} \frac{1/a}{-1/a^2} = \lim_{a\to 0^+} -a = 0.$$

Therefore,

$$\lim_{a\to 0^+} a\ln a - 1 + a = -1.$$

Problem 2 (12 points)**.** *Compute the following limits (4 points each):*

(1)
$$\lim_{x\to 0^+} \frac{e^x - 1}{x^{3/2}},$$

(2)
$$\lim_{x\to\infty} \frac{\ln x + x^2}{e^x},$$

(3)
$$\lim_{x\to 0^+} \frac{x}{\ln x}.$$

Solution (1) We use L'Hôpital's rule:

$$\lim_{x\to 0^+} \frac{e^x - 1}{x^{3/2}} = \lim_{x\to 0^+} \frac{e^x}{(3/2)x^{1/2}} = \infty.$$

(2) One could use L'Hôpital's rule, or simply note that e^x grows more quickly than $\ln x$ or x^2. Thus,

$$\lim_{x\to\infty} \frac{\ln x + x^2}{e^x} = \lim_{x\to\infty} \frac{\ln x}{e^x} + \frac{x^2}{e^x} = 0 + 0 = 0.$$

(3) We should not use L'Hôpital's rule, as this is not an indeterminate form. As $x \to 0^+$, $x \to 0$ and $\ln x \to -\infty$. Therefore,

$$\lim_{x\to 0^+} \frac{x}{\ln x} = 0.$$

Problem 3 (16 points)**.** *Using the test of your choice, determine whether the following series converge or diverge (4 points each):*

(1)
$$\sum_{n=1}^{\infty} \frac{n^2 + 3n + 1}{n^4 + 1},$$

(2)
$$\sum_{n=1}^{\infty} \frac{n}{2^n},$$

(3)
$$\sum_{n=1}^{\infty} ne^{-n^2},$$

(4)
$$\sum_{n=2}^{\infty} (-1)^n \ln n.$$

Solution

(1) We use the limit comparison test and compare to $\sum 1/n^2$.

$$\lim_{n\to\infty} \frac{(n^2+3n+1)/(n^4+1)}{1/n^2} = \lim_{n\to\infty} \frac{n^4+3n^3+n^2}{n^4+1} = 1.$$

Therefore, since $\sum 1/n^2$ converges (p-series where $p=2$), so does

$$\sum_{n=1}^{\infty} \frac{n^2+3n+1}{n^4+1}.$$

(2) We use the ratio test. Let $a_n = n/2^n$. Then

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)/2^{n+1}}{n/2^n} = \lim_{n\to\infty} \frac{n+1}{n}\frac{2^n}{2^{n+1}} = \frac{1}{2}\lim_{n\to\infty} \frac{n+1}{n} = \frac{1}{2}.$$

Since $\rho < 1$, the series converges.

(3) We can use the Ratio Test again here. Or we can use the Integral Test. Let's do the latter. Our series converges if and only if

$$\int_1^{\infty} xe^{-x^2}\,dx$$

converges. This integral is the same as the first integral in Midterm 2 (with a 1 replacing the 2). See the solutions of that for how to evaluate the integral. One gets that the integral converges and therefore the series converges.

(4) This is an alternating series. By the alternating series test, the series converges if and only if $\ln n \to 0$ as $n \to \infty$. However,

$$\lim_{n\to\infty} \ln n = \infty.$$

Therefore, the series diverges (one could also just use the n-th term test for divergence).

Problem 4 (15 points)**.** *Find the convergence set and the radius of convergence of the following power series (5 points each):*

(1)

$$\sum_{n=0}^{\infty} \frac{3^n}{n!}x^n = 1 + 3x + \frac{9x^2}{2} + \frac{27x^3}{6} + \dots,$$

(2)

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots,$$

(3)

$$\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}}x^n = 2x + \frac{2}{\sqrt{2}}x^2 + \frac{8}{\sqrt{3}}x^3 + \dots.$$

Solution

(1) We use the Absolute Ratio Test. Let $a_n = (3^n x^n)/n!$. Then,

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|(3^{n+1}x^{n+1})/(n+1)!\right|}{|(3^n x^n)/n!|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n}\frac{3^{n+1}}{3^n}\frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{3\,|x|}{n+1} = 0.$$

Since $\rho < 1$ for all x, this power series converges for all x. So the convergence set is all of $\mathbb{R}$ and the radius of convergence is ∞.

(2) Again, the Absolute Ratio Test:

$$\rho = \lim_{n\to\infty} \frac{|x|^{n+1}/(n+1)}{|x|^n/n} = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n}\frac{n}{n+1} = \lim_{n\to\infty} |x| \frac{n}{n+1} = |x|.$$

Therefore, if $|x| < 1$, the series converges. If $|x| > 1$, the series diverges. To find out what happens when $x = 1$ or $x = -1$, we plug those values into our series.

When $x = 1$, our power series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges (harmonic series).

When $x = -1$, our power series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which converges (alternating harmonic series).

Therefore, the convergence set is $[-1, 1)$ and the radius of convergence is 1.

(3) The Absolute Ratio Test one more time.

$$\rho = \lim_{n\to\infty} \frac{(2^{n+1}|x|^{n+1})/\sqrt{n+1}}{(2^n|x|^n)/\sqrt{n}} = \lim_{n\to\infty} 2|x| \frac{\sqrt{n}}{\sqrt{n+1}} = 2|x|.$$

Therefore, the series converges when $2|x| < 1$ i.e. when $|x| < 1/2$ and diverges when $|x| > 1/2$. To find out what happens when $x = 1/2$ or $x = -1/2$, we plug those values into our series.

When $x = 1/2$, the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

which diverges (p-series where $p = 1/2$).

When $x = -1/2$ the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

which converges by the alternating series test.

Therefore, the convergence set is $[-1/2, 1/2)$ and the radius of convergence is $1/2$.

Problem 5 (8 points)**.** *Find the Taylor series expansion of* $f(x) = 1 + 2x^2 + 7x^3$ *at the point* 1.

Solution $f(x) = 1 + 2x^2 + 7x^3$; $f(1) = 10$ and so $c_0 = 10$.
$f'(x) = 4x + 21x^2$; $f'(1) = 25$ and so $c_1 = 25$.
$f''(x) = 4 + 42x$; $f''(1) = 46$ and so $c_2 = 23$.
$f'''(x) = 42$; $f'''(1) = 42$ and so $c_3 = 7$.
One can verify that all higher order derivatives will be 0, and therefore the Taylor series expansion at the point 1 is

$$f(x) = 10 + 25(x-1) + 23(x-1)^2 + 7(x-1)^3.$$

If you want to verify the above formula, you can expand out the right hand side and make sure it is equal to f (note this is an instance where the Taylor series is a finite sum instead of an infinte one).

Problem 6 (12 points)**.**

(1) *What does*

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

converge to? For what values of x is this valid (4 points)?

(2) *Using the previous result, find a power series representation for $1/(1+2x)$. For what values of x is it valid (4 points)?*

(3) *Using the previous result, find a power series representation for $x^2/(1+2x)$. For what values of x is it valid (4 points)?*

Solution

(1) This converges to $1/(1-x)$ and is valid for $-1 < x < 1$ (this is example 1 page 495).

(2) Note that

$$\frac{1}{1+2x} = \frac{1}{1-(-2x)} = 1+(-2x)+(-2x)^2+(-2x)^3+\dots = 1-2x+4x^2-8x^3+\dots.$$

This will be valid for $-1 < -2x < 1$, i.e. $-1/2 < x < 1/2$.

(3) Notice that

$$\frac{x}{1+2x} = x(\frac{1}{1+2x}) = x - 2x^2 + 4x^3 - 8x^4 + \dots.$$

This will also be valid for $-1/2 < x < 1/2$.

Problem 7 (12 points)**.**

(1) *Give the Maclaurin series of e^x. For what values of x is this valid? (4 points)*

(2) *Use the previous result to give the power series representation of $e^{x^2} - x^2 - 1$ (4 points).*

(3) *Use the previous result to find*

$$\lim_{x \to 0} \frac{e^{x^2} - x^2 - 1}{x^4}$$

without using L'Hôpital's Rule (4 points)

Solution

(1) This is number 4 page 495.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

and is valid for all x in $\mathbb{R}$.

(2) Using the above,

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \dots.$$

Therefore,

$$e^{x^2} - x^2 - 1 = \frac{x^4}{2} + \frac{x^6}{6} + \dots.$$

This will also be valid for all x.

(3)

$$\lim_{x\to 0} \frac{e^{x^2} - x^2 - 1}{x^4} = \lim_{x\to 0} \frac{1}{x^4}\left(\frac{x^4}{2} + \frac{x^6}{6} + \dots\right) = \lim_{x\to 0} \frac{1}{2} + \frac{x^2}{6} + \dots = \frac{1}{2}.$$