$1$-Measure equivalence rigidity of hyperbolic lattices

Uri Bader  Alex Furman  Roman Sauer

$^1$Technion

$^2$University of Illinois

$^3$University of Chicago

Banff, March 2008
$l^p$-Measure equivalence

Measure equivalence with $l^p$-condition

A **ME-coupling** $(\Omega, \mu)$ of $\Gamma$ and $\Lambda$ is a measure space with a $\mu$-preserving action of $\Gamma \times \Lambda$ such that $\Gamma$, $\Lambda$ both have a $\mu$-finite fundamental domain. If $\Gamma$ and $\Lambda$ admit a ME-coupling with $l^p$-integrable cocycles w.r.t. some fundamental domains then we call them **$l^p$-measure equivalent**.

$l^p$-cocycles

A measurable cocycle $\alpha : \Gamma \times (X, \mu) \rightarrow \Lambda$ is **$l^p$-integrable** if for every $\gamma \in \Gamma$

$$\int_X l(\alpha(\gamma, x))^p d\mu(x) < \infty,$$

where $l : \Lambda \rightarrow \mathbb{N}$ is the length function for some word metric on $\Lambda$.

- $l^p$-ME interpolates between $p = \infty$ ($\Rightarrow$ QI) and $p = 0$ ($\Leftarrow$ ME).
- $l^p$-ME is an equivalence relation on groups.
\( p \)-Measure equivalence

Measure equivalence with \( p \)-condition

A ME-coupling \((\Omega, \mu)\) of \(\Gamma\) and \(\Lambda\) is a measure space with a \(\mu\)-preserving action of \(\Gamma \times \Lambda\) such that \(\Gamma, \Lambda\) both have a \(\mu\)-finite fundamental domain. If \(\Gamma\) and \(\Lambda\) admit a ME-coupling with \(p\)-integrable cocycles w.r.t. some fundamental domains then we call them \(p\)-measure equivalent.

\( p \)-coycles

A measurable cocycle \(\alpha : \Gamma \times (X, \mu) \to \Lambda\) is \(p\)-integrable if for every \(\gamma \in \Gamma\)

\[
\int_X l(\alpha(\gamma, x))^p \, d\mu(x) < \infty,
\]

where \(l : \Lambda \to \mathbb{N}\) is the length function for some word metric on \(\Lambda\).

- \(p\)-ME interpolates between \(p = \infty\) (\(\Rightarrow\) QI) and \(p = 0\) (\(\Rightarrow\) ME).
- \(p\)-ME is an equivalence relation on groups.
Rigidity result for hyperbolic lattices

**Theorem (informal)**

Let $\Gamma$ be a lattice in $G = \text{Isom}(\mathbb{H}^n)$, $n \geq 3$. Then any $l^1$-ME-coupling of $\Gamma$ with another group basically comes from the standard example of lattices in $G$ or atomic couplings of commensurable groups.

- The standard coupling of hyperbolic lattices is $l^1$-integrable.
- A corresponding rigidity result for orbit equivalence (OE) can be formulated.
- Analogous rigidity results (without any $l^1$-integrability condition) for lattices in higher rank Lie groups hold true [Furman, 2000].
- Lack of rigidity for $n = 2$: $\mathbb{Z}^2 \ast \mathbb{Z}^2$ OE to $\mathbb{Z} \ast \mathbb{Z}$.
Theorem (informal)

Let \( \Gamma \) be a lattice in \( G = \text{Isom}(\mathbb{H}^n), n \geq 3 \). Then any \( l^1\)-ME-coupling of \( \Gamma \) with another group basically comes from the standard example of lattices in \( G \) or atomic couplings of commensurable groups.

- The standard coupling of hyperbolic lattices is \( l^1\)-integrable.
- A corresponding rigidity result for orbit equivalence (OE) can be formulated.
- Analogous rigidity results (without any \( l^1\)-integrability condition) for lattices in higher rank Lie groups hold true [Furman, 2000].
- Lack of rigidity for \( n = 2 \): \( \mathbb{Z}^2 \ast \mathbb{Z}^2 \) OE to \( \mathbb{Z} \ast \mathbb{Z} \).
Let $\Gamma$ be a lattice in $G = \text{Isom}(\mathbb{H}^n)$, $n \geq 3$. Let $(\Omega, \mu)$ be an ergodic, $l^1$-integrable ME-coupling with another group $\Lambda$. Then the following holds:

a) There exists a homomorphism $\rho : \Lambda \to G$ with finite kernel and image being a lattice in $G$.

b) There exists a $\Gamma \times \Lambda$-equivariant measurable map $\phi : \Omega \to G$; the push-forward measure $\phi_*\mu$ is the Haar measure corresponding either
   i) to $G$,
   ii) or to its index two subgroup $G^0 = \text{Isom}_+(\mathbb{H}^n)$,
   iii) or to a lattice $\Gamma' < G$.

In the latter case, $\Gamma'$ contains $\Gamma$ and $\rho(\Lambda)$ as subgroups of finite index.
Classical Mostow rigidity

**Theorem (Mostow rigidity – Lie-theoretic version)**

Any isomorphism $\Gamma \to \Lambda$ of lattices in $G = \text{Isom}(\mathbb{H}^n)$, $n \geq 3$, extends to an automorphism of $G$.

**Theorem (Mostow rigidity – topological version)**

Let $M$ and $N$ be closed hyperbolic $n$-dimensional manifolds. Then any homotopy equivalence $M \to N$ is homotopic to an isometry.

**topological version $\Rightarrow$ Lie-theoretic version:**

Extension of map

\[ \tilde{M} \to \tilde{N} \]

\[ \Gamma \to \Lambda \]
Classical Mostow rigidity

**Theorem (Mostow rigidity – Lie-theoretic version)**

Any isomorphism $\Gamma \to \Lambda$ of lattices in $G = \text{Isom}(\mathbb{H}^n)$, $n \geq 3$, extends to an automorphism of $G$.

**Theorem (Mostow rigidity – topological version)**

Let $M$ and $N$ be closed hyperbolic $n$-dimensional manifolds. Then any homotopy equivalence $M \to N$ is homotopic to an isometry.

topological version $\Rightarrow$ Lie-theoretic version:

**Extension of map**

\[
\begin{array}{c}
\tilde{M} \to \tilde{N} \\
\Uparrow \\
\Gamma \to \Lambda
\end{array}
\]

**Induction over skeleta**

\[
\bigsqcup_{S_i} \Gamma \times \partial \Delta^i \to \tilde{M}^{(i-1)} \\
\downarrow \\
\bigsqcup_{S_i} \Gamma \times \Delta^i \to \tilde{M}^{(i)} \to \tilde{N}
\]
Classical Mostow rigidity

**Theorem (Mostow rigidity – Lie-theoretic version)**

Any isomorphism $\Gamma \rightarrow \Lambda$ of lattices in $G = \text{Isom}(\mathbb{H}^n)$, $n \geq 3$, extends to an automorphism of $G$.

**Theorem (Mostow rigidity – topological version)**

Let $M$ and $N$ be closed hyperbolic $n$-dimensional manifolds. Then any homotopy equivalence $M \rightarrow N$ is homotopic to an isometry.

topological version $\Rightarrow$ Lie-theoretic version:

**Extension of map**

$\tilde{M} \rightarrow \tilde{N}$

$\Gamma \rightarrow \Lambda$

**Induction over skeleta**

$\bigsqcup_{s_i \{1\}} \times \partial \Delta^i \rightarrow \tilde{M}^{(i-1)}$

$\bigsqcup_{s_i \{1\}} \times \Delta^i \rightarrow \tilde{M}^{(i)} \rightarrow \tilde{N} \simeq \text{pt}$
Classical Mostow rigidity

Theorem (Mostow rigidity – Lie-theoretic version)

Any isomorphism \( \Gamma \rightarrow \Lambda \) of lattices in \( G = \text{Isom}(\mathbb{H}^n) \), \( n \geq 3 \), extends to an automorphism of \( G \).

Theorem (Mostow rigidity – topological version)

Let \( M \) and \( N \) be closed hyperbolic \( n \)-dimensional manifolds. Then any homotopy equivalence \( M \rightarrow N \) is homotopic to an isometry.

topological version \( \Rightarrow \) Lie-theoretic version:

Extension of map

\[
\begin{array}{c}
\tilde{M} \\
\uparrow \downarrow \uparrow \downarrow \\
\Gamma \rightarrow \Lambda \\
\end{array}
\rightarrow
\begin{array}{c}
\tilde{N} \\
\end{array}
\]

Induction over skeleta

\[
\bigsqcup_{S_i} \Gamma \times \partial \Delta^i \rightarrow \tilde{M}^{(i-1)} \\
\downarrow \\
\bigsqcup_{S_i} \Gamma \times \Delta^i \rightarrow \tilde{M}^{(i)} \\
\downarrow \\
\rightarrow \tilde{N}
\]
Thurston’s proof of (topological) Mostow rigidity

Proof for closed manifolds

Step 1) \( f : M \simeq N \Rightarrow \|M\| = \|N\| \Rightarrow \text{vol}(M) = \text{vol}(N) \) [Gromov-Thurston].

Step 2) \( \tilde{f} : \mathbb{H}^n \to \mathbb{H}^n \) is a quasi-isometry, thus induces a homeomorphism \( \partial_\infty \tilde{f} : \partial_\infty \mathbb{H}^n \xrightarrow{\simeq} \partial_\infty \mathbb{H}^n \).

Step 3) Regular, ideal \( n \)-simplices are exactly the geodesic \( n \)-simplices with maximal volume [Haagerup-Munkholm]. \( \partial_\infty \tilde{f} \) preserves regular, ideal simplices.

Step 4) Hyperbolic geometry: \( \partial_\infty \tilde{f} \) induced by an isometry.

Modification for finite volume manifolds

Only from volume considerations, Thurston constructs a measurable \( \partial_\infty \tilde{f} \) that preserves regular, ideal \( n \)-simplices almost everywhere.
Thurston’s proof of (topological) Mostow rigidity

Proof for closed manifolds

Step 1) $f : M \xrightarrow{\sim} N \Rightarrow \|M\| = \|N\| \Rightarrow \text{vol}(M) = \text{vol}(N)$ [Gromov-Thurston].

Step 2) $\tilde{f} : H^n \rightarrow H^n$ is a quasi-isometry, thus induces a homeomorphism $\partial_\infty \tilde{f} : \partial_\infty H^n \xrightarrow{\sim} \partial_\infty H^n$.

Step 3) Regular, ideal $n$-simplices are exactly the geodesic $n$-simplices with maximal volume [Haagerup-Munkholm]. $\partial_\infty \tilde{f}$ preserves regular, ideal simplices.

Step 4) Hyperbolic geometry: $\partial_\infty \tilde{f}$ induced by an isometry.

Modification for finite volume manifolds

Only from volume considerations, Thurston constructs a measurable $\partial_\infty \tilde{f}$ that preserves regular, ideal $n$-simplices almost everywhere.
Let $\Gamma$ be a lattice in $G = \text{Isom}(\mathbb{H}^n)$, and $\Lambda$ be an arbitrary group ME to $\Gamma$ via the coupling $(\Omega, m)$. Let $(\Sigma, n) = (\Omega, m) \times_\Lambda (\Omega^\text{op}, m)$ be the corresponding self-coupling of $\Gamma$. Assume that there exists a measurable $\Gamma \times \Gamma$-equivariant map $\Phi : \Sigma \to G$ ("untwisting map"), i.e. $n$-a.e.

$$\Phi([[\gamma x, \gamma'y]]) = \gamma \Phi([x, y])\gamma'^{-1} \quad (\gamma, \gamma' \in \Gamma).$$

Then there exist measurable maps $f : \Omega \to G$ and a homomorphism $\rho : \Lambda \to G$ so that

$$f((\gamma, \lambda)x) = \gamma f(x)\rho(\lambda)^{-1}.$$  

Then elementary observations (for lattice image) and an application of Ratner’s theorems (for identifying $\Phi_* n$) eventually yield the main theorem.

…How do we get the untwisting map?
Setting

– Let $X \subset \Sigma$ be a common fundamental domain of both copies of $\Gamma$, and $\alpha : \Gamma \times X \to X$ be the corresponding OE-cocycle.

– We may assume that $\Gamma \subset \text{Isom}(\mathbb{H}^n)$ is co-compact. Let $M = \Gamma \backslash \mathbb{H}^n$.

Proof of main theorem – outline

Step 1) Extend $\alpha : X \to \text{map}(\Gamma, \Gamma)$ to a $\alpha$-equivariant, measurable map $
\psi : X \to \text{map}(\tilde{M}, \tilde{M}).$

Step 2) Show that $\psi$ induces a measurable, $\alpha$-equivariant map $\partial_{\infty} \psi : X \to \mathcal{M}(\partial \mathbb{H}^n, \partial \mathbb{H}^n)$ that preserves regular, ideal $n$-simplices.

Step 3) Hyperbolic geometry $\Rightarrow \partial_{\infty} \psi$ comes from of a $\alpha$-equivariant map $\phi : X \to \text{Isom}(\tilde{M}) = G$ (cocycle Mostow rigidity).

Step 4) $\phi$ is a coboundary for $\alpha$; thus we can also untwist $\Sigma$. 
A crucial step in the proof – controlling volume

**Lemma**

For any geodesic simplex $\sigma$ with $\text{vol}(\sigma) \approx v_{\text{max}}$ we have

$$\int_X \int_{\Gamma \backslash G} \text{vol}(\psi_X(g\sigma)) d\mu_X(x) d\mu_{\Gamma \backslash G}(g) \approx \text{vol}(\sigma).$$

**Volume and degree 1 maps**

Let $f : M \to M$ be a degree 1 map. Let $c = \sum a_i \sigma_i$ be an $n$-cycle. Then:

$$\sum a_i \text{vol}^{\text{or}}(\sigma_i) = \sum a_i \text{vol}^{\text{or}}(f(\sigma)).$$

- Find suitable homology theories for our situation.
- Show that $\psi : X \to \text{map}(\tilde{M}, \tilde{M})$ is of degree 1.
- View left side of lemma as the evaluation of a homology class at the volume form.
A crucial step in the proof – controlling volume

Lemma

For any geodesic simplex $\sigma$ with $\text{vol}(\sigma) \approx v_{\text{max}}$ we have

$$\int_X \int_{\Gamma \backslash G} \text{vol}(\psi_x(g\sigma))d\mu_X(x)d\mu_{\Gamma \backslash G}(g) \approx \text{vol}(\sigma).$$

Volume and degree 1 maps

Let $f : M \to M$ be a degree 1 map. Let $c = \sum a_i \sigma_i$ be an $n$-cycle. Then:

$$\sum a_i \text{vol}^{\text{or}}(\sigma_i) = \sum a_i \text{vol}^{\text{or}}(f(\sigma)).$$

- Find suitable homology theories for our situation.
- Show that $\psi : X \to \text{map}(\tilde{M}, \tilde{M})$ is of degree 1.
- View left side of lemma as the evaluation of a homology class at the volume form.
A crucial step in the proof – controlling volume

Lemma

For any geodesic simplex $\sigma$ with $\text{vol}(\sigma) \approx \nu_{\max}$ we have

$$\int_X \int_{\Gamma \backslash G} \text{vol}(\psi_x(g \sigma))d\mu_X(x)d\mu_{\Gamma \backslash G}(g) \approx \text{vol}(\sigma).$$

Volume and degree 1 maps

Let $f : M \rightarrow M$ be a degree 1 map. Let $c = \sum a_i \sigma_i$ be an $n$-cycle. Then:

$$\sum a_i \text{vol}^{or}(\sigma_i) = \sum a_i \text{vol}^{or}(f(\sigma)).$$

- Find suitable homology theories for our situation.
- Show that $\psi : X \rightarrow \text{map}(\tilde{M}, \tilde{M})$ is of degree 1.
- View left side of lemma as the evaluation of a homology class at the volume form.
Maps induced by $\alpha : \Gamma \times X \to \Gamma$ and $\psi : X \to \text{map}(\tilde{M}, \tilde{M})$

$\mathbb{Z} \otimes \mathbb{Z}_\Gamma \ C_*(\Gamma) \xrightarrow{C_*(\alpha)} \mathbb{Z} \otimes \mathbb{Z}_\Gamma \ C^\text{geo}_*(\tilde{M})$

$L^1(X; \mathbb{Z}) \otimes \mathbb{Z}_\Gamma \ C_*(\Gamma) \xrightarrow{C^\text{geo}_*(\phi)} L^1(X; \mathbb{Z}) \otimes \mathbb{Z}_\Gamma \ C^\text{geo}_*(\tilde{M})$

Remarks

- $L^1(X; \mathbb{Z}) \otimes \mathbb{Z}_\Gamma \ C^\text{geo}_*(\tilde{M}) = \bigoplus_F L^1(X; \mathbb{Z})$
- Vertical maps are inclusions of orbits.
- $C_0(\alpha)(1 \otimes \gamma) = \sum \chi_{X_i} \otimes \gamma_i$ where $\alpha(x, \gamma) = \gamma_i$ constant on $x \in X_i$. 

ℒ-homology and induced maps
A new deformation-rigidity phenomenon

Integrality, Poincare duality, simplicial volume
We have by Poincare duality and ergodicity

\[ H_n(L^1(X; \mathbb{Z}) \otimes_{\mathbb{Z} \Gamma} C_{\mathrm{geo}}^*(\tilde{M})) \cong H^0(\tilde{M}; L^1(X; \mathbb{Z})) = L^1(X; \mathbb{Z})^\Gamma \cong \mathbb{Z}. \]

Since the simplicial volume of \( M \) is \( > 0 \) every Cauchy sequence of cycles in \( L^1(X; \mathbb{Z}) \otimes_{\mathbb{Z} \Gamma} C_{n}^\mathrm{geo}(\tilde{M}) \) is eventually constant!

Sobolev homology and \( l^1 \)-condition
Under \( l^1 \)-integrability, we show that \( H_n(\phi) \) already lands in

\[ H_n(L^1(X; \mathbb{Z}) \otimes_{\mathbb{Z} \Gamma} C_{*}^\mathrm{geo}(\tilde{M})) \subset H_n(\overline{L^1(X; \mathbb{Z}) \otimes_{\mathbb{Z} \Gamma} C_{*}^\mathrm{geo}(\tilde{M})}^1) \]

For this we use a new tool (Sobolev homology) and the ability to subdivide geodesic simplices in negative curvature very efficiently.
A new deformation-rigidity phenomenon

Integrality, Poincare duality, simplicial volume

We have by Poincare duality and ergodicity

\[ H_n(L^1(X; \mathbb{Z}) \otimes_{\mathbb{Z} \Gamma} C^\text{geo}_* (\tilde{M})) \cong H^0(\tilde{M}; L^1(X; \mathbb{Z})) = L^1(X; \mathbb{Z})^{\Gamma} \cong \mathbb{Z}. \]

Since the simplicial volume of \( M \) is \( > 0 \) every Cauchy sequence of cycles in \( L^1(X; \mathbb{Z}) \otimes_{\mathbb{Z} \Gamma} C^\text{geo}_n (\tilde{M}) \) is eventually constant!

Sobolev homology and \( l^1 \)-condition

Under \( l^1 \)-integrability, we show that \( H_n(\phi) \) already lands in

\[ H_n(L^1(X; \mathbb{Z}) \otimes_{\mathbb{Z} \Gamma} C^\text{geo}_* (\tilde{M})) \subset H_n(L^1(X; \mathbb{Z}) \otimes_{\mathbb{Z} \Gamma} C^\text{geo}_* (\tilde{M})^{l^1}) \]

For this we use a new tool (Sobolev homology) and the ability to subdivide geodesic simplices in negative curvature very efficiently.
### A new deformation-rigidity phenomenon

#### Integrality, Poincare duality, simplicial volume

We have by Poincare duality and ergodicity

\[
H_n(L^1(X; \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C^\text{geo}_*(\tilde{M})) \cong H^0(\tilde{M}; L^1(X; \mathbb{Z})) = L^1(X; \mathbb{Z})^\Gamma \cong \mathbb{Z}.
\]

Since the simplicial volume of \( M \) is \( > 0 \) every Cauchy sequence of cycles in \( L^1(X; \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C^\text{geo}_n(\tilde{M}) \) is eventually constant!

#### Sobolev homology and \( l^1 \)-condition

Under \( l^1 \)-integrability, we show that \( H_n(\phi) \) already lands in

\[
H_n(L^1(X; \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C^\text{geo}_*(\tilde{M})) \subset H_n(L^1(X; \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C^\text{geo}_*(\tilde{M})^1)
\]

For this we use a new tool (Sobolev homology) and the ability to subdivide geodesic simplices in negative curvature very efficiently.

... THANK YOU!