NOTES ON KLEINER’S PROOF OF GROMOV’S POLYNOMIAL GROWTH THEOREM

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Abstract. We present and explain Kleiner’s new proof of Gromov’s polynomial growth [Kle07] theorem which avoids the use of Montgomery-Zippin theory. We also stress the connection to Shalom’s property $H_{FD}$. No originality is claimed. These notes have not yet been proofread or polished.

1. Introduction

We present Bruce Kleiner’s proof [Kle07] of the following famous theorem of Mikhail Gromov.

**Theorem 1.1.** A finitely generated group with weak polynomial growth is virtually nilpotent.

A finitely generated group $\Gamma$ is equipped with the word metric with respect to some finite symmetric set of generators. Let $B_\Gamma(r) \subset \Gamma$ denote the $r$-ball around $e$ in $\Gamma$. One says that $\Gamma$ has *weak polynomial growth* if there is some $d > 0$ such that

$$\liminf_{r \to \infty} \frac{\# B_\Gamma(r)}{r^d} < \infty.$$

The crucial ingredient of the proof by Bruce Kleiner is the following theorem which is adapted from a corresponding theorem in the setting of Riemannian manifolds with non-negative Ricci curvature by Tobias Colding and William Minicozzi to that discrete groups.

**Theorem 1.2.** Let $\Gamma$ be a finitely generated group. Let $G = G(\Gamma, S)$ be the Cayley graph with respect to a finite, symmetric set $S$ of generators. If $\Gamma$ has weak polynomial growth then for every $d > 0$ the vector space

$$\mathcal{H}_d(G) = \{ f : G \to \mathbb{R}; \ f \text{ harmonic} , \exists C > 0 \forall r > 0 : |f(r)| < Cr^d \}$$

is finite dimensional.

See the beginning of Section 2 for some remarks on the definition of harmonicity.

2. Poincare inequalities for finitely generated groups

Let $\Gamma$ be a group with finite, symmetric generating set $S$. Let $G = G(\Gamma, S)$ the associated Cayley graph. We regard $G$ here as a 1-dimensional metric, simplicial complex as opposed to a combinatorial object. Further, $G$ carries a unique measure that restricts to the Lebesgue measure on each edge $e \cong [0, 1]$. Let $B(x, r) \subset G$ denote the ball of radius $r$ around $x \in G$. For $x \in \Gamma$ let $B_\Gamma(x, r) \subset \Gamma$ denote the
Here the sum runs over ordered pairs \((e,y,e)\) of edges contained in \(B(R)\). We say a function \(f\) on \(\mathcal{G}\) is smooth if \(f\) is smooth on the interior of every edge (in the usual sense). We denote its derivative by \(\nabla f : \mathcal{G} \to \mathbb{R}\) which is defined everywhere except at the vertices.

**Theorem 2.1.** Let \(R \in \mathbb{N}\). For every smooth \(f : B(3R) \to \mathbb{R}\) we have

\[
\sum_{x \in B(R)} |f - f_R|^2 \leq 8|S|^2 R^2 \frac{V(2R)}{V(R)} \sum_{x \in B(3R)} |\nabla f|^2.
\]

**Proof.** For \(s \in B(R)\) we obtain from Cauchy-Schwarz that

\[
(1) \quad |f(s) - f_R| \leq \frac{1}{V(R)} \int_{B(R)} 1 \cdot |f(s) - f(x)| dx
\]

\[
(2) \quad \leq \frac{V(R)^{1/2}}{V(R)} \left( \int_{B(R)} (f(s) - f(x))^2 dx \right)^{1/2}.
\]

Hence \(|f(s) - f_R|^2 \leq \int_{B(R)} (f(s) - f(x))^2 / V(R) dx\), and thus

\[
\int_{x \in B(R)} (f(x) - f_R)^2 \leq \frac{1}{V(R)} \int_{x,y \in B(R)} (f(x) - f(y))^2
\]

\[
\leq \frac{1}{V(R)} \sum_{(e_1,e_2) \subseteq B(R)} \int_{(x_1,x_2) \in e_1 \times e_2} (f(p_1) - f(p_2))^2 dp_1 dp_2.
\]

Here the sum runs over ordered pairs \((e_1,e_2)\) of edges contained in \(B(R)\). Next we estimate the integrand \(|f(p_1) - f(p_2)|\). Let’s fix \(e_1,e_2,p_1,p_2\) for the following consideration.

Let \(x_i \in e_i \cap \Gamma, i = 1,2\) be vertices such that \(d(x_1,x_2) \leq 2R - 2\). Let \(y = x^{-1}_1 x_2 \in B(2R - 2)\). Choose a shortest (vertex) path \(\gamma_y : 0,\ldots,d(y,e) \to \Gamma\) from \(e\) to \(y\). Let

\[\delta f : B(3R - 1) \to \mathbb{R}, \delta f(x) = \int_{B(x_1)} |\nabla f(w)|^2 dw.\]

We obviously have

\[|f(p_1) - f(p_2)| \leq \sum_{i=0}^{d(y,e)} \int_{B(x_1 \gamma_y(i),1)} |\nabla f(w)| dw\]

By Cauchy-Schwarz we obtain that

\[
\sum_{i=0}^{d(y,e)} \int_{B(x_1 \gamma_y(i),1)} 1 \cdot |\nabla f(w)| dw \leq (2R)^{1/2} \left( \sum_{i=0}^{d(y,e)} (\delta f(x_1 \gamma_y(i))) \right)^{1/2}.
\]

Note that the latter expression does not depend on \((p_1,p_2)\). Thus we have

\[
(3) \quad \int_{(p_1,p_2) \in e_1 \times e_2} |f(p_1) - f(p_2)|^2 dp_1 dp_2 \leq 2R \sum_{i=0}^{d(y,e)} (\delta f(x_1 \gamma_y(i))).
\]
Equations (2.1) and (2.3) and the fact that the map \((e_1, e_2) \mapsto (x_1, y)\) at most \(|S|^2\)-to-one yield

\[
\int_{B(R)} |f - f_R|^2 \leq 2R|S|^2/V(R) \sum_{x_1 \in B(R-1)} \sum_{y \in B(2R-2)} \sum_{i=0}^{\lfloor y \rfloor} (\delta f)(x_1 \gamma_y(i)).
\]

If \(y \in B(2R-2)\), then

\[
\sum_{x_1 \in B(R-1)} \sum_{i=0}^{\lfloor y \rfloor} (\delta f)(x_1 \gamma_y(i)) \leq 2R \sum_{z \in B(3R-1)} (\delta f)(z),
\]

since the map \(B(R-1) \times \{0, \ldots, d(y, e)\} \rightarrow B(3R-1)\) defined by \((x, i) \mapsto x \gamma_y(i)\) is at most \(2R\)-to-one. Thus, by (2.4),

\[
\int_{B(R)} |f - f_R|^2 \leq 4R^2|S|^2/V(R) \sum_{y \in B(2R-2)} \sum_{z \in B(3R-1)} (\delta f)^2(z)
\]

\[
\leq 4R^2|S|^2V(2R)/V(R) \sum_{y \in B(2R-2)} \sum_{z \in B(3R-1)} (\delta f)^2(z)
\]

\[
\leq 8R^2|S|^2V(2R)/V(R) \int_{B(3R)} |\nabla f|^2.
\]

\[\square\]

**Theorem 2.2** (Reverse Poincare inequality). There is a constant \(C > 0\) such that for any harmonic function \(f : G \rightarrow \mathbb{R}\) and every \(R > 0\) we have

\[
R^2 \int_{B(2R)} |\nabla f|^2 \leq C \int_{B(16R)} f^2.
\]

**Proof.**

\[\square\]

3. Finding good scales

This section serves as a preparation for the proof of Theorem 1.2. Throughout, let \(f\) be a monotone non-decreasing function \(\mathbb{N} \rightarrow \mathbb{R}_{>0}\).

We say that \(R = \{r_1, r_2, \ldots\} \subset \mathbb{N}\) is a set of good scales for \(f\) if \(R\) is infinite and there is a constant \(C > 0\) such that for every \(i \geq 1\)

\[
f(16r_i) < C f(r_i).
\]

When a constant \(C\) is fixed and (3.1) is satisfied we also say that \(f\) has the doubling property at scale \(r = r_i\).

Assume now that there is some \(d > 0\) with

\[
\liminf_{r \rightarrow \infty} \frac{f(r)}{r^d} < \infty.
\]

We prove as a warm-up the following little lemma.

**Lemma 3.1.** Let \(f\) satisfy (3.2). Then there is a set of good scales for \(f\). More precisely, we have \(f(16r) \leq 4 \cdot 16^{2d} f(r)\) for infinitely many \(r\).

\[1\text{For technical reasons, we have a } 16 \text{ here instead of a } 2.\]
Proof. Let \( M = \liminf_{r \to \infty} \frac{f(r)}{r^2} < \infty \), and let \((r_i)\) be a monotone sequence with \( M = \lim_{i \to \infty} \frac{f(r_i)}{r_i^2} \). Let \( C = 4 \cdot 16^{2d} \). Suppose there is no infinite sequence \((r_i)\) such that \( f(16r_i) \leq C f(r_i) \). Then there exists \( j \in \mathbb{N} \) so large that \( r_i^d/f(r_i) < 2M^{-1} \) and \( f(r_i)/r_i^d < 2M \) for \( l > j \) and for \( i > 1 \) we have
\[
 f(16^i r_j) > C^i f(r_j) .
\]
Choose \( i > 0 \) large enough so that there is \( k > j \) with \( 16^i r_j \leq r_k < 16^{i+1} r_j \). Then we obtain
\[
 C^i < f(16^i r_j)/f(r_j) \leq f(r_k)/f(r_j) < 2Mr_k^d/f(r_j) < 2M16^{d(i+1)}r_j^d/f(r_j) < 4 \cdot 16^d16^d ,
\]
which is a contradiction. \( \square \)

Now suppose we have two monotone non-decreasing functions \( f_1, f_2 \) satisfying (3.2) (with possibly different \( d \)'s). If we want a set that is simultaneously a set of good scales for \( f_1 \) and \( f_2 \) we simply apply the lemma above to the pointwise product \( f = f_1 f_2 \).

For the proof of Theorem 1.2, let \( V \subset \mathcal{H}_d(G) \) be an even-dimensional subspace. The goal is to bound the dimension of \( V \) by some constants. It turns out be essential in the proof to have a set \( R \) of scales such that the growth function and \( r \mapsto \int_{B(r)} u^2 \) for every \( u \in V \) (or, equivalently, for every \( u \) in a basis of \( V \)) have the doubling property at scales in \( R \).

The problem is that if we apply the lemma to the product of the growth function and all basis elements we will get constants that involve the dimension of \( V \) which prevents us from proving that this dimension is finite. That’s why Bruce Kleiner applies the lemma to the function
\[
 f(r) = V(r)(\det Q_r)^{1/\dim U}
\]
where \( Q \) is the symmetric form with \( Q_r(u, u) = \int_{B(r)} u^2 \). This provides not quite a set of scales for which all \( u \in V \) have the doubling property but one can show that there is subspace \( V' \subset V \) with \( \dim V' = \dim V/2 \) such that there is a set of scales for which the growth function and all \( Q_r(u), u \in V' \), have the doubling property with a constant only depending on \( d \) and the growth rate of the group.

Actually, Bruce Kleiner needs an even more sophisticated set of scales in the proof. We comment on that in the next section.

4. The proof of Theorem 1.2

Let \( V \) be a finite-dimensional subspace of \( \mathcal{H}_d(G) \). We want to bound the dimension of \( V \) by a constant only involving \( d \) and growth rate \( m \) of the group. We have seen in the previous section how to choose appropriate scales at which the growth function and the functions \( Q_r(u) \) for \( u \in V \) have doubling behaviour with such a constant.

For simplicity, we just assume that the growth function and the \( Q_r(u) \) for every \( u \in V \) have doubling behaviour at all scales. For convenience we assume that \( V(r')/V(r) < (r'/r)^m \) for all \( r' > r \). Furthermore, we can pick \( i_0 \) such that \( Q_r \) is positive definite on \( V \) for every \( r > i_0 \).

Now let \( R_2 > R_1 > i_0 \). Let \( x_{i \in I} \) be a maximal \( R_1 \)-separated net in \( B(R_2) \). Let \( B_i = B(x_i, R_i) \). Note that the balls \( \frac{1}{2} B_i = B(x_i, R_i/2) \) are disjoint, and the balls \( B_i \) cover \( B(R_2) \). By the volume doubling property, the multiplicity of \( 3B_i \) is bounded by a constant not depending on \( R_1 \) or \( R_2 \).
Consider the linear map

\[ \phi : V \rightarrow \mathbb{R}^{\vert I \vert}, f \mapsto \left( \frac{1}{V(R_1)} \int_{B_i} f \right)_{i \in I}. \]

We show that \( \phi \) is injective.

Let \( f \in \ker(\phi) \). It is enough to show that \( \int_{B(R_2)} f^2 = 0 \). In the following the constant \( C > 0 \) only depends on \( |S| \) and the growth rate of \( \Gamma \) but it may vary from line to line. From the Poincare inequalities of Section 2 and the bounded multiplicity we obtain that

\[
\int_{B(R_2)} f^2 \leq \sum_{i \in I} \int_{B_i} f^2 \leq CR_1^2 \sum_{i \in I} \int_{3B_i} (\nabla f)^2 \leq CR_1^2 \int_{B(2R_2)} (\nabla f)^2 \\
\leq C \left( \frac{R_1}{R_2} \right)^2 \int_{B(16R_2)} f^2 \\
\leq C \left( \frac{R_1}{R_2} \right)^2 \int_{B(R_2)} f^2,
\]

hence \( \int_{B(R_2)} f^2 = 0 \) follows provided \( R_1/R_2 = C^{-1/2}/2 \). But then \( \dim V \leq \vert I \vert \leq \left( \frac{R_2}{R_1} \right)^m = 2^m C^{m/2} \) which finishes the proof.

5. The proof of Gromov’s polynomial growth theorem

Yehuda Shalom introduced the following property \((H_{FD})\) of groups [Sha04].

**Definition 5.1.** A group \( \Gamma \) has property \((H_{FD})\) if every unitary \( \Gamma \)-representation \( \mathcal{H} \) with \( \bar{H}^1(\Gamma; \mathcal{H}) \neq 0 \) has a finite dimensional sub-representation.

The idea to use property \((H_{FD})\) to deduce Gromov’s theorem is due to Yehuda Shalom [Sha04]. Bruce Kleiner does not deal with first cohomology and proves Gromov’s theorem using harmonic maps instead (which is actually a slightly quicker than the presentation here) but we wanted to highlight the connection to property \((H_{FD})\) and the equivalence of the approach with harmonic maps versus the one with first cohomology.

**Theorem 5.2.** Let \( \Gamma \) be a group with weak polynomial growth and property \((H_{FD})\). Then \( \Gamma \) is virtually nilpotent.

**Proof.** If \( \Gamma \) is finite, this is trivial. Let \( \Gamma \) be infinite. Since \( \Gamma \) is infinite and amenable, in particular has not property \((T)\), Theorem 6.4 implies that there exists a unitary \( \Gamma \)-representation \( \mathcal{H} \) with \( \bar{H}^1(\Gamma; \mathcal{H}) \neq 0 \). Since \( \mathcal{H} \) decomposes as a direct integral of irreducible representations and

\[
\bar{H}^1(\Gamma; \int_{(Y, \nu)} \mathcal{H}_y) = 0 \iff \bar{H}^1(\Gamma; \mathcal{H}_y) \text{ for } \nu\text{-almost every } y \in Y
\]

we may and will assume that \( \mathcal{H} \) is irreducible. Property \((H_{FD})\) implies that \( \mathcal{H} \) is finite dimensional. Consider a non-zero 1-cocycle in \( H^1(\Gamma; \mathcal{H}) = \bar{H}^1(\Gamma; \mathcal{H}) \). It corresponds to an affine action on \( \mathcal{H} \) without fixed points. Thus we obtain a homomorphism

\[ \phi : \Gamma \rightarrow G := \text{Aff}(\mathcal{H}) \]

into the linear group of affine automorphisms that has infinite image. Finite image would imply that the cocycle had a fixed point.
From now on the argument is the same as in [Gro81, end of section 4] (see also [vdDW84, Lemma 2.1]). By a theorem of Jordan and Tits (see [Sha98] for an easier proof of this consequence), a finitely generated group that is linear over a field of characteristic 0 and amenable is virtually solvable. So, upon replacing \( \Gamma \) by a subgroup of finite index, we can assume that \( \phi(\Gamma) \) is solvable and infinite. Hence \( \Gamma/[\Gamma, \Gamma] \) is infinite, finitely generated, and abelian, and so there is a surjection \( p: \Gamma \to \mathbb{Z} \).

The kernel \( K = \ker(p) \) is finitely generated. This follows from the fact that \( \Gamma \) has sub-exponential growth (it is false otherwise!). See [Gro81, (b) on p. 61] for the argument.

If \( \Gamma \) has growth rate \( \leq d \) then \( K \) has growth rate \( \leq d - 1 \): Let \( \gamma_0 \in \Gamma \) map to the generator \( t \in \mathbb{Z} \), and let \( S_K = \{ \gamma_1, \ldots, \gamma_m \} \) be a finite symmetric generating set of \( K \). Then \( S_\Gamma = \{ \gamma_0, \gamma_0^{-1} \} \cup S_K \) is a finite, symmetric generating set of \( \Gamma \). For every \( r > 0 \) the multiplication map

\[
B_{S_K}(r) \times \{ t^{-r}, t^{-r+1}, \ldots, t^r \} \to B_{S_\Gamma}(2r)
\]

is injective which implies that \( K \) has growth rate at most \( d - 1 \).

By an inductive assumption we can assume that \( K \) is virtually nilpotent. Let \( K' \subset K \) be a normal nilpotent subgroup of finite index. Let \( \gamma_0 \in \Gamma \) be such that \( p(\gamma) \) maps to a generator of \( \mathbb{Z} \). Then the subgroup \( \Gamma' \) generated by \( K' \) and \( \gamma_0 \) is solvable: One checks that

\[
1 \to K' \to \Gamma' \xrightarrow{p} \mathbb{Z} \to 1
\]

is exact. Hence \( \Gamma \) is virtually solvable. By a theorem of Milnor and Wolf, a finitely generated virtually solvable group with polynomial growth is virtually nilpotent, thus the proof is finished.

Bruce Kleiner’s proof yields that groups with polynomial growth have property \((H_{FD})\) using Theorem 1.2.

**Theorem 5.3.** A finitely generated group with polynomial growth has property \((H_{FD})\).

**Proof.** Let \((\mathcal{H}, \pi)\) be a linear \( \Gamma \)-representation with \( \hat{H}^1(\Gamma; \mathcal{H}) \neq 0 \). By Lemma 6.6 there is an affine \( \Gamma \)-action on \( \mathcal{H} \) with linear part \( \pi \) and an equivariant, harmonic, non-constant map \( f: \mathcal{G} \to \mathcal{H} \). For \( v \in \mathcal{H} \) the map is \( x \mapsto \langle f(x), v \rangle \) is Lipschitz (but not equivariant anymore!). Consider the following linear map

\[
\phi: \mathcal{H} \to \mathcal{H}_1(\mathcal{G}), v \mapsto \langle f(\gamma), v \rangle \in \mathbb{R}.
\]

Since \( \mathcal{H}_1(\mathcal{G}) \) is finite dimensional by Theorem 1.2, \( \ker(\phi) \) is finite dimensional. By definition, \( \text{im}(f) \subset \ker(\phi) \). Choose \( w \in \text{im}(f) \). Then the linear span of \( \text{im}(f) - w \) is finite dimensional and invariant under the linear \( \Gamma \)-action.

6. A characterization of property \((T)\)

We want to characterize property \((T)\) in terms of harmonic maps. As in Riemannian geometry we can detect harmonic maps by looking for at critical points of the energy functional. An equivariant, piecewise linear\(^2\) map \( f: \mathcal{G} \to \mathcal{H} \) is

\(^2\)Harmonics maps are linear along an edge according to our definition.
determined by its value at \( e \in \Gamma \). Consequently, our energy functional \( E \) is a map from \( G \) to \( H \):
\[
E(x) = \sum_{s \in S} d^2(sx, x).
\]

One verifies that the derivative in direction \( v \in H \) is given by
\[
DE(x)(v) = 4 \sum_{s \in S} \langle x - sx, v \rangle.
\]

With convexity of \( E \) it follows the

**Lemma 6.1.** The following statements are equivalent:

a) \( x \in H \) is a critical point of \( E \).

b) \( x \) is a minimum of \( E \).

c) The map \( f : \Gamma \to H, f(\gamma) = \gamma x \) extends to an equivariant map \( f : G \to H \).

We briefly review the notion of ultrafilter.

**Definition 6.2.** A non-principal ultrafilter is a finitely additive probability measure \( \omega \) defined on all subsets of \( \mathbb{N} \) such that

a) \( \omega(S) \in \{0, 1\} \).

b) \( \omega(S) = 1 \) if \( S \) is finite.

One can show the existence of non-principal ultrafilters using Zorn’s lemma. A non-principal ultrafilter allows one to define a limit \( \lim_{\omega}(x_k) \) for every bounded sequence \((x_k)\) of real numbers:

Assume \( x_k \in [0, 1] \). In a first step, if \( \omega(k; x_k < 1/2) = 1 \), pick the interval \([0, 1/2]\). Otherwise pick \([1/2, 1]\). Divide the interval you have chose in two halves and proceed in the same way. The sequence of intervals defines a real number which we take as a definition for \( \lim_{\omega}(x_k) \).

**Lemma 6.3.** Let \((H_k, x_k)\) be a sequence of pointed Hilbert spaces. Let \( H_\omega \) be the quotient of \( \{(y_k); y_k \in H_k, \exists C > 0: d_k(y_k, x_k) < C\} \subset \prod_k H_k \) by the submodule of sequences \((y_k)\) with \( \lim_{\omega} d(y_k, x_k) = 0 \). Let \( x_\omega = [(x_k)] \in \lim_{\omega}(H_k, x_k) \). Then \((H_\omega, x_\omega)\) is a natural way a pointed Hilbert space.

**Theorem 6.4.** Let \( \Gamma \) be a finitely generated group and \( G \) its Cayley graph. The following statements are equivalent:

a) The group \( \Gamma \) has property \((T)\).

b) There exists no isometric \( \Gamma \)-action on a Hilbert space \( H \) such that \( E \) attains a positive minimum.

c) For every isometric action of \( \Gamma \) on a Hilbert space \( H \) and every \( x \in H \) there is \( x' \in B(x, D\sqrt{E(x)}) \) such that \( E(x') \leq E(x) \).

**Proof.** a) \( \Rightarrow \) b) is obvious. Further, c) \( \Rightarrow \) a) is easy: One obtains a sequence \((x_k)\) of points in \( H \) such that \( E(x_k) \leq 4^k E(x_0) \to 0 \) and \( d(x_{k+1}, x_k) \leq 4(1/2)^{k/2} E(x_0)^{1/2} \). The latter implies that \( (x_k) \) is Cauchy with limit \( x_\infty \) satisfying \( E(x_\infty) = 0 \).

Ad b) \( \Rightarrow \) c): Assume that c) fails. Then for every \( k \in \mathbb{N} \) we can find an isometric action \( \Gamma \rtimes H_k \) and \( x_k \in H_k \) such that
\[
E(y) > (1 - 1/k)E(x_k)
\]
for every \( y \in B(x_k, k\sqrt{E(x_k)}) \). Rescale the scalar product of \( \mathcal{H}_k \) by \( 1/E(x_k) \). With this new scalar product we have
\[
E(y) > 1 - 1/k
\]
for all \( y \in B(x_k, k) \). Then any ultralimit of the sequence \((\mathcal{H}_k, x_k)\) of pointed Hilbert spaces with isometric \( \Gamma \)-actions is a pointed Hilbert space \((\mathcal{H}_\omega, x_\omega)\) with an isometric \( \Gamma \)-action and \( E(x_\omega) = 1 \) and \( E(y) \geq 1 \) for every \( y \in \mathcal{H}_\omega \). \( \square \)

The following theorem combines work of Korevaar-Schoen, Kleiner, Fisher-Margulis, and Shalom.

**Theorem 6.5.** Let \( \Gamma \) be a finitely generated group and \( G \) its Cayley graph. The following statements are equivalent:

a) \( \Gamma \) does not have property \((T)\).

b) There exists a non-constant, harmonic, equivariant map \( G \to \mathcal{H} \) to a Hilbert space with an isometric \( \Gamma \)-action.

c) There exists a unitary \( \Gamma \)-representation such that \( \hat{H}^1(\Gamma; \mathcal{H}) \neq 0 \).

**Proof.** The equivalence of a) and b) follows immediately from Lemma 6.1 and Theorem 6.4. The equivalence of b) and c) follows from the following lemma. \( \square \)

**Lemma 6.6.** Let \((\mathcal{H}, \pi)\) be a \( \Gamma \)-representation. Let \( \mathcal{H}(G; \pi) \) be the vector space of harmonic functions \( G \to \mathcal{H} \) with \( f(e) = 0 \) for which there is an affine \( \Gamma \)-action on \( \mathcal{H} \) such that its linear part is \( \pi \) and \( f \) is equivariant. Then there is an isomorphism
\[
\hat{H}^1(\Gamma; \mathcal{H}) \cong \mathcal{H}(G; \pi).
\]

**Proof.** Let \( X \) be a contractible \( \Gamma \)-simplicial complex whose 1-skeleton is \( G \). So, \( X \) is a model of \( E\Gamma \). For convenience, we assume that its 2-skeleton is \( \Gamma \)-cocompact. We describe the necessary modifications for the general case in the end. Let \( C_i(X) \) denote the simplicial chain complex, which is a complex of modules over the group ring \( \mathbb{Z}\Gamma \). Note that for \( i = 0, 1, 2 \) the module
\[
M_i = \text{hom}_{\mathbb{Z}\Gamma}(C_i(X), \mathcal{H})
\]
is a Hilbert space, and the differential \( d_i : M_i \to M_{i+1} \) is a bounded operator. In fact, a finite set \( S \) of \( \Gamma \)-representatives of \( i \)-simplices defines an isomorphism \( H_i \cong l^2(\Gamma)^{|S|} \). We denote the adjoint by \( d_i^* \). The reduced cohomology of \( M_\ast \) is \( \hat{H}^\ast(\Gamma; \mathcal{H}) \). Let \( \Delta = d_0d_0^* + d_1^*d_1 : M_1 \to M_1 \) be the combinatorial Laplacian (in degree 1). One verifies that \( \Delta(f) = 0 \) if and only if \( d_0^*(f) = 0 \) and \( d_1(f) = 0 \). Moreover, the inclusion \( \ker \Delta \subset \ker d_1 \) is the orthogonal complement of \( \text{im} d_0 \). Thus, \( \ker \Delta = \hat{H}^1(\Gamma; \mathcal{H}) \). The cocycles in \( \ker \Delta \) are called harmonic.

We obtain a map \( \ker \Delta \to \mathcal{H}(G; \mathcal{H}) \) by integrating: Note that \( C_1(X) \) is the free module over the set of edges of \( G \). A harmonic cocycle \( f : C_1(X) \to \mathcal{H} \) defines an equivariant, harmonic map \( F : G \to \mathcal{H} \) by setting \( F(e) = 0 \) and \( F(x) = \sum_{w \in e} f(w) \) where \( w \) runs through the edges of a geodesic from \( e \) to \( x \). This map is the desired isomorphism. That \( F \) is well defined, i.e. independent of the choice of the geodesic, follows from \( d_1(f) = 0 \). That \( F \) is harmonics follows from \( d_0^*(f) = 0 \).

Although \( M_2 \) is no Hilbert space anymore if \( X \) has no \( \Gamma \)-finite 2-skeleton the definition of \( \Delta \) still makes sense and one can run the same argument. \( \square \)
References


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