Chapter 4. The Basics of the Geometric Theory

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This chapter introduces the reader to the basics of the geometric theory of functions of a complex variable. We will consider here the main problems of the theory of conformal mappings as well as the geometric principles that concern the most general properties of holomorphic functions.

1 The Geometric Principles

1.1 The Argument Principle

Let the function f be holomorphic in a punctured neighborhood $\{0 < |z-a| < r\}$ of a point $a \in \mathbb{C}$. We assume also that f does not vanish in this neighborhood. The logarithmic residue of the function f at the point a is the residue of the logarithmic derivative

$$\frac{f'(a)}{f(a)} = \frac{d}{dz}\mathrm{Ln} z \tag{1.1}$$

of this function at the point a.

Apart from isolated singular points the function f may have a non-zero logarithmic residue at its zeros. Let $a \in \mathbb{C}$ be a zero of order n of a function f holomorphic at a. Then we have $f(z) = (z-a)^n \phi(z)$ in a neighborhood U_a of a with the function ϕ holomorphic and different from zero in U_a. Therefore we have in U_a

$$\frac{f'(z)}{f(z)} = \frac{n(z-a)^{n-1}\phi(z) + (z-a)^n \phi'(z)}{(z-a)^n \phi(z)} = \frac{1}{z-a} \cdot \frac{n\phi(z) + (z-a)\phi'(z)}{\phi(z)}$$

with the second factor holomorphic in U_a. Hence it may be expanded into the Taylor series with the zero order term equal to n. Therefore we have in U_a

$$\frac{f'(z)}{f(z)} = \frac{1}{z-a}\left\{n + c_1(z-a) + c_2(z-a)^2 + \dots\right\} = \frac{n}{z-a} + c_1 + c_2(z-a) + \dots \tag{1.2}$$

This shows that the logarithmic derivative has a pole of order one with residue equal to n at the zero of order n of f: *the logarithmic residue at a zero of a function is equal to the order of this zero.*

If a is a pole of f of the order p then $1/f$ has a zero of order p at this point. Observing that

$$\frac{f'(z)}{f(z)} = -\frac{d}{dz}\mathrm{Ln}\frac{1}{f(z)},$$

and using (1.2) we conclude that the logarithmic derivative has residue equal to $-p$ at a pole of order p: *the logarithmic residue at a pole is equal to the order of this pole with the minus sign.*

Those observations allow to compute the number of zeros and poles of meromorphic functions. We adopt the convention that a pole and a zero are counted as many times as their order is.

Theorem 1.1 *Let the function f be meromorphic in a domain $D \subset \mathbb{C}$ and let G be a domain properly contained in D with the boundary ∂G that is a continuous curve. Let us assume that ∂G contains neither poles nor zeros of f and let N and P be the total number of zeros and poles of f in the domain G, then*

$$N - P = \frac{1}{2\pi i}\int_{\partial G}\frac{f'(z)}{f(z)}dz. \tag{1.3}$$

Proof. The function f has only finitely many poles $a_1, \ldots, a_l$ and zeros $b_1, \ldots, b_m$ in G since G is properly contained in D. The function $g = f'/f$ is holomorphic in a neighborhood of ∂G since the boundary of G does not contain poles or zeros. Applying the Cauchy theorem on residues to g we find

$$\frac{1}{2\pi i}\int_{\partial G}\frac{f'}{f}dz = \sum_{\nu=1}^{l}\mathrm{res}_{a_\nu}g + \sum_{\nu=1}^{m}\mathrm{res}_{b_\nu}g. \tag{1.4}$$

However, according to our previous remark,

$$\mathrm{res}_{a_\nu}g = n_\nu, \quad \mathrm{res}_{b_\nu}g = p_\nu.$$

Here n_ν and p_ν are the order of zero a_ν and pole b_ν, respectively. Using this in (1.4) and counting the multiplicities of zeros and poles we obtain (1.3) since $N = \sum n_\nu$ and $P = \sum p_\nu$. $\square$

Exercise 1.2 Let the function f satisfy assumptions of Theorem 1.1 and let g be holomorphic in $\bar{G}$. Show that then

$$\frac{1}{2\pi i}\int_{\partial G}g(z)\frac{f'(z)}{f(z)}dz = \sum_{k=1}^{l}g(a_k) - \sum_{k=1}^{m}g(b_m), \tag{1.5}$$

where the first some includes all the zeros and the second all the poles of f in G. This generalizes Theorem 1.1 that follows from (1.5) when $g \equiv 1$.

The theorem that we have just proved has a geometric interpretation. Let us parameterize ∂G as $z = z(t)$, $\alpha \le t \le \beta$ and denote by $\Phi(t)$ the anti-derivative of $\dfrac{f'}{f}$ along this path. The Newton-Leibnitz formula implies that

$$\int_{\partial G} \frac{f'(z)}{f(z)} dz = \Phi(\beta) - \Phi(\alpha). \tag{1.6}$$

However, clearly, $\Phi(t) = \ln[f(z(t))]$, where ln denotes any branch of the logarithm that varies continuously along the path ∂G. It suffices to choose a branch of $\arg f$ that varies continuously along ∂G since $\mathrm{Ln} f = \ln|f| + i\mathrm{Arg} f$ and the function $\ln|f|$ is single-valued. The increment of $\ln|f|$ along a closed path ∂G is equal to zero and thus

$$\Phi(\beta) - \Phi(\alpha) = i\{\arg f(z(\beta)) - \arg f(z(\alpha))\}.$$

We denote the increment of the argument of f in the right side by $\Delta_{\partial G} f$ and re-write (1.6) as

$$\int_{\partial G} \frac{f'}{f} dz = i\Delta_{\partial G} \arg f.$$

Theorem 1.1 may now be expressed as

Theorem 1.3 *(The argument principle) Under the assumptions of Theorem 1.1 the difference between the number of zeros N and the number of poles P of a function f in a domain G is equal to the increment of the argument of this function along the oriented boundary of G divided by 2π:*

$$N - P = \frac{1}{2\pi} \Delta_{\partial G} \arg f. \tag{1.7}$$

Geometrically the right side of (1.7) is the total number of turns the vector $w = f(z)$ makes around $w = 0$ as z varies along ∂G. Let us denote by ∂G^* the image of ∂G under the map f, that is, the path $w = f(z(t))$, $\alpha \le t \le \beta$. Then this number is equal to the total number of times the vector w rotates around $w = 0$ as it varies along ∂G^*. This number is called the winding number of ∂G^* around $w = 0$, we will denote it by $\mathrm{ind}_0 \partial G^*$. The argument principle states that

$$N - P = \frac{1}{2\pi} \Delta_{\partial G} \arg f = \mathrm{ind}_0 \partial G^*. \tag{1.8}$$

Remark 1.4 *We may consider the a-points of f, solutions of $f(z) = a$ and not only its zeros: it suffices to replace f by $f(z) - a$ in our arguments. If ∂G contains neither poles nor a-points of f then*

$$N_a - P = \frac{1}{2\pi i} \int_{\partial G} \frac{f'(z)}{f(z) - a} dz = \frac{1}{2\pi} \Delta_{\partial G} \arg\{f(z) - a\}, \tag{1.9}$$

where N_a is the number of a-points of f in the domain D. Passing to the plane $w = f(z)$ and introducing the index of the path ∂G^* around the point a we may re-write (1.9) as

$$N_a - P = \frac{1}{2\pi} \Delta_{\partial G} \arg\{f(z) - a\} = ind_a \partial G^*. \tag{1.10}$$

The next theorem is an example of the application of the argument principle.

Theorem 1.5 *(Rouche[1]) Let the functions f and g be holomorphic in a closed domain $\bar{G}$ with a continuous boundary ∂G and let*

$$|f(z)| > |g(z)| \quad \text{for all } z \in \partial G. \tag{1.11}$$

Then the functions f and $f+g$ have the same number of zeros in G.

Proof. Assumption (1.11) shows that neither f nor $f+g$ vanish on ∂G and thus the argument principle might be applied to both of these functions. Moreover, since $f \neq 0$ on ∂G, we have $f+g = f\left(1+\dfrac{g}{f}\right)$ and thus we have with the appropriate choice of a branch of the argument:

$$\Delta_{\partial G} \arg(f+g) = \Delta_{\partial G} \arg f + \Delta_{\partial G} \arg\left(1+\frac{g}{f}\right). \tag{1.12}$$

However, since $\left|\dfrac{g}{f}\right| < 1$ on ∂G, the point $\omega = \dfrac{g}{f}$ lies in $\{|\omega| < 1\}$ for all $z \in \partial G$. Therefore the vector $w = 1+\omega$ may not turn around zero and hence the second term in the right side of (1.12) vanishes. Therefore, $\Delta_{\partial G} \arg(f+g) = \Delta_{\partial G} \arg f$ and the argument principle implies the statement of the theorem. □

The Rouche theorem is useful in counting the zeros of holomorphic functions. In particular it implies the main theorem of algebra in a very simple way.

Theorem 1.6 *Any polynomial P_n of degree n has exactly n roots in $\mathbb{C}$.*

Proof. All zeros of P_n must lie in a disk $\{|z| < R\}$ since P_n has a pole at infinity. Let $P_n = f+g$ where $f = a_0 z^n$, $a_0 \neq 0$ and $g = a_1 z^{n-1} + \cdots + a_n$, then, possibly after increasing R, we may assume that $|f| > |g|$ on $\{|z| = R\}$ since $|f| = |a_0| R^n$ while g is a polynomial of degree less than n. The Rouche theorem implies that P_n has as many roots in $\{|z| < R\}$ as $f = a_0 z^n$, that is, exactly n of them. □

Exercise 1.7 1. Find the number of roots of the polynomial $z^4 + 10z + 1$ in the annulus $\{1 < |z| < 2\}$.
2. Show that any polynomial with real coefficients may be decomposed as a product of linear and quadratic factors with real coefficients.

1.2 The Open Mapping Theorem

This is the name of the following basic

Theorem 1.8 [2] *If a function f holomorphic in a domain D is not equal identically to a constant then the image $D^* = f(D)$ is also a domain.*

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[1] Eugene Rouche (1832-1910) was a French mathematician.

[2] This theorem was proved by Riemann in 1851.

Proof. We have to show that D^* is connected and open. Let w_1 and w_2 be two arbitrary points in D^* and let z_1 and z_2 be some pre-images of w_1 and w_2, respectively. Since the domain D is path-wise connected there exists a path $\gamma : [\alpha, \beta] \to D$ that connects z_1 and z_2. Its image $\gamma^* = f \circ \gamma$ connects w_1 and w_2 and is a path since the function f is continuous. Moreover, it is clearly contained in D^* and hence the set D^* is path-wise connected.

Let w_0 be an arbitrary point in D^* and let z_0 be a pre-image of w_0. There exists a disk $\{|z - z_0| < r\}$ centered at z_0 that is properly contained in D since D is open. After decreasing r we may assume that $\{|z - z_0| \leq r\}$ contains no other w_0-points of f except z_0: since $f \neq \text{const}$ its w_0 points are isolated in D. We denote by $\gamma = \{|z - z_0| = r\}$ the boundary of this disk and let

$$\mu = \min_{z \in \gamma} |f(z) - w_0|. \tag{1.13}$$

Clearly $\mu > 0$ since the continuous function $|f(z) - w_0|$ attains its minimum on γ, so that if $\mu = 0$ then there would exist a w_0-point of f on γ contrary to our construction of the disk.

Let us now show that the set $\{|w - w_0| < \mu\}$ is contained in D^*. Indeed, let w_1 be an arbitrary point in this disk, that is, $|w_1 - w_0| < \mu$. Then we have

$$f(z) - w_1 = f(z) - w_0 + (w_0 - w_1), \tag{1.14}$$

and, moreover, $|f(z) - w_0| \geq \mu$ on γ. Then, since $|w_0 - w_1| < \mu$, the Rouche theorem implies that the function $f(z) - w_1$ has as many roots inside γ as $f(z) - w_0$. Hence it has at least one zero (the point z_0 may be a zero of order higher than one of $f(z) - w_0$). Thus the function f takes the value w_1 and hence $w_1 \in D^*$. However, w_1 is an arbitrary function in the disk $\{|w - w_0| < \mu\}$ and hence this whole disk is contained in D^* so that D^* is open. □

Exercise 1.9 Let f be holomorphic in $\{\text{Im} z \geq 0\}$, real on the real axis and bounded. Show that $f \equiv \text{const}$.

A similar but more detailed analysis leads to the solution of the problem of local inversion of holomorphic functions. This problem is formulated as follows.

A holomorphic function $w = f(z)$ is defined at z_0, find a function $z = g(w)$ analytic at $w_0 = f(z_0)$ so that $g(w_0) = z_0$ and $f(g(w)) = w$ in a neighborhood of w_0.

We should distinguish two cases in the solution of this problem:

I. The point z_0 is not a critical point: $f'(z_0) \neq 0$. As in the proof of the open mapping theorem we choose a disk $\{|z - z_0| \leq r\}$ that contains no w_0-points except z_0, and define μ according to (1.13). Let w_1 be an arbitrary point in the disk $\{|w - w_0| < \mu\}$. Then the same argument (using (1.14) and the Rouche theorem) shows that the function f takes the value w_1 as many times as w_0. However, the value w_0 is taken only once and, moreover, z_0 is a simple zero of $f(z) - w_0$ since $f'(z_0) \neq 0$.

Therefore the function f takes all values in the disk $\{|w - w_0| < \mu\}$ once in the disk $\{|z - z_0| < r\}$. In other words, the function f is a local bijection at z_0.

Then the function $z = g(w)$ is defined in the disk $\{|w - w_0| < r\}$ so that $g(w_0) = z_0$ and $f \circ g(w) = w$. Furthermore, derivative $g'(w)$ exists at every point of the disk $\{|w - w_0| < r\}$:

$$g'(w) = \frac{1}{f'(z)} \tag{1.15}$$

and thus g is holomorphic in this disk[3].

II. The point z_0 is a critical point: $f'(z_0) = \cdots = f^{(p-1)}(z_0) = 0$, $f^{(p)} \neq 0$, $p \geq 2$. Repeating the same argument as before choosing a disk $\{|z - z_0| < r\}$ that contains neither w_0-points of f nor zeros of the derivative f' (we use the uniqueness theorem once again). As before, we choose $\mu > 0$, take an arbitrary point w_1 in the disk $\{|w-w_0| < \mu\}$ and find that f takes the value w_1 as many times as w_0. However, in the present case the w_0-point z_0 has multiplicity p: z_0 is a zero of order p of $f(z) - w_0$. Furthermore, since $f'(z) \neq 0$ for $0 < |z - z_0| < r$ the value w_1 has to be taken at p different points. Therefore, the function f takes each value p times in $\{|z - z_0| < r\}$.

The above analysis implies the following

Theorem 1.10 *Condition $f'(z_0) \neq 0$ is necessary and sufficient for the local invertibility of a holomorphic function f at the point z_0.*

Remark 1.11 The general inverse function theorem of the real analysis implies that the assumption $f'(z_0) \neq 0$ is sufficient for the local invertibility since the Jacobian $J_f(z) = |f'(z)|^2$ of the map $(x, y) \to (u, v)$ is non-zero at this point. However, for an arbitrary differentiable map to be locally invertible one needs not $J_f(z) \neq 0$ to hold. This may be seen on the example of the map $f = x^3 + iy$ that has Jacobian equal to zero at $z = 0$ but that is nevertheless one-to-one.

Remark 1.12 The local invertibility condition $f'(z) \neq 0$ for all $z \in D$ is not sufficient for the global invertibility of the function in the whole domain D. This may be seen on the example of $f(z) = e^z$ that is locally invertible at every point in $\mathbb{C}$ but is not one-to-one in any domain that contains two points that differ by $2k\pi i$ where $k \neq 0$ is an integer.

We have described above a qualitative solution of the problem of local invertibility. Methods of the theory of analytic functions also allow to develop an effective quantitative solution of this problem. Let us consider for simplicity the case $f'(z_0) \neq 0$.

Let us construct as before the disks $\{|z - z_0| \leq r\}$ and $\{|w - w_0| < \mu\}$. Given a fixed w in the latter we consider the function $h(\zeta) = \dfrac{\zeta f'(\zeta)}{f(\zeta) - w}$. It is holomorphic everywhere in the former disk except possibly at the point $z = g(w)$ where g is the inverse of the function f. The residue of h at this point (a pole of multiplicity one) is equal to z. Therefore, according to the Cauchy theorem on residues we have

$$z = \frac{1}{2\pi i} \int_\gamma \frac{\zeta f'(\zeta)}{f(\zeta) - w} d\zeta, \tag{1.16}$$

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[3] Expression (1.15) shows that in order for derivative to exist we need $f' \neq 0$. Using continuity of f' we may conclude that $f' \neq 0$ in the disk $\{|z - z_0| < r\}$, possibly decreasing r if needed.

where $\gamma = \{|\zeta - z_0| = r\}$.

The integral in the right side depends on w so that we have obtained the integral representation of the inverse function $g(w)$. We may use it in order to obtain the Taylor expansion of the function g in the same way as we used the Cauchy integral formula in order to obtain the Taylor expansion of a holomorphic function. We have

$$\frac{1}{f(\zeta) - w} = \frac{1}{f(\zeta) - w_0} \cdot \frac{1}{1 - \dfrac{w - w_0}{f(\zeta) - w_0}} = \sum_{n=0}^{\infty} \frac{(w - w_0)^n}{(f(\zeta) - w_0)^{n+1}}.$$

This series converges uniformly in ζ on the circle γ (we have $|f(\zeta) - w_0| \geq \mu$ while $|w - w_0| < \mu$). Multiplying this expansion by $\dfrac{\zeta f'(\zeta)}{2\pi i}$ and integrating term-wise along γ we obtain

$$z = g(w) = \sum_{n=0}^{\infty} d_n (w - w_0)^n, \tag{1.17}$$

where

$$d_n = \frac{1}{2\pi i} \int_\gamma \frac{\zeta f'(\zeta) d\zeta}{(f(\zeta) - w_0)^{n+1}}, \quad n = 0, 1, \ldots$$

We clearly have $d_0 = z_0$, while we may integrate by parts in the above integral when $n \geq 1$ to get

$$d_n = \frac{1}{2\pi i n} \int_\gamma \frac{d\zeta}{(f(\zeta) - w_0)^n}.$$

The integrand has pole of order n at the point z_0. We may find the residue at this point to obtain the final expression for the coefficients:

$$d_0 = z_0, \quad d_n = \frac{1}{n!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left(\frac{z - z_0}{f(z) - w_0} \right)^n, \quad n = 1, 2, \ldots \tag{1.18}$$

The series (1.17) may be effectively used to invert holomorphic functions.

Example 1.13 Let us find the inverse function of $f(z) = ze^{-az}$ at the point $w_0 = 0$ that corresponds to $z_0 = 0$. Using expression (1.18) we obtain

$$d_n = \frac{1}{n!} \lim_{z \to 0} \frac{d^{n-1}}{dz^{n-1}} \left(\frac{z}{f(z)} \right)^n = \frac{(an)^{n-1}}{n!}.$$

The inverse function has the representation

$$g(w) = \sum_{n=1}^{\infty} \frac{(an)^{n-1}}{n!} w^n.$$

1.3 The maximum modulus principle and the Schwartz lemma

The maximum modulus principle is expressed by the following theorem.

Theorem 1.14 *If the function f is holomorphic in a domain D and its modulus $|f|$ achieves its (local) maximum at a point $z_0 \in D$ then f is constant.*

Proof. We use the open mapping theorem. If $f \neq \text{const}$ then it maps z_0 into a point w_0 of the domain D^*. There exists a disk $\{|w - w_0| < \mu\}$ centered at w_0 that is contained in D^*. There must be a point w_1 in this disk so that $|w_1| > |w_0|$. The value w_1 is taken by the function f in a neighborhood of the point z_0 which contradicts the fact that $|f|$ achieves its maximum at this point. $\square$

Taking into account the properties of continuous functions on a closed set the maximum modulus principle may be reformulated as

Theorem 1.15 *If a function f is holomorphic in a domain D and continuous in $\bar{D}$ then $|f|$ achieves its maximum on the boundary ∂D.*

Proof. If $f = \text{const}$ in D (and hence in $\bar{D}$ by continuity) the statement is trivial. Otherwise if $f \neq \text{const}$ then $|f|$ may not attain its maximum at the points of D. However, since this maximum is attained in $\bar{D}$ it must be achieved on ∂D. $\square$

Exercise 1.16 1. Let $P(z)$ be a polynomial of degree n in z and let $M(r) = \max\limits_{|z|=r} |P(z)|$. Show that $M(r)/r^n$ is a decreasing function.

2. Formulate and prove the maximum principle for the real part of a holomorphic function.

A similar statement for the minimum of modulus is false in general. This may be seen on the example of the function $f(z) = z$ in the disk $\{|z| < 1\}$ (the minimum of $|f|$ is attained at $z = 0$). However, the following theorem holds.

Theorem 1.17 *Let a function f be holomorphic in a domain D and not vanish anywhere in D. Then $|f|$ may attain its (local) minimum in D only if $f = \text{const}$.*

For the proof of this theorem it suffices to apply Theorem 1.14 to the function $g = 1/f$ that is holomorphic since $f \neq 0$.

Exercise 1.18 1. Let a function f be holomorphic in $U = \{|z| < 1\}$ and continuous in $\bar{U}$. Assume also that $f \neq 0$ anywhere in U and, moreover, that $|f| = 1$ on $\{|z| = 1\}$. Show that then $f = \text{const}$.

2. Let the functions f and g be holomorphic in U and continuous in $\bar{U}$. Show that $|f(z)| + |g(z)|$ attains its maximum on $\{|z| = 1\}$. Hint: consider the function $h = e^{i\alpha} f + e^{i\beta} g$ with suitably chosen constants α and β.

A simple corollary of the maximum modulus principle is

Lemma 1.19 *(The Schwartz lemma[4]) Let a function f be holomorphic in the unit disk $\{|z| \leq 1\}$, satisfy $|f(z)| \leq 1$ for all $z \in U$ and $f(0) = 0$. Then we have*

$$|f(z)| \leq |z| \tag{1.19}$$

for all $z \in U$. Moreover, if the equality in (1.19) holds for at least one $z \neq 0$ then it holds everywhere in U and in this case $f(z) = e^{i\alpha}z$, where α is a real constant.

Proof. Consider the function $\phi(z) = f(z)/z$, it is holomorphic in U since $f(0) = 0$. Let $U_r = \{|z| < r\}$, $r < 1$ be an arbitrary disk centered at zero. The function $\phi(z)$ attains its maximum in U_r on its boundary $\gamma_r = \{|z| = r\}$ according to Theorem 1.15. However, we have $|\phi| \leq 1/r$ on γ_r since $|f| \leq 1$ by assumption. Therefore we have

$$|\phi(z)| \leq 1/r \tag{1.20}$$

everywhere in U_r. We fix $z \in U$ and observe that $z \in U_r$ for $r > |z|$. Therefore (1.20) holds for any given z with all $r > |z|$. We let $r \to 1$, and passing to the limit $r \to 1$ we obtain $|\phi(z)| \leq 1$ or $|f(z)| \leq |z|$. This proves the inequality (1.19).

Let us assume that equality in (1.19) holds for some $z \in U$, then $|\phi|$ attains its maximum equal to 1 at this point. Then ϕ is equal to a constant so that $\phi(z) = e^{i\alpha}$ and $f(z) = e^{i\alpha}z$. $\square$

The Schwartz lemma implies that a holomorphic map f that maps the disk $\{|z| < 1\}$ into the disk $\{|w| < 1\}$ and that takes the center to the center, maps any circle $\{|z| = r\}$ inside the disk $\{|w| < r\}$. The image of $\{|z| = r\}$ may intersect $\{|w| = r\}$ if and only f is a rotation around $z = 0$.

Exercise 1.20 *1. Show that under the assumptions of the Schwartz lemma we have $|f'(0)| \leq 1$ and equality is attained if and only if $f(z) = e^{i\alpha}z$.*
2. Let $f \in \mathcal{O}(D)$, $f : U \to U$ and $f(0) = \cdots = f^{(k-1)}(0) = 0$. Show that then $|f(z)| \leq |z|^k$ for all $z \in U$.

2 The Riemann Theorem

Any holomorphic one-to-one function defined in a domain D defines a conformal map of this domain since the above assumptions imply that f has no critical points in D. We have encountered such maps many times before. Here we consider a more difficult and important for practical purposes problem:

Given two domains D_1 and D_2 find a one-to-one conformal map $f : D_1 \to D_2$ of one of these domains onto the other.

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[4] Hermann Schwartz (1843-1921) was a German mathematician, a student of Weierstrass. This important lemma has appeared in his papers of 1869-70.

2.1 Conformal isomorphisms and automorphisms

Definition 2.1 *A conformal one-to-one map of a domain* D_1 *onto* D_2 *is said to be a (conformal) isomorphism, while the domains* D_1 *and* D_2 *that admit such a map are isomorphic (or conformally equivalent). Isomorphism of a domain onto itself is called a (conformal) automorphism.*

It is easy to see that the set of all automorphisms $\phi: D \to D$ of a domain D forms a group that is denoted $\mathrm{Aut}D$. The group operation is the composition $\phi_1 \circ \phi_2$, the unity is the identity map and the inverse is the inverse map $z = \phi^{-1}(w)$.

The richness of the group of automorphisms of a domain allows to understand the richness of the family of the conformal maps onto it of a different domain, as may be seen from the next

Theorem 2.2 *Let* $f_0: D_1 \to D_2$ *be a fixed isomorphism. Then any other isomorphism of* D_1 *onto* D_2 *has the form*

$$f = \phi \circ f_0 \tag{2.1}$$

where ϕ *is an automorphism of* D_2.

Proof. First, it is clear that all maps of the form of the right side of (2.1) are isomorphisms from D_1 onto D_2. Furthermore, if $f: D_1 \to D_2$ is an arbitrary isomorphism then $\phi = f \circ f_0^{-1}$ is a conformal map of D_2 onto itself, that is, an automorphism of D_2. Then (2.1) follows. □

In the sequel we will only consider simply connected domains D. We will distinguish three special domains that we will call canonical: the closed plane $\overline{\mathbb{C}}$, the open plane $\mathbb{C}$ and the unit disk $\{|z| < 1\}$. We have previously found the group of all fractional-linear automorphisms of those domains. However, the following theorem holds.

Theorem 2.3 *Any conformal automorphism of a canonical domain is a fractional-linear transformation.*

Proof. Let ϕ be automorphism of $\overline{\mathbb{C}}$. There exists a unique point z_0 that is mapped to infinity. Therefore ϕ is holomorphic everywhere in $\mathbb{C}$ except at z_0 where it has a pole. This pole has multiplicity one since in a neighborhood of a pole of higher order the function ϕ could not be one-to-one. Therefore since the only singularities of ϕ are poles ϕ is a rational function. Since it has only one simple pole, ϕ should be of the form $\phi(z) = \dfrac{A}{z - z_0} + B$ if $z_0 \neq \infty$ and $\phi(z) = Az + B$ if $z_0 = \infty$. The case of the open complex plane $\mathbb{C}$ is similar.

Let ϕ be an arbitrary automorphism of the unit disk U. Let us denote $w_0 = \phi(0)$ and consider a fractional linear transformation

$$\lambda: \ w \to \frac{w - w_0}{1 - \bar{w}_0 w}$$

of the disk U that maps w_0 into 0. The composition $f = \lambda \circ \phi$ is also an automorphism of U so that $f(0) = 0$. Moreover, $|f(z)| < 1$ for all $z \in U$. Therefore the Schwartz lemma implies that $|f(z)| \leq |z|$ for all $z \in U$. However, the inverse map $z = f^{-1}(w)$ also satisfies the assumptions of the Schwartz lemma and hence $|f^{-1}(w)| \leq |w|$ for all $w \in U$ that in turn implies that $|z| \leq |f(z)|$ for all $z \in U$. Thus $|f(z)| = |z|$ for all $z \in U$ so that the Schwartz lemma implies that $f(z) = e^{i\alpha}z$. Then $\phi = \lambda^{-1} \circ f = \lambda^{-1}(e^{i\alpha}z)$ is also a fractional-linear transformation. □

Taking into account our results from Chapter 1 we obtain the complete description of all conformal automorphisms of the canonical domains.

(I) The closed complex plane:

$$\mathrm{Aut}\overline{\mathbb{C}} = \left\{ z \to \frac{az+b}{cz+d},\ ad - bc \neq 0 \right\}. \tag{2.2}$$

(II) The open plane:

$$\mathrm{Aut}\mathbb{C} = \{ z \to az + b,\ a \neq 0 \}. \tag{2.3}$$

(III) The unit disk:

$$\mathrm{Aut}U = \left\{ z \to e^{i\alpha}\frac{z-a}{1-\bar{a}z},\ |a| < 1, \alpha \in \mathbb{R} \right\}. \tag{2.4}$$

It is easy to see that different canonical domains are not isomorphic to each other. Indeed, the closed complex plane $\overline{\mathbb{C}}$ is not even homeomorphic to $\mathbb{C}$ and U and hence it may not be mapped conformally onto these domains. The domains $\mathbb{C}$ and U are homeomorphic but there is no conformal map of $\mathbb{C}$ onto U since such a map would have to be realized by an entire function such that $|f(z)| < 1$ which has then to be equal to a constant by the Liouville theorem.

A domain that has no boundary (boundary is an empty set) coincides with $\overline{\mathbb{C}}$. Domains with boundary that consists of one point are the plane $\overline{\mathbb{C}}$ without a point which are clearly conformally equivalent to $\mathbb{C}$ (even by a fractional linear transformation). The main result of this section is the Riemann theorem that asserts that any simply connected domain D with a boundary that contains more than one point (and hence infinitely many points since boundary of a simply connected domain is connected) is conformally equivalent to the unit disk U.

This theorem will be presented later while at the moment we prove the uniqueness theorem for conformal maps.

Theorem 2.4 *If a domain D is conformally equivalent to the unit disk U then the set of all conformal maps of D onto U depends on three real parameters. In particular there exists a unique conformal map f of D onto U normalized by*

$$f(z_0) = 0,\ \arg f'(z_0) = \theta, \tag{2.5}$$

where z_0 is an arbitrary point of D and θ is an arbitrary real number.

Proof. The first statement follows from Theorem 2.2 since the group $\mathrm{Aut}U$ depends on three real parameters: two coordinates of the point a and the number α in (2.4).

In order to prove the second statement let us assume that there exist two maps f_1 and f_2 of the domain D onto U normalized as in (2.5). Then $\phi = f_1 \circ f_2^{-1}$ is an automorphism of U such that $\phi(0) = 0$ and $\arg f'(0) = 0$. Expression (2.4) implies that then $a = 0$ and $\alpha = 0$, that is $\phi(z) = z$ and $f_1 = f_2$.

Exercise 2.5 Show that there exists no more than one conformal map of a domain D onto the unit disk U that is continuous in $\bar{D}$ and is normalized by one of the following two conditions: (i) the images of one internal and one boundary point in D are prescribed, and (ii) the images of three boundary points of D are prescribed.

In order to prove the Riemann theorem we need to develop some methods that are useful in other areas of the complex analysis.

2.2 The compactness principle

Definition 2.6 *A family $\{f\}$ of functions defined in a domain D is locally uniformly bounded if for any domain K properly contained in D there exists a constant $M = M(K)$ such that*

$$|f(z)| \leq M \text{ for all } z \in K \text{ and all } f \in \{f\}. \tag{2.6}$$

A family $\{f\}$ is locally equicontinuous if for any $\varepsilon > 0$ and any domain K properly contained in D there exists $\delta = \delta(\varepsilon, K)$ so that

$$|f(z') - f(z'')| < \varepsilon \tag{2.7}$$

for all $z', z'' \in K$ so that $|z' - z''| < \delta$ and all $f \in \{f\}$.

Theorem 2.7 *If a family $\{f\}$ of holomorphic functions in a domain D is locally uniformly bounded then it is locally equicontinuous.*

Proof. Let K be a domain properly contained in D. Let us denote by 2ρ the distance between the closed sets $\bar{K}$ and ∂D[5] and let

$$K^{(\rho)} = \cup_{z_0 \in K} \{z : \ |z - z_0| < \rho\}$$

be a ρ-enlargement of K. The set $K^{(\rho)}$ is properly contained in D and thus there exists a constant M so that $|f(z)| \leq M$ for all $z \in K^{(\rho)}$ and $f \in \{f\}$. Let z' and z'' be arbitrary points in K so that $|z' - z''| < \rho$. The disk $U_\rho = \{z : \ |z - z'| < \rho\}$ is contained in $K^{(\rho)}$ and hence $|f(z) - f(z')| < 2M$ for all $z \in U_\rho$. The mapping $\zeta = \dfrac{1}{\rho}(z - z')$ maps U_ρ onto the disk $|\zeta| < 1$ and the function $g(\zeta) = \frac{1}{2M}\{f(z' + \zeta\rho) - f(z')\}$ satisfies the assumptions of the Schwartz lemma.

This lemma implies that $g(\zeta)| \leq |\zeta|$ for all ζ, $|\zeta| < 1$, which means

$$|f(z) - f(z')| \leq \frac{2M}{\rho}|z - z'| \text{ for all } z \in U_\rho. \tag{2.8}$$

---

[5] Note that ρ is positive except when $D = \mathbb{C}$ or $\overline{\mathbb{C}}$ when the statement of the theorem is trivial.

Given $\varepsilon > 0$ we choose $\delta = \min\left(\rho, \frac{\varepsilon\rho}{2M}\right)$ and obtain from (2.8) that $|f(z') - f(z'')| < \varepsilon$ for all $f \in \{f\}$ provided that $|z' - z''| < \delta$. □

Definition 2.8 *A family of functions $\{f\}$ defined in a domain D is compact in D if any sequence f_n of functions of this family has a subsequence f_{n_k} that converges uniformly on any domain K properly contained in D.*

Theorem 2.9 *(Montel[6]) If a family of functions $\{f\}$ holomorphic in a domain D is locally uniformly bounded then it is compact in D.*

Proof. (a) We first show that if a sequence $f_n \subset \{f\}$ converges at every point of an everywhere dense set $E \subset D$ then it converges uniformly on every compact subset K of D. We fix $\varepsilon > 0$ and the set K. Using equicontinuity of the family $\{f\}$ we may choose a partition of D into squares with sides parallel to the coordinate axes and so small that that for any two points $z', z'' \in K$ that belong to the same square and any $f \in \{f\}$ we have

$$|f(z') - f(z'')| < \frac{\varepsilon}{3}. \tag{2.9}$$

The set K is covered by a finite number of such squares q_p, $p = 1, \dots, P$. Each q_p contains a point $z_p \in E$ since the set E is dense in D. Moreover, since the sequence$\{f_n\}$ converges on E there exists N so that

$$|f_m(z_p) - f_n(z_p)| < \frac{\varepsilon}{3} \tag{2.10}$$

for all $m, n > N$ and all z_p, $p = 1, \dots, P$.

Let now z be an arbitrary point in K. Then there exists a point z_p that belongs to the same square as z. We have for all $m, n > N$:

$$|f_m(z) - f_n(z)| \le |f_m(z) - f_m(z_p)| + |f_m(z_p) - f_n(z_p)| + |f_n(z_p) - f_n(z)| < \varepsilon$$

due to (2.9) and (2.10). The Cauchy criterion implies that the sequence $\{z_n\}$ converges for all $z \in K$ and convergence is uniform on K.

(b) Let us show now that any sequence $\{f_n\}$ has a subsequence that converges at every point of a dense subset E of D. We choose E as the set $z = x + iy \in D$ with both coordinates $x$and y rational numbers. This set is clearly countable and dense in D, let $E = \{z_\nu\}_{\nu=1}^{\infty}$.

The sequence $f_n(z_1)$ is bounded and hence it has a converging subsequence $f_{k1} = f_{n_k}(z_1)$, $k = 1, 2, \dots$. The sequence $f_{n1}(z_2)$ is also bounded so we may extract its subsequence $f_{k2} = f_{n_k 1}$, $k = 1, 2, \dots$. The sequence f_{n2} converges at least at the points z_1 and z_2. Then we extract a subsequence $f_{k3} = f_{n_k 2}$ of the sequence $f_{n2}(z_3)$ so that f_{n3} converges at least at z_1, z_2 and z_3. We may continue this procedure indefinitely. It remains to choose the diagonal sequence

$$f_{11}, f_{22}, \dots, f_{nn}, \dots$$

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[6] Paul Montel (1876-1937) was a French mathematician.

This sequence converges at any point $z_p \in E$ since by construction all its entries after index p belong to the subsequence f_{np} that converges at z_p.

Parts (a) and (b) together imply the statement of the theorem. $\square$

The Montel theorem is often called the compactness principle.

Exercise 2.10 Show that any sequence $\{f_n\}$ of functions holomorphic in a domain D with $\mathrm{Re} f_n \geq 0$ everywhere in D has a subsequence that converges locally uniformly either to a holomorphic function or to infinity.

Definition 2.11 *A functional J of a family $\{f\}$ of functions defined in a domain D is a mapping $J : \{f\} \to \mathbb{C}$, that is, $J(f)$ is a complex number. A functional J is continuous if given any sequence of functions $f_n \in \{f\}$ that converges uniformly to a function $f_0 \in \{f\}$ on any compact set $K \subset D$ we have*

$$\lim_{n\to\infty} J(f_n) = J(f_0).$$

Example 2.12 Let $\mathcal{O}(D)$ be the family of all functions f holomorphic in D and let a be an arbitrary point in D. Consider the p-th coefficient of the Taylor series in a:

$$c_p(f) = \frac{f^{(p)}(a)}{p!}.$$

This is a functional on the family $\mathcal{O}(D)$. Let us show that it is continuous. if $f_n \to f_0$ uniformly on every compact set $K \subset D$ we may let K be the circle $\gamma = \{|z - a| = r\} \subset D$. Then given any $\varepsilon > 0$ we may find N so that $|f_n(z) - f_0(z)| < \varepsilon$ for all $n > N$ and all $z \in \gamma$. The Cauchy formula for c_p

$$c_p = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz$$

implies that

$$|c_p(f_n) - c_p(f_0)| \leq \frac{\varepsilon}{r^n}$$

for all $n > N$ which in turn implies the continuity of the functional $c_p(f)$.

Definition 2.13 *A compact family of functions $\{f\}$ is sequentially compact if the limit of any sequence f_n that converges uniformly on every compact subset $K \subset D$ belongs to the family $\{f\}$.*

Theorem 2.14 *Any functional J that is continuous on a sequentially compact family $\{f\}$ is bounded and attains its lowest upper bound. That is, there exists a function $f_0 \in \{f\}$ so that we have*

$$|J(f_0)| \geq |J(f)|$$

for all $f \in \{f\}$.

Proof. We let $A = \sup_{f\in\{f\}} |J(f)|$ - this is a number that might be equal to infinity. By definition of the supremum there exists a sequence $f_n \in \{f\}$ so that $|J(f_n)| \to A$. Since $\{f\}$ is a sequentially compact family there exists a subsequence f_{n_k} that converges to a function $f_0 \in \{f\}$. Continuity of the functional J implies that

$$|J(f_0)| = \lim_{k\to\infty} |J(f_{n_k})| = A.$$

This means that first $A < \infty$ and second, $|J(f_0)| \geq |J(f)|$ for all $f \in \{f\}$. □

We will consider below families of univalent functions in a domain D. The following theorem is useful to establish sequential compactness of such families.

Theorem 2.15 *(Hurwitz[7]) Let a sequence of functions f_n holomorphic in a domain D converge uniformly on any compact subset K of D to a function $f \neq$ const. Then if $f(z_0) = 0$ then given any disk $U_r = \{|z - z_0| < r\}$ there exists N so that all functions f_n vanish at some point in U_r when $n > N$.*

Proof. The Weierstrass theorem implies that f is holomorphic in D. The uniqueness theorem implies that there exists a punctured disk $\{0 < |z - z_0| \leq \rho\} \subset D$ where $f \neq 0$ (we may assume that $\rho < r$). We denote $\gamma = \{|z - z_0| = \rho\}$ and $\mu = \min_{z\in\gamma} |f(z)|$, and observe that $\mu > 0$. However, f_n converges uniformly to f on γ and hence there exists N so that

$$|f_n(z) - f(z)| < \mu$$

for all $z \in \gamma$ and all $n > N$. The Rouche theorem implies that for such n the function $f_n = f + (f_n - f)$ has as many zeros (with multiplicities) as f inside γ, that is, f_n has at least one zero inside U_ρ. □

Corollary 2.16 *If a sequence of holomorphic and univalent functions f_n in a domain D converges uniformly on every compact subset K of D then the limit function f is either a constant or univalent.*

Proof. Assume that $f(z_1) = f(z_2)$ but $z_1 \neq z_2$, $z_{1,2} \in D$ and] $f \neq$ const. Consider a sequence of functions $g_n(z) = f_n(z) - f_n(z_2)$ and a disk $\{|z - z_1| < r\}$ with $r < |z_1 - z_2|$. The limit function $g(z)$ vanishes at the point z_1. Hence according to the Hurwitz theorem all functions f_n starting with some N vanish in this disk. This, however, contradicts the assumption that $f_n(z)$ are univalent. □

2.3 The Riemann theorem

Theorem 2.17 *Any simply connected domain D with a boundary that contains more than one point is conformally equivalent to the unit disk U.*

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[7] Adolf Hurwitz (1859-1919) was a German mathematician, a student of Weierstrass.

Proof. The idea of the proof is as follows. Consider the family S of holomorphic and univalent functions f in D bounded by one in absolute value, that is, those that map D into the unit disk U. We fix a point $a \in D$ and look for a function f that maximizes the dilation coefficient $|f'(a)|$ at the point a. Restricting ourselves to a sequentially compact subset S_1 of S and using continuity of the functional $J(f) = |f'(a)|$ we may find a function f_0 with the maximal dilation at the point a. Finally we check that f_0 maps D onto U and not just into U as other functions in S.

Such a variational method when one looks for a function that realizes the extremum of a functional is often used in analysis.

(i) Let us show that there exists a holomorphic univalent function in D that is bounded by one in absolute value. By assumption the boundary ∂D contains at least two points α and β. The square root $\sqrt{\dfrac{z-\alpha}{z-\beta}}$ admits two branches ϕ_1 and ϕ_2 that differ by a sign. Each one of them is univalent in D[8] since the equality $\phi_\nu(z_1) = \phi_\nu(z_2)$ ($\nu = 1$ or 2) implies

$$\frac{z_1-\alpha}{z_1-\beta} = \frac{z_2-\alpha}{z_2-\beta} \tag{2.11}$$

which implies $z_1 = z_2$ since fractional linear transformations are univalent. The two branches ϕ_1 and ϕ_2 map D onto domains $D_1^* = \phi_1(D)$ and $D_2^* = \phi_2(D)$ that have no overlap. Otherwise there would exist two points $z_{1,2} \in D$ so that $\phi_1(z_1) = \phi_2(z_2)$ which would in turn imply (2.11) so that $z_1 = z_2$ and then $\phi_1(z_1) = -\phi_2(z_2)$. This is a contradiction since $\phi_\nu 9z) \neq 0$ in D.

The domain D_2^* contains a disk $\{|w - w_0| < \rho\}$. Hence ϕ_1 does not take values in this disk. Therefore the function

$$f_1(z) = \frac{\rho}{\phi_1(z) - w_0} \tag{2.12}$$

is clearly holomorphic and univalent in D and takes values inside the unit disk: we have $|f_1(z)| \leq 1$ for all $z \in D$.

(ii) Let us denote by S the family of functions that are holomorphic and univalent in D, and are bounded by one in absolute value. This family is not empty since it contains the function f_1. It is compact by the Montel theorem. The subset S_1 of the family S that consists of all functions $f \in S$ such that

$$|f'(a)| \geq |f_1'(a)| > 0 \tag{2.13}$$

at some fixed point $a \in D$ is sequentially compact. Indeed Corollary 2.16 implies that the limit of any sequence of functions $f_n \in S_1$ that converges on any compact subset K of D may be only a univalent function (and hence belong to S_1) or be a constant but the latter case is ruled out by (2.13).

Consider the functional $J(f) = |f'(a)|$ defined on S_1. It is a continuous functional as was shown in Example 2.12. Therefore there exists a function $f_0 \in S$ that attains its

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[8] In general we may define a univalent branch of $\sqrt{\dfrac{z-a}{z-b}}$ in a domain D if neither a nor b are in D.

maximum, that is, such that

$$|f'(a)| \le |f_0'(a)| \tag{2.14}$$

for all $f \in S$.

(iii) The function $f_0 \in S_1$ maps D conformally into the unit disk U. Let us show that $f_0(a) = 0$. Otherwise, the function

$$g(z) = \frac{f_0(z) - f_0(a)}{1 - \overline{f_0(a)} f_0(z)}$$

would belong to S_1 and have $|g'(a)| = \frac{1}{1-|f_0(a)|^2}|f_0'(a)| > |f_0'(a)|$, contrary to the extremum property (2.14) of the function f.

Finally, let us show that f_0 maps D onto U. Indeed, let f_0 omit some value $b \in U$. Then $b \neq 0$ since $f_0(a) = 0$. However, the value $b^* = 1/b$ is also not taken by f_0 in D since $|b^*| > 1$. Therefore one may define in D a single valued branch of the square root

$$\psi(z) = \sqrt{\frac{f_0(z) - b}{1 - \bar{b} f_0(z)}} \tag{2.15}$$

that also belongs to S: it is univalent for the same reason as in the square root in part (i), and $|\psi(z)| \le 1$. However, then the function

$$h(z) = \frac{\psi(z) - \psi(a)}{1 - \overline{\psi(a)}\psi(z)}$$

also belongs to S. We have $|h'(a)| = \dfrac{1+|b|}{2\sqrt{|b|}}|f_0'(a)|$. However, $1 + |b| > 2\sqrt{|b|}$ since $|b| < 1$ and thus $h \in S_1$ and $|h'(a)| > |f_0'(a)|$ contrary to the extremal property of f_0. $\square$

The Riemann theorem implies that any two simply connected domains D_1 and D_2 with boundaries that contain more than one point are conformally equivalent. Indeed, as we have shown there exist conformal isomorphisms $f_j : D_j \to U$ of these domains onto the unit disk. Then $f = f_2^{-1} \circ f_1$ is a conformal isomorphism between D_1 and D_2. Theorem 2.4 implies that an isomorphism $f : D_1 \to D_2$ is uniquely determined by a normalization

$$f(z_0) = w_0, \quad \arg f'(z_0) = \theta, \tag{2.16}$$

where $z_0 \in D_1$, $w_0 \in D_2$ and θ is a real number.

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