Homework # 4.

1. Recall that

$$Mf(x) = \sup_{\delta>0} \frac{1}{\mu(B(y,\delta))} \int_{B(y,\delta)} |f(y)|dy.$$

with the supremum taken over all balls $B(x,\delta)$ and

$$\tilde{M}f(x) = \sup_{\delta>0} \frac{1}{\mu(B(y,\delta))} \int_{B(y,\delta)} |f(y)|dy.$$

with the supremum taken over all balls $B(y,\delta)$ such that $x \in B(y,\delta)$. Show that if $f \in L^1$ and $f \neq 0$ identically then there exist $C, R > 0$ so that $\tilde{M}f(x) \geq C|x|^{-n}$ (here n is the space dimension) for all x with $|x| \geq R$. Hence $m(\{x : \tilde{M}(x) > \alpha\}) \geq C'/\alpha$. Show also that $M(x) \leq \tilde{M}(x) \leq 2^n M(x)$.

2. Show that (i) if $\nu \ll \mu \ll \lambda$ then

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda};$$

and (ii) if $\nu \ll \mu$ and $\mu \ll \mu$ then

$$\frac{d\nu}{d\mu} = \left(\frac{d\mu}{d\nu}\right)^{-1}.$$

3. Suppose that $\{g_n\}$ is a sequence of positive continuous functions on $[0,1]$, μ is a positive Borel measure on $[0,1]$ and that (i) $\lim_{n\to\infty} g_n(x) = 0$ a.e., (ii) $\int_0^1 g_n dx = 1$ for all n and (iii) $\lim_{n\to\infty} \int_0^1 f g_n dx = \int_0^1 f d\mu$ for every continuous function $f \in C[0,1]$. Does it follow that μ is mutually singular with respect to the Lebesgue measure?

4. Show that $|f(x)| \leq Mf(x)$ at every Lebesgue point of f if $f \in L^1(\mathbb{R}^n)$ (here Mf is the maximal function defined in problem 1).

5. Assume that a continuous function f has two periods s and t so that s/t is irrational. Show that $f = \text{const}$.

6. Suppose that μ_n is a sequence of Borel measures and

$$\mu(E) = \sum_{n=1}^{\infty} \mu_n(E).$$

Show that this series may be differentiated pointwise with respect to the Lebesgue measure, that is

$$D\mu(x) = \sum_{n=1}^{\infty} D\mu_n(x).$$

7. Let

$$\phi_0(t) = \begin{cases} 1, & x \in [0,1] \\ -1, & x \in [1,2) \end{cases}.$$

and define $\phi_n(t) = \phi_0(2^n t)$, $n \in \mathbb{N}$. Assume that $\sum |c_n|^2 < \infty$ and show that the series

$$\sum_{n=1}^{\infty} c_n \phi_n(t)$$

converges for almost every t. Hint: $\{\phi_n\}$ form an orthonormal set, hence the series converges to a function f in L^2 (not necessarily pointwise yet!). Then define $a = j/2^N$, $b = (j+1)/2^N$, $a < t < b$ and $s_N = c_1\phi_+ \cdots + c_N\phi_N$, then for $n > N$ we have

$$s_N(t) = \frac{1}{b-a}\int_a^b s_N dx = \frac{1}{b-a}\int_a^b s_n dx.$$

The right side converges to $\int_a^b f dx$ as $n \to \infty$. Show that our series converges then at all Lebesgue points of f.

8. A measurable function f is said to be in weak L^1 denoted by L^1_w if $m(\lambda) = \lambda \times |\{x : \ |f(x)| > \lambda\}|$ is a bounded function of $\lambda \geq 0$. Show that L^1_w contains L^1 but is larger than L^1.

9. Let $f(x) \in C(\mathbb{R})$, $f(x) > 0$ for $0 < x < 1$ and $f(x) = 0$ otherwise. Show that the function $h_c(x) = \sup_n\{n^c f(nx)\}$ is (i) in $L^1(\mathbb{R})$ if $c \in (0,1)$, (ii) is in $L^1_w(\mathbb{R})$ but not in $L^1(\mathbb{R})$ if $c = 1$, (ii) not in L^1_w if $c > 1$.