Final Exam

Books and notes are permitted but consulting other students is not allowed. If you use a source you should cite it precisely, and also put down the whole argument by yourself, as I do not necessarily have access to it.

1. Suppose that the function $u(x, t; K, T)$ solves the Black-Scholes equation (in (x, t)) with terminal data $u(x, T; K, T) = (x - K)_+^2$. Derive a PDE (and either the terminal or initial condition) for the function $v(K, T) = u(x, t; K, T)$ in the variables K and T.

2. Consider the following terminal value problem for $w(x, t)$:

$$\begin{aligned} w_t + w_{xx} &= 0, \quad \text{for } t < T,\ x \in [1, 2] \\ w(x, T) &= \phi(x), \quad x \in [1, 2] \quad \text{(terminal condition)} \\ w(1, t) = 0 &= w(2, t), \quad t \leq T \quad \text{(boundary condition)} \end{aligned}$$

   Suppose $\phi(x)$ is smooth and compactly supported within $(1, 2)$. Use the method of separation of variables to find a series representation for $w(x, t)$.

3. For $s \geq t \geq 0$ and $x \in \mathbb{R}$, let $Y_s(\omega) : [t, T] \times \Omega \to \mathbb{R}$ satisfy

$$\begin{aligned} dY_s &= (\alpha_s + v(Y_s))\, ds + \sigma dB_s, \quad s \geq t \\ Y_t &= x \end{aligned} \tag{0.1}$$

   where $\alpha_s(\omega) \in \mathcal{A}_{t,T}$ is a control process. $\mathcal{A}_{t,T}$ is the set of admissible controls, which are stochastic processes adapted to the Brownian filtration and taking values in the compact set $A \subset \mathbb{R}$. B_s is a standard one-dimensional Brownian motion, $\sigma > 0$ is a constant. Consider the value function

$$u(x, t) := \max_{\alpha \in \mathcal{A}_{t,T}} E\left[\int_t^T -\frac{1}{4}(\alpha_s)^6\, ds + g(Y_T) \mid Y_t = x\right] \tag{0.2}$$

   Formally derive the Hamilton-Jacobi-Bellman equation and terminal condition for $u(x, t)$.

4. Consider the eigenvalue problem

$$-\varepsilon u'' + V(x)u = \lambda u, \quad -1 \leq x \leq 1,$$

   with the boundary conitions $u(-1) = u(1) = 0$. Here $\varepsilon \in (0, 1)$ is a small parameter. The general theory tells you that the eigenfunction corresponding to the smallest (also called principal) eigenvalue λ is non-negative. (i) Assume that $V(x) = 1$ for $|x| < 1/2$ and $V(x) = 0$ otherwise. Find the smallest eigenvalue λ and the corresponding eigenfunction explicitly and describe how they depend on the small parameter $\varepsilon > 0$. (ii) Assume that $V(x)$ is an even non-negative function that vanishes outside of the interval $(-1/2, 1/2)$, attains its maximum at $x = 0$, where $V(0) = 1$, and is decreasing on the interval $(0, 1/2)$. Find the asymptotic behavior of the principal eigenvalue and eigenfunction as $\varepsilon \to 0$. Hint: use the variational principle for λ:

$$\lambda = \inf_\phi \frac{\varepsilon \int |\nabla\phi|^2 dx + \int V(x)\phi^2(x)dx}{\int \phi^2(x)dx}.$$

5. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $\phi(x)$ a given smooth function that is compactly supported in Ω, and let $u(x)$ be a solution of the Dirichlet problem

$$-\Delta u + \nabla \phi(x) \cdot \nabla u = f(x), \quad x \in \Omega,$$

with the boundary condition $u = 0$ on $\partial\Omega$. Find a variational principle for $u(x)$.