

## Homework 1

Notation: given  $1 \leq p < +\infty$  we say that  $f \in L^p(\mathbb{R}^n)$  if the integral  $\int_{\mathbb{R}^n} |f(x)|^p dx < +\infty$  and denote  $\|f\|_p = (\int_{\mathbb{R}^n} |f(x)|^p dx)^{1/p}$ . For  $p = +\infty$  we do as follows: for a bounded function  $g(x)$  we set  $\|g\|_\infty = \sup_{x \in \mathbb{R}^n} |g(x)|$ . For a function  $u(x, t)$  we denote  $\|u(t)\|_p = (\int_{\mathbb{R}^n} |u(x, t)|^p dx)^{1/p}$ .

- Suppose that  $g(x) \geq 0$  is continuous and bounded, but not constant. Suppose that  $u(x, t)$  solves the initial value problem

$$\begin{aligned} u_t &= u_{xx}, & x \in \mathbb{R}, t > 0 \\ u(x, 0) &= g(x), & x \in \mathbb{R} \end{aligned} \tag{0.1}$$

- Show that  $u(x, t) < \|g\|_\infty$  (strict inequality) for all  $x \in \mathbb{R}$  and  $t > 0$ .
  - Assume that  $g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ . Show that  $\|u(t)\|_2 < \|g\|_2$  for all  $t > 0$ .
  - Assume in addition that  $g(x)$  changes sign. Show that  $\|u(t)\|_1 < \|g\|_1$  for all  $t > 0$ .
- Sometimes it is possible to solve a PDE by transforming the equation using a change of variables. These exercises demonstrate some simple transformations of the heat equation. Suppose that  $u(x, t)$  satisfies the heat equation

$$u_t = u_{xx}, \quad x \in \mathbb{R}, t > 0 \tag{0.2}$$

- What equation does the function  $v(x, t) = u(\gamma x, \delta t)$  satisfy? Here  $\gamma$  and  $\delta$  are positive constants. Under what condition on  $\gamma$  and  $\delta$  does  $v$  satisfy the same equation as  $u$  (other than the obvious case  $\gamma = \delta = 1$ )?
- What equation does the function  $v(x, t) = e^{\alpha x} u(x, t)$  satisfy? Here  $\alpha$  is a constant.
- What equation does the function  $v(x, t) = e^{\beta t} u(x, t)$  satisfy? Here  $\beta$  is a constant.
- Suppose  $\lambda$  and  $c$  are constants. Use ideas from part (i), (ii), and (iii) to solve the initial value problem

$$\begin{aligned} v_t &= \sigma v_{xx} + \lambda v_x + cv, & x \in \mathbb{R}, t > 0 \\ v(x, 0) &= g(x) \end{aligned} \tag{0.3}$$

Here  $\sigma, \lambda, c$  are constants with  $\sigma > 0$ . Remember that transformations effect the initial data, too.

- Suppose that  $u(x, t)$  satisfies the heat equation

$$u_t = u_{xx}, \quad x \in \mathbb{R}, t > 0 \tag{0.4}$$

with initial condition  $u(x, 0) = g(x) > 0$  for all  $x \in \mathbb{R}$ .

- What nonlinear equation and initial condition does the function  $v(x, t) = \log(u(x, t))$  satisfy?
- Suppose that  $u(x, t) = v(e^{\lambda x}, t)$  for some function  $v(y, t)$  where  $y = e^{\lambda x}$ . What initial value problem does  $v(y, t)$  satisfy? Specify the PDE, the initial condition, and the domain in terms of  $y$  and  $t$ .

4. Suppose that  $u(x, t)$  satisfies

$$\begin{aligned}u_t &= u_{xx} - 2u, & x \in (0, \pi), t \in (0, T] \\u(x, 0) &= \sin(x) + 3, & \forall x \in [0, \pi] \\u(0, t) &= 3 = u(\pi, t) & \forall t \in [0, T]\end{aligned}$$

Can  $u(x, t) = 0$  for some  $x \in (0, \pi)$  and  $t \in (0, T)$ ? Justify your answer.

5. (a) Solve the boundary value problem

$$\begin{aligned}u_t &= u_{xx}, & x > 0, t > 0 \\u(x, 0) &= 0, & x > 0 \\u(0, t) &= 1\end{aligned}$$

Notice that  $x$  varies on the half line  $\{x > 0\}$  and that the boundary condition is at  $x = 0$ . Write the solution as an integral expression.

(b) For the solution in part (a), describe  $u(x, t)$  qualitatively. Show that for any  $x > 0$ ,

$$|u(x, t) - 1| \leq \frac{C}{\sqrt{t}}$$

for some constant  $C > 0$ , depending on  $x$  but not on  $t$ .

Interlude: the Fourier transform of a function  $f(x)$  is

$$\hat{f}(\xi) = \int e^{-2\pi i x \cdot \xi} f(x) dx.$$

The Plancherel identity says that

$$\int |\hat{f}(\xi)|^2 d\xi = \int |f(x)|^2 dx.$$

6. Let  $f \in L^1 \cap L^2(\mathbb{R}^n)$  and assume that  $\nabla f \in L^2(\mathbb{R}^n)$ .

(i) Show that  $|\hat{f}(\xi)| \leq \|f\|_1$  for all  $\xi \in \mathbb{R}^n$ .

(ii) Show that the Fourier transform of  $\nabla f$  is  $2\pi i \xi \hat{f}(\xi)$ . Use the Plancherel identity to show that

$$\int |\nabla f(x)|^2 dx = \int |2\pi \xi|^2 |\hat{f}(\xi)|^2 d\xi.$$

(iii\*) Use the Plancherel identity to show that for any  $R > 0$  we have

$$\int |f(x)|^2 dx \leq C_n R^n \left( \int |f(x)| dx \right)^2 + \frac{1}{(2\pi R)^2} \int |\nabla f(x)|^2 dx.$$

Here  $C_n$  is the volume of the  $n$ -dimensional sphere.

(iv\*) Use (iii) with an optimal  $R > 0$  to show that there exists a constant  $D_n > 0$  that depends only on the dimension  $n$  so that

$$\int |\nabla f|^2 dx \geq D_n \left( \int |f|^2 dx \right)^{1+2/n} \left( \int |f| dx \right)^{-4/n}.$$

7. Consider the heat equation with a variable diffusion coefficient  $a(x)$ , with a differentiable function  $a(x)$  such that  $1 < a(x) < 2$  for all  $x \in \mathbb{R}^n$

$$\frac{\partial u}{\partial t} = \nabla \cdot (a(x)\nabla u), \quad x \in \mathbb{R}^n$$

with the initial data  $u(0, x) = u_0(x) \in C_c^\infty(\mathbb{R}^n)$  with  $u_0(x) \geq 0$ .

- (i) Show that

$$\frac{1}{2} \frac{d}{dt} \int |u|^2 dx = - \int a(x) |\nabla u|^2 dx.$$

- (ii) Show that  $\int u(t, x) dx$  is independent of  $t$ .

- (iii\*) Use part (i) and the result of Problem 6(iv) to show that there exists a constant  $B_n > 0$  that depends only on dimension  $n$  so that

$$\|u(t)\|_{L^2(\mathbb{R}^n)}^2 \leq \frac{B_n}{t^{n/2}} \|u_0\|_{L^1(\mathbb{R}^n)}.$$

- (iv\*\*) Use the previous result to show that actually

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{B'_n}{t^{n/2}} \|u_0\|_{L^1(\mathbb{R}^n)},$$

with another constant  $B'_n$  which depends only on the spatial dimension.