Math 227 Winter 2009

Solutions to HW #1

Exercise 1.

i. Following the notes, we can assume the solution is unique and thus given by the integral discussed in class. Hence for any fixed (x,t):

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}} \Phi(x-y,t)\, g(y)\, dy \\ &= \int_{\mathbb{R}} \Phi(y,t)\, g(x-y)\, dy \\ &< \int_{\mathbb{R}} \Phi(y,t)\, \|g\|_\infty\, dy \\ &= \|g\|_\infty \int_{\mathbb{R}} \Phi(y,t)\, dy \\ &= \|g\|_\infty \end{aligned}$$

where the inequality follows from the fact that if $f\colon \mathbb{R} \to \mathbb{R}^+$ is a continuous, non-constant function, then $\int_{\mathbb{R}} f(y)\, dy > 0$. [Can you see that? Since it is not constant, there must exist an $a > 0$ such that $f(x) = a$ for some $x \in \mathbb{R}$. Then, by continuity, there is a $\delta > 0$ such that $f(y) \geqslant a/2$ for $y \in [x - \delta, x + \delta]$ and so $\int_{\mathbb{R}} f(y)\, dy \geqslant \delta\, a > 0$.] We took $f(y) = \Phi(y,t)\, \{\|g\|_\infty - g(x-y)\} \geqslant 0$ here.

ii. Similarly:

$$\begin{aligned} \|u(t)\|_2^2 = \int_{\mathbb{R}} u(x,t)^2\, dx &= \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} \Phi(y,t)\, g(x-y)\, dy \right\}^2 dx \\ &= \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} \Phi(y_1,t)\, g(x-y_1)\, dy_1 \right\} \left\{ \int_{\mathbb{R}} \Phi(y_2,t)\, g(x-y_2)\, dy_2 \right\} dx \\ &= \int_{\mathbb{R}} \Phi(y_1,t)\ \Phi(y_2,t) \int_{\mathbb{R}} g(x-y_1)\, g(x-y_2)\, dx\, dy_1\, dy_2 \\ &< \int_{\mathbb{R}} \Phi(y_1,t)\ \Phi(y_2,t)\ \|g\|_2^2\, dy_1\, dy_2 \\ &= \|g\|_2^2 \end{aligned}$$

where the inequality follows from an argument similar to that in (i), since by the Cauchy-Schwarz inequality

$$f(y_1,y_2) = \int g(x-y_1)\, g(x-y_2)\, dx - \int g(x)^2\, dx \leqslant 0$$

and moreover f is non-constant and clearly continuous,

iii. Here

$$\begin{aligned}
\|u(t)\|_1 = \int_{\mathbb{R}} |u(x,t)|\,dx &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \Phi(y,t)\, g(x-y)\,dy \right| dx \\
&< \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi(y,t)\, |g(x-y)| dy\, dx \\
&= \int_{\mathbb{R}} \Phi(y,t)\ dy\, \|g\|_1 \\
&= \|g\|_1
\end{aligned}$$

where the inequality follows from that fact that $|\int f(x)\,dx| < \int |f(x)|\,dx$ for any f that changes sign.

Exercise 2. All four sub-problems below require the same approach, namely first expressing derivatives of v in terms of derivatives of u and then eliminating the latter using the PDE that u satisfies.

i. Observe that $v_s(y,s) = \delta\, u_t(\gamma y, \delta s)$ and $v_{yy}(y,s) = \gamma^2\, u_{xx}(\gamma y, \delta s)$, hence

$$v_t = \frac{\delta}{\gamma^2}\, v_{xx}$$

and this equation is the same as that for u if and only if $\delta = \gamma^2$.

ii. Here $v_t(x,t) = e^{\alpha x}\, u_t(x,t)$, $v_x(x,t) = \alpha\, v(x,t) + e^{\alpha x} u_x(x,t)$, and differentiating again

$$v_{xx}(x,t) = \alpha^2\, v(x,t) + 2\,\alpha\, e^{\alpha x}\, u_x(x,t) + e^{\alpha x}\, u_{xx}(x,t)\,.$$

Hence $e^{\alpha x}\, u_x(x,t) = v_x(x,t) - \alpha\, v(x,t)$ and putting all of this together gives

$$v_t\,(x,t) = e^{\alpha x}\, u_t(x,t) = e^{\alpha x}\, u_{xx}(x,t) = v_{xx}(x,t) - \alpha^2\, v(x,t) - 2\,\alpha\, [v_x(x,t) - \alpha\, v(x,t)],$$

finally yielding

$$v_t = v_{xx} - 2\,\alpha\, v_x + \alpha^2\, v.$$

iii. Here $v_t(x,t) = \beta\, v(x,t) + e^{\beta t}\, u_t(x,t)$ and $v_{xx}(x,t) = e^{\beta t}\, u_{xx}(x,t)$, so

$$v_t = v_{xx} + \beta\, v.$$

iv. We begin by simplifying the PDE

$$v_t = \sigma\, v_{xx} + \lambda\, v_x + c\, v.$$

Our idea is to somehow reverse the transformations above, i.e. to use (ii) to get rid of v_x, then (iii) to eliminate v, and finally (i) to remove the constant, thus reducing this PDE to the heat equation.

First, we set $u^{(1)}(x,t) = e^{-\alpha x}\, v(x,t)$ for some constant α to be specified soon. Then

$$\begin{aligned}
u_t^{(1)}(x,t) = e^{-\alpha x}\, v(x,t) &= e^{-\alpha x}\, \{\sigma\, v_{xx} + \lambda\, v_x + c\, v\,\} \\
&= e^{-\alpha x} \Big\{ \sigma[\alpha^2\, v(x,t) + 2\,\alpha\, e^{\alpha x}\, u_x^{(1)}(x,t) + e^{\alpha x}\, u_{xx}^{(1)}(x,t)] \\
&\qquad + \lambda\, [\alpha\, v(x,t) + e^{\alpha x} u_x^{(1)}(x,t)] + c\, v(x,t) \Big\}
\end{aligned}$$

and we see that if $2\,\sigma\,\alpha = -\,\lambda$, then the first derivative disappears and we are left with

$$u_t^{(1)}(x,t) = \ \sigma\, u_{xx}^{(1)}(x,t) + (\sigma\alpha^2 + \lambda\,\alpha + c)\, u^{(1)}(x,t),$$

where $\alpha = -\frac{\lambda}{2\,\sigma}$.

Second, we let $u^{(2)}(x,t)=e^{-\beta t}\,u^{(1)}(x,t)$, again for some constant β, and obtain

$$u_t^{(2)}(x,t)=e^{-\beta t}\Big[u_t^{(1)}(x,t)-\beta\,u(x,t)\Big] \;=\; e^{-\beta t}\Big\{\sigma\,u_{xx}^{(1)}(x,t)+(\sigma\alpha^2+\lambda\,\alpha+c-\beta)\,u^{(1)}(x,t)\Big\}.$$

For $\beta=\sigma\,\alpha^2+\lambda\,\alpha+c=\sigma\,\frac{\lambda^2}{4\,\sigma^2}-\frac{\lambda^2}{2\,\sigma}+c=c-\frac{\lambda^2}{4\,\sigma}$ the above simplifies to

$$u_t^{(2)}(x,t)=\sigma\,u_{xx}^{(2)}(x,t).$$

Finally, we let $u(x,t)=u^{(2)}(x,\delta\,t)$. Thus $u_t(x,t)=\delta\,\sigma\,u_{xx}(x,t)$ and setting $\delta=\sigma^{-1}$ we recover the heat equation

$$u_t=u_{xx}.$$

Working back to express u in terms of the original v gives

$$u(x,t)=e^{-\beta t/\sigma-\alpha x}\,v(x,t/\sigma)$$

and more interestingly

$$v(x,t)=e^{\beta t+\alpha x}\,u(x,\sigma\,t)=\exp\bigg\{\bigg(c-\frac{\lambda^2}{4\,\sigma}\bigg)t-\frac{\lambda}{2\,\sigma}\,x\bigg\}\,u(x,\sigma\,t).$$

Now, the last step in determing v is to find u, which will have to satisfy

$$\begin{cases} u_t=u_{xx} & x\in\mathbb{R}, t>0\\ u(x,0)=e^{-\alpha x}\,v(x,0)=e^{-\alpha x}\,g(x) & \end{cases}$$

Fortunately, the solution to this is given in the notes as

$$u(x,t)=\int_{\mathbb{R}}\Phi(x-y,t)e^{-\alpha y}\,g(y)\,d\,y\,,$$

leading us to conclude that

$$v(x,t)=\exp\bigg\{\bigg(c-\frac{\lambda^2}{4\,\sigma}\bigg)t-\frac{\lambda}{2\,\sigma}\,x\bigg\}\int_{\mathbb{R}}\Phi(x-y,\sigma\,t)\,e^{\lambda y/(2\sigma)}\,g(y)\,d\,y\,.$$

Exercise 3. The idea here is almost the same as in the previous exercise.

i. Since $u=e^v$, differentiating gives

$$\begin{aligned} u_t &= e^v\,v_t\\ u_x &= e^v\,v_x\\ u_{xx} &= e^v\,(v_x)^2+e^v\,v_{xx} \end{aligned}$$

Combining these with the heat equation for u and cancelling the exponentials gives $v_t=v_{xx}+(v_x)^2$ and the initial condition becomes $v(x,0)=\log\left(g(x,0)\right)$.

ii. Differentiating gives

$$\begin{aligned} u_t(x,t) &= v_t(e^{\lambda x},t)\\ u_x(x,t) &= \lambda\,e^{\lambda x}\,v_x(e^{\lambda x},t)\\ u_{xx}(x,t) &= \lambda^2\,e^{\lambda x}\,v_x(e^{\lambda x},t)+(\lambda\,e^{\lambda x})^2\,v_{xx}(e^{\lambda x},t). \end{aligned}$$

Combining these and substituting $y=e^{\lambda x}$ gives $v_t(y,t)=\lambda^2\,y\,v_y(y,t)+(\lambda\,y)^2\,v_{yy}(y,t)$. The domain $\mathbb{R}\times\mathbb{R}^+$ is mapped onto $\mathbb{R}^+\times\mathbb{R}^+$ and thus v satisfies the following initial value problem:

$$\begin{cases} v_t=(\lambda\,y)^2\,v_{yy}+\lambda^2\,y\,v_y & y>0, t>0\\ v(y,0)=g\bigg(\dfrac{\log y}{\lambda},0\bigg) & y>0 \end{cases}$$

Exercise 4. Observe that, by **Exercise 2(iii)** we have $u(x,t)=e^{-2t}\,v(x,t)$ where v satsifies

$$\begin{cases} v_t = v_{xx} & x\in(0,\pi), t\in(0,T] \\ v(x,0)=\sin(x)+3 & x\in[0,\pi] \\ v(0,t)=3\,e^{2t}=v(\pi,t) & t\in[0,T] \end{cases}$$

Notice that v satsifies the maximum principle (as stated in the notes), so in particular it achieves its minimum on the boundary. Since v is always positive on the boundary (i.e. $v(x,0)>0$ and $v(0,t)=v(\pi, t)>0$), it follows that $v(x,t)>0$ for all $x\in(0,\pi), t\in(0,T]$ and the same holds true for u, being a positive multiple of v. Thus we cannot have $u(x,t)=0$ in the domain.

Exercise 5.

a) Define $v:=u-1$ which thus satisfies

$$\begin{cases} v_t = v_{xx} & x>0, t>0 \\ v(x,0)=-1 & x>0 \\ v(0,t)=0 & t>0 \end{cases}$$

Thus using the integral representation from the notes for v we can represent it as

$$u(x,t)=1-\frac{1}{\sqrt{4\pi t}}\int_0^\infty\left(e^{-\frac{|x-y|^2}{4t}}-e^{-\frac{|x+y|^2}{4t}}\right)dy \tag{1}$$

b) Although it is not clear how to evaluate the above integral, we can still recognize that

$$u(x,t)=1-\mathbb{P}\left(X>-\frac{x}{\sqrt{2t}}\right)+\mathbb{P}\left(X>\frac{x}{\sqrt{2t}}\right)=\mathbb{P}\left(|X|>\frac{x}{\sqrt{2t}}\right)$$

for a random variable $X\sim N(0,1)$. [Can you see this? Work out ALL the details, so that you are completely comfortable switching between the heat kernel and the probabilistic interpretations.]

From this representation we easily show that

$$|u(x,t)-1|=\mathbb{P}\left(|X|<\frac{x}{\sqrt{2t}}\right)=2\int_0^{x/\sqrt{2t}}\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy\leqslant 2\,\frac{x}{\sqrt{2t}}\,\frac{1}{\sqrt{2\pi}}=\frac{C(x)}{\sqrt{t}}$$

as required.

It is instructive, however, to see how you can also deduce this straight from the original expression. Observe that rearranging (1) we get

$$\begin{aligned} 1-u(x,t) &= \frac{1}{\sqrt{4\pi t}}\int_0^\infty\left(e^{-\frac{|x-y|^2}{4t}}-e^{-\frac{|x+y|^2}{4t}}\right)dy \\ &= \frac{1}{\sqrt{4\pi t}}\left\{\int_0^\infty e^{-\frac{|x-y|^2}{4t}}\,dy-\int_{-\infty}^0 e^{-\frac{|x-y|^2}{4t}}\,dy\right\} \\ &= \int_{\mathbb{R}}\Phi(x-y,t)\,g(y)\,dy \end{aligned}$$

where $g(y)=1$ for $y\geqslant 0$ and $g(y)=-1$ for $y<0$. Since the funtion $y\mapsto\Phi(x-y,t)$ is symmetric about $y=x\geqslant 0$, we have

$$\int_{-\infty}^0\Phi(x-y,t)\,g(y)\,dy=-\int_{-\infty}^0\Phi(x-y,t)\,dy=-\int_{2x}^\infty\Phi(x-y,t)\,dy=-\int_{2x}^\infty\Phi(x-y,t)\;g(y)dy$$

and so

$$|u(x,t)-1|=\int_0^{2x}\Phi(x-y,t)\,g(y)\,dy\leqslant\frac{2\,x}{\sqrt{4\pi t}}=\frac{C(x)}{\sqrt{t}}$$

as before.

Exercise 6.

i. Clearly

$$|\hat{f}(\xi)| = \left| \int e^{-2\pi i x\cdot\xi} f(x)\,dx \right| \leqslant \int |e^{-2\pi i x\cdot\xi}|\,|f(x)|\,dx = \int |f(x)|\,dx = \|f\|_1$$

ii. Looking at the jth component and integrating by parts gives

$$\begin{aligned}[\widehat{\nabla f}(\xi)]_j = \int e^{-2\pi i x\cdot\xi}\partial_{x_j} f(x)\,dx &= e^{-2\pi i x\cdot\xi} f(x)\big|_{x=-\infty}^{x=+\infty} - \int (-2\pi i\,\xi_j)\,e^{-2\pi i x\cdot\xi} f(x)\,dx \\ &= 2\pi i\,\xi_j \int e^{-2\pi i x\cdot\xi} f(x)\,dx = 2\pi i\,\xi_j \hat{f}(x)\end{aligned}$$

where we use the fact that $f(x)\to 0$ as $|x|\to\infty$. Hence

$$\int |\nabla f(x)|^2 dx = \int |\widehat{\nabla f}(\xi)|^2 d\xi = \int |2\pi\xi\,\hat{f}(\xi)|^2 d\xi$$

as desired, where the first equality is an application of the Plancherel identity.

iii. Here

$$\begin{aligned}\int |f(x)|^2 dx = \int |\hat{f}(\xi)|^2 d\xi &= \int_{|\xi|<R} |\hat{f}(\xi)|^2 d\xi + \int_{|\xi|\geqslant R} |\hat{f}(\xi)|^2 d\xi \\ &\leqslant \int_{|\xi|<R} \|f\|_1^2\,d\xi + \int \frac{|2\pi\xi|^2}{|2\pi R|^2}|\hat{f}(\xi)|^2 d\xi \\ &= C_n R^n\,\|f\|_1^2 + \frac{1}{(2\pi R)^2}\int |\nabla f(x)|^2 dx\end{aligned}$$

where we used the fact that from (i) $\sup_\xi |\hat{f}(\xi)| \leqslant \|f\|_1$.

iv. It follows that

$$\int |\nabla f(x)|^2 dx \;\geqslant\; (2\pi)^2\left\{\|f\|_2^2\,R^2 - C_n\,\|f\|_1^2\,R^{n+2}\right\},$$

so equating the derivative of the RHS to zero gives

$$2\,\|f\|_2^2\,\overline{R} - (n+2)\,C_n\|f\|_1^2\,\overline{R}^{\,n+1} = 0$$

and so

$$\overline{R}^{\,n} = \frac{2\,\|f\|_2^2}{(n+2)\,C_n\,\|f\|_1^2}.$$

Evaluating the inequality at this particular R yields

$$\begin{aligned}\int |\nabla f(x)|^2 dx &\geqslant (2\pi)^2\left\{\|f\|_2^2 - C_n\,\|f\|_1^2\frac{2\,\|f\|_2^2}{(n+2)\,C_n\,\|f\|_1^2}\right\}\left(\frac{2\,\|f\|_2^2}{(n+2)\,C_n\,\|f\|_1^2}\right)^{2/n} \\ &= D_n\,\|f\|_2^2\left(\frac{\|f\|_2^2}{\|f\|_1^2}\right)^{2/n} \\ &= D_n\|f\|_2^{2(1+2/n)}\,\|f\|_1^{-4/n}\end{aligned}$$

as desired.

Exercise 7.

i. Interchanging the derivative and the integral

$$\frac{1}{2}\frac{d}{d\,t}\int\;|u|^2\,dx=\int u\,\partial_t\,u\,dx\;\;=\;\;\int u\,\nabla\cdot(a(x)\,\nabla u)\,dx$$

$$=\;\int\;\nabla\cdot(u\,a(x)\,\nabla u)\,dx-\int\;\nabla u\cdot(a(x)\,\nabla u)\,dx$$

$$=\;-\int a(x)\,|\nabla u|^2\,dx$$

where the last step follows by Divergence Theorem, since both u and ∇u vanish at infinity sufficiently fast.

ii. Notice that by the Divergence Theorem again

$$\frac{d}{d\,t}\int u(x,t)dx=\int\partial_t\,u(x,t)\,dx\;\;=\;\;\int\;\nabla\cdot(a(x)\,\nabla u)\,dx=0,$$

hence $\int\;u(x,t)\,dx$ cannot depend on t.

iii. Applying the result from **Exercise 6(iv)** to u and using the fact that from (ii) we must have $\|u(t)\|_1\geqslant|\int\;u(x,t)\,dx|=|\int\;u_0(x)\,dx|=\|u_0\|_1$ (since $u_0\geqslant0$) for all t yields

$$\int\;|\nabla u|^2dx\geqslant D_n\|u\|_2^{2(1+2/n)}\,\|u_0\|_1^{-4/n}.$$

Hence since $1<a(x)<2$ we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2\leqslant-\int\;|\nabla u|^2dx\leqslant-D_n\|u\|_2^{2(1+2/n)}\,\|u_0\|_1^{-4/n}$$

and so

$$\|u\|_2^{2(-1-2/n)}\,\frac{d}{dt}\|u\|_2^2\;\leqslant\;-2\,D_n\,\|u_0\|_1^{-4/n}$$

and integrating with respect to t gives

$$-\frac{n}{2}\left\{\|u(t)\|_2^{2(-2/n)}-\|u_0\|_2^{2(-2/n)}\right\}\leqslant-2\,D_n\,\|u_0\|_1^{-4/n}\,t$$

so that

$$\|u(t)\|_2^{2(-2/n)}\geqslant\frac{4}{n}\,D_n\,\|u_0\|_1^{-4/n}\,t+\|u_0\|_2^{2(-2/n)}\geqslant\frac{4}{n}\,D_n\,\|u_0\|_1^{-4/n}\,t$$

and

$$\|u(t)\|_2^2\leqslant\frac{B_n}{t^{n/2}}\,\|u_0\|_1^2,$$

which after taking square root becomes

$$\|u(t)\|_2\leqslant\sqrt{\frac{B_n}{t^{n/2}}}\|u_0\|_1\,.$$

iv. Consider the linear operator $S(t)$: $L_1\rightarrow L_2$ given by $u_0\mapsto u(t)$. The last inequality of part (iii) shows that this is a bounded operator between the two spaces, so its adjoint $S(t)^*$: $L_2\rightarrow L_\infty$ is a bounded operator as well with the same norm. Now the adjoint satisfies

$$\int u_0\,S(t)^*\,[\varphi]dx\;\;=\;\;\int S(t)[u_0]\,\varphi\,dx=\int u(t)\,\varphi\,dx$$

where $\varphi \in L^2$. Since $u_0 \in C_c^\infty(\mathbb{R}^n)$, we can take $\varphi = u_0$ from which follows that in fact $S(t)^*[u_0] = u(t)$, so that $S(t)^*$ is the same operator as $S(t)$, which is thus bounded as an operator from L_2 to L_∞. Finally

$$S(t)[u_0] \;=\; S(t/2)\,[S(t/2)[u_0]]$$

so that

$$\begin{aligned}
\|S(t)\|_{L_1\to L_\infty} &\leqslant \|S(t/2)\|_{L_1\to L_2}\,\|S(t/2)\|_{L_2\to L_\infty} \\
&\leqslant \sqrt{\frac{B_n}{(t/2)^{n/2}}}\sqrt{\frac{B_n}{(t/2)^{n/2}}} \\
&= \frac{B_n'}{t^{n/2}}.
\end{aligned}$$